Citation:
Abrahamson, D., & Howison, M. L. (2008, December). *Kinemathics: Kinetically induced mathematical learning.* Presentation and workshop at the UC Berkeley Gesture Study Group (Eve Sweetser, Director), December 5, 2008. http://edrl.berkeley.edu/projects/kinemathics/MIT.mov

Kinemathics: Kinetically Induced Mathematical Learning—Overview of Rationale[1]

Dor Abrahamson & Mark Howison
Embodied Design Research Laboratory
University of California, Berkeley

We are investigating conjectures central to research on embodied mathematical cognition. Specifically, we are carrying out an empirical study designed to evaluate the plausibility of theoretical models that argue for an embodied basis of mathematical learning and reasoning. Prior arguments for embodied mathematics either build on theoretical analyses of human reasoning (Barsalou, 1999; Goldin, 1987; Lakoff & Núñez, 2000), empirical studies of human's activity in general (Barsalou, 2008; Hatano, Miyake, & Binks, 1977), interpretations specifically of mathematics student behaviors (Fuson & Abrahamson, 2005; Nemirovsky, Tierney, & Wright, 1998), or, in particular, evidence of gestures accompanying speech acts in the solution of mathematical problems (Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999; Edwards, Radford, & Arzarello, 2009). Whereas these studies furnish strong support for the potential viability of the embodied conjecture, and whereas they have demonstrated the plausibility of an embodied substrate for working memory (in-the-moment cognitive operations), they have not established conclusively a *sine qua non* role of multi-modal imagery in the ontological development of mathematical concepts. Namely, it has yet to be shown compelling that imagery plays more than a supportive or epiphenomenal role in the communication of essentially abstract concepts.[2]

To the extent that the embodied conjecture obtains, a central challenge in evaluating the roles of imagery in mathematical reasoning has been that these psychological constructs are currently inaccessible for measurement. That is, because we cannot see people's "pictures in the head," we instead rely on indirect means of investigating these images. For instance, to examine the nature of procedural fluency deeply rooted in object-based visual–kinesthetic experiences, Hatano et al. (1977) had expert abacus users solve arithmetic problems in the absence of an actual abacus yet while performing interfering finger-tapping tasks (all but master users' performance at "mental abacus" degraded). Such experimental designs are well geared for studying mathematical imagery, because the researcher can confidently identify the experiential basis of the mental operations, and especially because the experience is based upon interaction with a known object. However, cross-sectional studies, such as the above, do not examine individual learning trajectories. Abrahamson (2004) introduced new mathematical objects into a middle-school

[1] This work is funded in part by a University of California, Berkeley Committee-on-Research Faculty Research Grant to Dor Abrahamson.
[2] See also D. L. Schwartz and Black (1999) for investigations of imagistic quantitative reasoning.

classroom and examined students' gestural acts as a window onto the imagery underlying their mathematical learning. Yet, it is difficult to appraise whether these gestures, performed in authentic classroom discussions, were pivotal to the students' reasoning or only epiphenomenal manifestations inherent to discursive norms. Therefore, a study is needed that both monitors individual object-based learning and does so through an experimental design that confidently implicates this object as the source of elicited imagery. Thus, rather than seeking out an existing mediating physical artifact, such as the abacus, or waiting for spontaneous gestures to emerge, such as in mathematical argumentation, we take a more proactive approach in this study by directly inducing an unfamiliar image then asking participants to begin reasoning with it mathematically. Prior research has not examined the *initial* construction of imagery from multi-modal embodied experiences nor how it begins to support mathematical reasoning.

Our research begins with the premise that mathematical learning is indeed image-based and mathematical reasoning is image-simulated (Case & Okamoto, 1996; Dahaene, 1997; Freudenthal, 1986; Lakoff & Núñez, 2000; Martin, 2008; Nemirovsky & Borba, 2004; Nemirovsky & Ferrara, in press; Nemirovsky et al., 1998; Pirie & Kieren, 1994; Presmeg, 2006). We further assume that people who demonstrate difficulty in understanding a particular mathematical concept nevertheless possess the cognitive wherewithal for leveraging imagery toward mathematical understanding, and conjecture that what these people lack is instead the initial, requisite image for that concept. Implicit in our approach is the view that teaching is a means of creating opportunities for students to develop a family of images that become resources for reasoning about a class of mathematical situations. Our conjecture is that *some mathematical concepts may be difficult to learn precisely because our everyday experience fails to provide adequate imagery for those specific concepts.* To evaluate this conjecture, we are conducting a study in which we first provide students with a "ready-made" visual–kinesthetic basis for developing imagery pertaining to a difficult mathematical concept and then measure the effects of such provision on their understanding of the targeted mathematical concept. Thus, our approach—which we describe as "image-before-concept"—is to kindle and investigate the incipient processes by which multi-modal imagery is developed, then to monitor how that imagery is subsequently leveraged as a cognitive resource for constructing and applying mathematical knowledge. So doing, we are also problematizing claims that students' "everyday sensual experience" (Botzer & Yerushalmi, 2008) is a critical developmental resource for embodied knowing, as the experience we provide our participants is extra-ordinary, contrived, and perhaps necessarily so due to the scarcity or even absence of its everyday embodied manifestation.

The specific content we target is the notion of a *proportional progression*, e.g., a sequence of equivalent fractions whose numerators and denominators respectively increase by fixed amounts at each count (i.e. the sequence 2/3, 4/6, 6/9, etc.). Our work builds on a pilot study conducted by Fuson and Abrahamson (2005), which demonstrated that students' conceptual difficulties in understanding such progressions coincided with their physical inability to act out the progressions with their hands (e.g., one hand continuously "grows" by 2 units every second *while* the other hand simultaneously grows by 3). Many students maintained a constant distance between their moving hands, acting out an additive rather than multiplicative relationship—a physical performance error that is reflected in mathematical errors reported throughout the literature on children's development of rational number concepts (e.g., De Bock, 2002; Gelman

& Gallistel, 1978; Karplus, Pulos, & Stage, 1983; Lamon, 2007; Noelting, 1980a, 1980b; J. L. Schwartz, 1988; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004; Vergnaud, 1983). We conjecture that students' difficulties with proportionality, and in particular their additive "same difference" errors, stem from their lack of a suitable dynamic image in which to ground their understanding of proportionality. Furthermore, it is unlikely that students construct the relevant imagery spontaneously because of the challenge of ambidextrously executing the *physical* actions themselves. Therefore, we decided to create an opportunity for study participants to undergo this ambidextrous multi-modal embodied experience—initially experiencing the physical concept passively and then gradually assuming physical command of the motion—thus allowing us to subsequently monitor any personal construction of meaning for this physical action.

The initial product of our experimental paradigm is a *gesturoid*—an embodied artifact for mathematical inquiry—that enables us to study how meaning for a mathematical notion emerges from, or vis-à-vis, an embodied sense of that notion. We refer to our artifact as a gesturoid because it has gesture-like properties yet is initially devoid of representational intentionality: it is instigated as a passive, physically-induced experience that has no function beyond the immediate discursive context, yet a participant may then assume agency in performing this physical action and it may acquire meaning within a familiar system of practice. One difference between a gesturoid and a physical manipulatable artifact such as the abacus is that the gesturoid does not admit straight-forward indexical semiotic acts. That is, pointing to parts of the gesturoid is difficult because it is moving and ephemeral, and anyway both hands are involved in performing this particular gesturoid so that neither hand it available for deixis. In contrast, an abacus functions almost solely on simple dislocational–indexical operations, configuring beads to indicate number, which effect spatial encodings that are cognitively ergonomic for performing arithmetic. A crucial research question is,

- What, if any, is the relation between a capacity to perform a physical action and the construction of the mathematical meaning it may be taken to represent?;

Moreover, because the gesturoid is inherently and a priori embodied, it throws into question the learning affordances of various instructional media. In particular,

- Is the body, as a medium, more conducive for constructing a multi-modal mathematical image as compared to artifacts embedded in other media, such as a video or textbook?

We hypothesize that just because a physical activity is designed to embody a mathematical notion, we cannot assume that performing this activity necessarily implies an understanding of the notion, just as walking or breathing do not in-and-of-themselves bespeak rhyme, rate, or reason. This is an alternative a naïve strong claim that performance indicates understanding, or that doing is knowing. However, we believe that, at the very least, practicing the physical activity creates a context for constructing mathematical meaning through the guided re-inventing of the mathematical notion and systematic coordination with normative mathematical semiotic means, such as diagrams and symbolical notation. More pointedly, the gesturoid is first conducive for taking on meaning when it has become a gesture, gesticulation, or—semantics aside—an intentional choreographed coordination whose systematicity can be articulated.

How does our gesturoid activity relate to extant mathematical conceptualizations of proportionality, as seen in standard curricular units? There are quite a number of instructional designs for the topic of proportionality (Lamon, 2007). Whereas these range along many superficial parameters, they differ with respect to how proportional relations are modeled and the consequence of this modeling for the possibility of an embodied experience of proportion. In particular, these models differ with respect to their construction of the equivalence between the two ratios involved in the proportion—whether this equivalence is rooted in an acknowledgement of a constant *process* (e.g., same pair of quantities added over and over) or in identical sensory perceptions of the *product* (e.g., same sweetness).

Vergnaud (1983) differentiates between two types of situated X:Y ratios. He names *isomorphism of measure* cases in which X and Y are functionally related but phenomenologically separate, such that there is no immediate meaning to a 'mix' of the two measures. Such is the case of a 3:5 ratio between the number of boys and girls in a classroom. In this classical "recipe" model—e.g., 3 units of X per 5 units of Y—two separate pools of objects increase as the consequence of iterated adding of fixed quotas, yielding 3 & 5, then 6 & 10, then 9 & 15, etc. Proportional equivalence between two ratios is then defined as co-membership in a class of ratios, e.g., the 3 & 5 class. This co-membership or equivalence can be determined either directly, through iterated adding/subtracting of the base ratio or through a multiplication shortcut, or through common kinship with a base ratio, e.g., 6:10 and 9:15 are related transitively through their respective relations with 3 & 5. An isomorphism-of-measure introduction to proportional equivalence would thus lean on an essentially algorithmic criterion relating to situations of iterated adding or scaling up of a recipe. Note, in particular, that there is no immediate perceptual basis for evaluating the proportionality of two ratios of this type, e.g., in comparing 6:10 and 9:15.

Vergnaud (1983) also speaks of *product of measures*, e.g., sweetness, area, slope, or chance—cases in which the multiplicative relation of two measures is instantiated in terms of an ontologically distinct phenomenon. For situations embodying products of measures, humans may have a sensory basis for evaluating proportionality, even before performing measurement and calculation. For example, if two liquids of different volume have the same sweetness, we can directly experience equivalence per se—though not yet *proportional* equivalence—through our gustatory sense. That is, a liquid solution of 6 units of sugar and 10 units of water would have the identical flavor as a respective mix of 9 and 15. Later, one might attend to these volume measures and re-articulate these pairs of numerical measures as ratios between indicated quantities of sugar and water.

Our ambidextrous activity might be viewed as a hybrid of these previous meta-designs for proportionality, carrying elements both of isomorphism and product of measures. Indeed, the activity a priori recruits an embodied sense of continuity for each of two individual progressions (the respective motions of the right hand and the left hand) rather than focusing on the ratio between the quantities, let alone the equivalence of two ratios. Thus, *within each* such continuity, we might experience constant products of measure, velocity being expressed as the ratio of distance and time. However, the two moving hands are a priori spatially and ontologically separate entities—a case of isomorphism of measure. Moreover, whereas the hand motion is inherently continuous, these two motions can be discretized and rendered as a recipe, e.g., rising 3 inches here for every 5 inches there, over and over. In fact, we expect that the process of discretizing the motion will be imperative for describing and reasoning with the gesturoid—this process would hence provide a context for re-describing the gesturoid symbolically.

We have presented here an artifact for mathematical investigation that is novel along at least two dimensions that, in turn, correspond to two study phases. First, the artifact induces and makes available for simulation a kinesthetic image that is potentially conducive for mathematical reasoning about proportional progression. Second, the artifact invites an instructional design for proportional progression that combines aspects of previous designs, providing both an isomorphism of measure and a perceptual basis for evaluating proportionality. With this artifact "in hand," we aim to test the claim that embodied imagery is a prerequisite for certain conceptual mathematical understandings, while also demonstrating a new design for proportionality that draws on such imagery. Our future publications will present the supplementary technological devices we are creating for this project[3] as well as our rationale, experimental design, and instruments for measuring and interpreting any learning gains that may be fostered by articulating the gesturoid.

References

Abrahamson, D. (2004). Embodied spatial articulation: A gesture perspective on student negotiation between kinesthetic schemas and epistemic forms in learning mathematics. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the Twenty Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 791 – 797). Toronto, Ontario: Preney.

Alibali, M. W., Bassok, M., Olseth, K. L., Syc, S. E., & Goldin-Meadow, S. (1999). Illuminating mental representations through speech and gesture. *Psychological Science, 10*, 327-333.

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences, 22*, 577–660.

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617-645.

Botzer, G., & Yerushalmy, M. (2008). Embodied semiotic activities and their role in the construction of mathematical meaning of motion graphs. *International Journal of Computers for Mathematical Learning, 13*(2), 111-134.

Case, R., & Okamoto, Y. (1996). The role of central conceptual structures in the development of children's thought. In *Monographs of the society for research in child development: Serial No. 246* (Vol. 61). Chicago: University of Chicago Press.

Dahaene, S. (1997). *The number sense*. Oxford: Oxford University Press.

De Bock, D. (2002). *The illusion of linearity: An empirical analysis of secondary school students' improper proportional reasoning in geometry problems*. Unpublished doctoral dissertation. Leuven, Belgium: University of Leuven.

Edwards, L., Radford, L., & Arzarello, F. (Eds.). (2009). Gestures and multimodality in the construction of mathematical meaning [Special Issue]. *Educational Studies in Mathematics* 70(2).

Freudenthal, H. (1986). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Fuson, K. C., & Abrahamson, D. (2005). Understanding ratio and proportion as an example of the Apprehending Zone and Conceptual-Phase problem-solving models. In J. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 213-234). New York: Psychology Press.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

[3] See http://edrl.berkeley.edu/projects/kinemathics/MIT.mov (and thank you, Becky Blessing!)

Goldin, G. A. (1987). Levels of language in mathematical problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 59-65). Hillsdale, NJ: Lawrence Erlbaum Associates.
Hatano, G., Miyake, Y., & Binks, M. (1977). Performance of expert abacus operators. *Cognition, 5*, 57-71.
Karplus, R., Pulos, S., & Stage, E. K. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 45-89). New York: Academic Press.
Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*: Basic Books.
Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age Publishing.
Martin, L. C. (2008). Folding back and the dynamical growth of mathematical understanding: Elaborating the Pirie-Kieren theory. *The Journal of Mathematical Behavior, 27*(1), 64-85.
Nemirovsky, R., & Borba, M. C. (2004). PME Special Issue: Bodily activity and imagination in mathematics learning. *Educational Studies in Mathematics, 57*, 303-321.
Nemirovsky, R., & Ferrara, F. (in press). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*.
Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119-172.
Noelting, G. (1980a). The development of proportional reasoning and the ratio concept Part 1--Differentiation of stages. *Educational Studies in Mathematics, 11*, 217-253.
Noelting, G. (1980b). The development of proportional reasoning and the ratio concept Part II--Problem-structure at successive stages: problem-solving strategies and the mechanism of adaptive restructuring. *Educational Studies in Mathematics, 11*(331-363).
Pirie, S. E. B., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*, 165-190.
Presmeg, N. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future* (pp. 205-235). Rotterdam: Sense Publishers.
Schwartz, D. L., & Black, T. (1999). Inferences through imagined actions: Knowing by simulated doing. *Journal of Experimental Psychology, 25*(1), 116-136.
Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41–52). Hillsdale, NJ: Erlbaum.
Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2004). Remedying secondary school students' illusion of linearity: a teaching experiment aiming at conceptual change. *Learning and Instruction, 14*, 485-501.
Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 127–174). New York: Academic Press.