Hooks and Shifts in Instrumented Mathematics Learning

José Gutiérrez, Dragan Trninic, Rosa Lee, & Dor Abrahamson

University of California, Berkeley

Doctoral Student
Graduate School of Education
University of California, Berkeley
Berkeley, CA 94720-1670
USA
e-mail: josefrancisco@berkeley.edu

Key phrases: cognition, conceptual change, embodied interaction, guided reinvention, mathematics education, proportion, proportional reasoning, socioculturalism, symbolic artifact.
Abstract
This paper presents and analyzes vignettes selected from empirical data gathered in a design-based research study investigating the microgenesis of proportional reasoning through technology-mediated embodied interaction. 22 Grades 4-6 students participated in individual or paired semi-structured clinical interviews in which they engaged in problem-solving activities involving remote manipulation of virtual objects on a computer screen. At different points throughout these interviews, the interviewer introduced various symbolic artifacts into the problem space, such as a Cartesian grid and y-axis numerals. In our on-going microgenetic analyses, we have observed that participants appropriated the artifacts as strategic or discursive means of accomplishing their goals; yet in the course of enacting these goal-oriented tasks, the participants recognized certain other embedded affordances in these symbolic artifacts as supporting the development of a more sophisticated strategy. We have characterized this two-step mediated-discovery process as: (a) hooks—appropriating an artifact as enactive, explanatory, or evaluative means; and (b) shifts—spontaneously reconfiguring current or emerging strategy to avail of the artifact’s newly discovered affordances. This paper stems from this on-going work and extends the findings. Specifically, I here provide quantitative data about the frequency of participants’ hooking/shiftin the grid and numerals and elaborate on interaction parameters enabling or hindering hooks and shifts in artifact-mediated discovery-based learning.
1. Introduction and Objectives

This paper is about “hooks and shifts” in mathematics education. In our previous work, we have presented and empirically demonstrated the dual construct of hook and shift to illuminate nuances of instructional interaction (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, under review). Specifically, this ontological innovation (diSessa & Cobb, 2004) was developed in an attempt to characterize emergent, unanticipated behavioral patterns we discerned during our collaborative microgenetic analysis of videographed empirical data. These data, from a design-based research study of mathematical learning in an embodied-interaction design for proportionality, documented students’ spontaneous bootstrapping of mathematical forms through appropriating artifacts in the process of problem-solving a challenging task. A hook is a subjective and contextual affordance of a mathematical artifact that students recognize as they become cognizant of the artifact’s availability in the course of solving a problem and communicating their solution. Students engage the artifact as a utility for better implementing their practical competence or a semiotic means of objectifying their unarticulated strategy (Radford, 2003; Sfard, 2002, 2007). A shift is the students’ consequent, unpremeditated and undemonstrated yet more mathematically sophisticated analysis of the problem situation that emerges through using the artifact (see Schön, 1992, on "see...move...see").

In this paper, we adopt a hooks-and-shifts perspective to re-analyze and expand the scope of focal excerpts selected from this same videographed data corpus. These data consists of a set of semi-structured, task-based, individual or paired clinical tutorial interviews (diSessa, 2007; Ginsburg, 1997; Goldin, 2000; Piaget, 1952). During these interviews, 22 Grades 4-6 student-
participants engaged in problem-solving activities involving remote manipulation of virtual objects on a computer display monitor. At different points along these interviews, the researcher introduced various symbolic artifacts into the problem space, such as a virtual Cartesian grid and \( y \)-axis numerals overlain onto the display. The interview protocol thus structures the intervention into a sequence of forward-feeding episodes, each launched by the researcher. The interviewer intends for the student to interpret this gesture as a pragmatic cue to engage, use, and/or appropriate the artifact as a means of advancing the shared objective of completing the task. If this appropriation is within the participant’s reach, s/he experiences a modicum of conceptual change manifest either as a new solution strategy or refinement of a previous one. When the tutor evaluates this reorientation as stabilized, the next episode ensues (cf. Stevens & Hall, 1998).

Our analyses indicated that some participants indeed appropriated the artifacts as strategic or discursive means of accomplishing their goals; in the course of enacting these goal-oriented tasks, moreover, the participants were likely to recognize certain other embedded affordances in these symbolic artifacts as supporting the development of a more sophisticated strategy. We have characterized this two-step mediated-discovery process as: (a) hooks—appropriating an artifact as enactive, explanatory, or evaluative means; and (b) shifts—spontaneously reconfiguring current or emerging strategy to avail of the artifact’s newly discovered affordances. Furthermore, in previous publications emerging from this project, we identified a set of interaction parameters that may enable or hinder hooks and shifts in artifact-mediated discovery-based learning. Whereas our previous analyses were not exhaustive, they did suggest the importance of these interaction parameters for students’ learning experiences; this paper substantiates our claim by casting an analytic net on the entire data corpus and not only the illustrative vignettes. Specifically, we here provide a quantitative grip on the innovative construct
“hook and shift” by documenting the frequency of all participants’ hooking/shifting and elaborating each of the proposed interaction parameters with data excerpts. We interpret the implications of these new findings for instructional designers and practitioners wishing to avail of students’ apparent inclination toward, and resources for creative mathematical agency within organized pedagogical regimes, as captured by the construct.

2. Theoretical Perspectives

Our work in the Embodied Design Research Laboratory (Dor Abrahamson, Director) involves the design, testing, and refinement of specially crafted pedagogical artifacts, as well as the development of theoretical models pertaining to the expansion of mathematical cognition though interacting with said artifacts, such as our hook-and-shift construct. This work draws heavily from the interdisciplinary field of the learning sciences, which is informed broadly by two predominate theoretical perspectives. On the one hand, a cognitivist view of learning, typified by Jean Piaget and Neo-Piagetian scholars, posits that a child’s intelligence is an intact, integrated-yet-evolving system located within the child and therefore novel cognitive structures are conceptualized as emerging from essentially endogenous processes (Greeno, Collins, & Resnick, 1996). That is, cognitivists view learning as an *intrapersonal* process that evolves of the child’s own agency as she engages and tries to make sense of her environment. On the other hand, a sociocultural view of learning, typified by Lev Vygotsky and Neo-Vygotskian scholars, conceptualizes a child’s intelligent behavior as manifesting the internalization of regulated social participation in the complex cultural practice of a socio-technological system located in or “smeared” across available media, tools and other objects; competence is explicitly or just implicitly the production of a network distributed across people, artifacts, and time (cf. Hutchins,
1995). From this perspective, therefore, novel cognitive structures are conceptualized as resulting from essentially exogenous processes. That is, sociocultural theorists view learning as an inherently interpersonal process whereby the child engages, uses, or appropriates cultural tools toward accomplishing emergent goals within a social context (Saxe, 2004).

Furthermore, different pedagogical frameworks for teaching and learning in the disciplines stem from the respective positions of cognitivist and sociocultural theory. For example, in the realm of mathematics education, radical constructivists emphasize instruction that enables students to “discover” mathematical knowledge without any explication of the targeted concepts or direct instruction of solution procedures (von Glasersfeld, 1987). In contrast, sociocultural theorists might advocate that teachers “funnel” students toward mathematical knowledge by offering them conceptual/procedural scaffolds at key points along a desired learning trajectory, while at the same time socializing them to adopt socio-mathematical norms (Voigt, 1995).

Clearly, though, the truth lies somewhere in between (Cole & Wertsch, 1996). Toward a reconciliation of these two apparently vying positions, some scholars have argued that a “dialectical” approach to the study of cognition and learning is needed (diSessa, 2008)—an approach that respects the need to account both for cognitive and socio-contextual factors in student learning. Within mathematics educational research, studies attempting dialectical synthesis are becoming increasingly prevalent (Abrahamson, 2009; diSessa, Philip, Saxe, Cole, & Cobb, 2010), and the larger study from which this particular report stems is inspired by and builds on this stream of research. In particular, the larger study draws from our on-going efforts to understand, from a dialectical perspective, the microgenesis of mediated mathematical learning in instrumented activity situations (see Vérillon & Rabardel, 1995). Thus we offer the
dual construct of “hook and shift” as an analytic tool that could potentially account for both endogenous and exogenous forces at interplay in the microgenesis of mathematics learning (Abrahamson, et al., under review).

In analyzing our data, we adopt the general Vygotskian view that cognitive change occurs through collaborative activity in instrumented situations, wherein students are guided to reformulate their naïve views of situations into analytic perspectives of the same situations (Bartolini Bussi & Mariotti, 2008; Newman, Griffin, & Cole, 1989). At the same time, in our analyses we have observed that the students rely on domain-general heuristics for learning (Gelman & Williams, 1998; Giggenzer & Brighton, 2009; Tversky & Kahneman, 1974; Xu & Vashti, 2008). Importantly for the construct of “hook and shift,” the instructor never demonstrated or even insinuated some of the new situated practices that students discovered as the symbolic artifacts’ implicit interactive potential, such that the students experienced guided reinvention of logico–linguistic structures familiar to experts as mathematical or proto-mathematical (Gravemeijer, 1999; Resnick, 1992).

Indeed, we characterize this guided-reinvention as a two-stepped process, elaborated as: (1) hook – when a symbolic artifact is first introduced into the interaction space, problem-solvers engage it because they recognize its contextual utility for enhancing the enactment, explanation, or evaluation of their current solution strategy—they interpret the enunciated introduction of the artifact as bearing a strong pragmatic cue to carefully consider the object; then (2) shift – in the course of implementing these new affordances, problem-solvers notice in these artifacts additional embedded properties as affording a new or reconfigured strategy that better meets domain-general criteria of conciseness, precision, prediction, and—in the case of co-production with a peer—communication, coordination, and collaboration. These new strategies are then
sanctioned by instructors, to the extent that they view the shift as advancing the child’s process of mathematization closer to disciplinary structures and procedures, in accord with the intervention’s pedagogical objectives.

Furthermore, this study is inspired by previous research that explores the microgenesis of cultural artifacts and practices. For example, Saxe (2004) has defined the “form-function” dialectic as the interplay over time between certain cultural forms and their evolving purposes (i.e., function) in response to emerging goals in collective practices. In contrast to Saxe’s authentic research contexts, wherein the activity’s overall goals, and not only the artifacts’ perceived structure and utility, were all in flux, the “contrived” setting examined in this educational study did not lend agency to the student to adapt the activity’s goals (neither the interaction task goals nor the tutor’s didactical goals). Thus, what we call a “shift” is an new instrumentilization of a given artifact toward the same objective (cf. Vérillon & Rabardel, 1995).

Studying mathematics learning through the lens of hooks and shifts, we maintain, is valuable to educational research and practice. If supported, the construct could inform the development of the “dialectical” approach (diSessa, 2008) that negotiates cognitivist and sociocultural perspectives on mathematics and science learning processes (Abrahamson, 2009; Cole & Wertsch, 1996; diSessa, et al., 2010; Greeno & van de Sande, 2007; Halldén, Scheja, & Haglund, 2008). On the one hand, the proposed hook-and-shift mechanism casts human learning as deeply dependent on cultural artifacts, but on the other hand, the construct also suggests that conceptual reconfiguration from naïve to expert forms of reasoning is not as discontinuous as often characterized in the sociocultural literature. The construct of hooks and shifts could also inform the development of pedagogical frameworks for instructional design and classroom regimes. Namely, the hook-and-shift perspective envisions a compatibility of ostensibly
orthogonal stances, by which: (1) students need to “discover” mathematical knowledge (von Glasersfeld, 1987); and (2) teachers need to “funnel” student inquiry (Voigt, 1995).

3. Methods

Our broad research objective is to draw out general observations on the nature of mathematics learning in artifact-mediated instrumented situations. Our investigations are conducted through a process of collaborative, intensive microgenetic analyses (Schoenfeld, Smith, & Arcavi, 1991). In these analyses, we apply principles of grounded theory (Glaser & Strauss, 1967) as a means of identifying patterns in students’ behavior around instances when various symbolic artifacts were introduced into the problem space. So doing, the hypothetical constructs that we eventually named hooks and shifts emerged as inscribing a stable, ubiquitous phenomenon, whose variations, we suspect, co-vary with dimensions of learning that we have delineated in previous research yet are substantiating in this expansion study.

Our data corpus consists of videotapes and field notes gathered during 20 clinical interviews with 4th – 6th grade students who participated voluntarily in the implementation of an experimental design for proportion (Abrahamson & Howison, 2010a; Reinholz, Trninic, Howison, & Abrahamson, 2010). The research team included Dor Abrahamson, the PI and director of the Embodied Design Research Laboratory (EDRL), as well as the author and two other graduate student researchers.

Our research project was conducted in accord with the design-based research approach (DBR), in which learning theory and instructional materials are co-developed simultaneously, interdependently, reciprocally, and iteratively (Collins, 1992; Confrey, 2005; Edelson, 2002; Engeström, 2008; Kelly, 2003; Sandoval & Bell, 2004). Typical of DBR studies, our
experimental design was driven by a conjecture (Confrey, 1998); that is, we hypothesized a certain cognitive mechanism that is usually dormant, at least within instructional contexts designed to support the development of our targeted concept, yet whose activity, when properly activated, could potentially support the desired learning. Drawing inspiration from the embodied/enactive approach (Barsalou, 1999; Lakoff & Núñez, 2000; Nemirovsky, 2003; Núñez, Edwards, & Matos, 1999), our conjecture was that some mathematical concepts are difficult to learn because our everyday experiences do not occasion opportunities to embody and rehearse the body-based dynamic schemes underlying those specific concepts. Specifically, we conjectured that students’ canonically incorrect solutions for rational-number problems—“additive” solutions (e.g., "2:3 = 4:5" or "2/3 = 4/5" - Behr, Harel, Post, & Lesh, 1993)—indicate students’ lack of multimodal kinesthetic–visual action images to ground proportion-related concepts (Goldin, 1987; Pirie & Kieren, 1994).

Accordingly, we engineered an embodied-interaction computer-supported inquiry activity for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the Mathematical Imagery Trainer (MIT; see Figure 1, below).

![Figure 1. The Mathematical Imagery Trainer (MIT) set at a 1:2 ratio, so that the right hand needs to be twice as high along the monitor as the left hand. A paradigmatic interaction trajectory: (a) incorrect performance (red feedback); (b) almost correct performance (yellow feedback); (c) correct performance (green feedback); and (d) another instance of correct performance.](image-url)
Using remote-action sensor technology (see Figure 2a, below), the MIT device measures the heights of the users’ hands above the desk. When these heights (e.g., 10” and 20”) match the unknown ratio set on the interviewer’s console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display at an appropriate rate (so, always 1:2), the screen will remain green; otherwise, such as if she maintains a fixed distance between her hands while moving them away from a “green spot,” the screen will turn yellow then red. Study participants were tasked first to make the screen green and then, once they had done so, to maintain a green screen even as they moved their hands.

*Figure 2.* The Mathematical Imagery Trainer: (a) top view of the system featuring the earlier MIT version, in which students held tennis balls with reflective tape. Figures 2b – 2e are schematic representations of different display configurations, beginning with (b) a blank screen, and then featuring a set of symbolical objects incrementally overlain onto the display: (c) crosshairs; (d) a grid; and (e) numerals along the y-axis of the grid.

At first, the condition for green was set as a 1:2 ratio, and no feedback other than the background color was given (see Figure 2b; this challenging condition was used only in the last six interviews). Then, crosshairs were introduced that “mirrored” the location of participants’ hands (see Figure 2c). Next, a grid was overlain on the display monitor to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions (see Figure 2d). In time, the numerical labels “1, 2, 3,…” were overlain on the grid’s vertical axis on the left of the screen to help students construct further meanings by more readily
recruiting arithmetic knowledge and skills and more efficiently distributing the problem-solving task (see Figure 2e).¹

Participants included 22 students from a private K–8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students; one student participated twice). Students participated either individually (17 of the 20 interviews) or paired (the last 3 interviews) in a semi-structured interview (duration: mean 70 min.; SD 20 min.). For this study, we drew on the last 15 interviews, wherein our protocol had stabilized.

Throughout the various stages of our analyses, our overarching, complex research question has been:

*What heuristic, semiotic, discursive, and pragmatic mechanisms facilitate and modulate learners’ conceptual change in artifact-mediated embodied-interaction design?*

Toward addressing this research question, our initial publications reported on students’ apparent learning trajectories through the interview protocol (Abrahamson & Howison, 2010b; Reinholz, et al., 2010). Those publications focused on the range of mathematical meanings students generated as they engaged the problem-solving inquiry activity. In a recent study, we moved from the “whether” to the “how,” specifically in an attempt to account both for cognitive and socio-cultural factors affecting student learning in the context of embodied-interaction design, a study that led to conceptualizing the hook-and-shift construct (Abrahamson, et al., under review). With this present study, we have moved farther along our analytic trajectory in two ways. First, we are now delving into the “how” by comprehensively illustrating and empirically supporting the interaction dimensions we had previously conjectured as predicating

---

¹ Not treated in this paper yet key to the designed learning trajectory is yet another structure layered onto the screen, namely an interactive ratio table for effecting green numerically rather than gesturally (Reinholz, et al., 2010). Also note that in addition to the 1:2 beginner ratio, we worked with students on 1:3 and 2:3 ratios not reported here.
hooks and shifts. Second, we are inquiring as to “how often” by determining and interpreting the frequencies that students did or did not hook to, and shift with the various symbolic artifacts.²

4. Findings

The semi-structured clinical interview protocol used in this study guided the interviewer to sequentially introduce the following set of symbolic artifacts into the problem space: (1) a pair of crosshairs mirroring the users’ hand positions; (2) a Cartesian grid; and (3) numerals rising from zero along the grid’s $y$-axis (see Methods section, above, for further details). In the sections that follow, I first present a set of empirical episodes (4.1) selected so as to review and elaborate the hypothetical constructs of hook and shift. Next I present case studies of several hook-and-shift episodes (4.2), focusing on several interaction dimensions identified as contributing to variation across participants with respect to their behavior when the artifacts were first introduced. Finally, I focus specifically on the Cartesian grid and $y$-axis numerals and present quantitative data about the frequency of participants’ hooking/shifting the grid and/or numerals across all 15 interviews (4.3).

4.1 Paradigmatic Cases of Hook and Shift


Following the pair of crosshairs, the Cartesian grid is the second symbolic artifact layered onto the computer display. In this first section we present and analyze video data from a paired-student interview, in which the dyad collectively hooks to, and then shifts by using the grid.

² Note that a frequency metric of hooks and shifts need not be taken as an index of the quality of the design. First, coming into this pilot study—our first implementation of the MIT design—we had not foreseen this behavioral pattern. Second, we have first to understand better the nature and potential of this behavioral pattern toward future implementations.
Whereas the primary rhetorical objective of this section is to enrich and ground our definitions of hook-and-shift, the dyadic interaction of the specific data episode selected for this section offers a supplementary, if complicating, dimension. Yet we believe this added complexity is ultimately rewarding, because its unique discursive qualities bring out in relief the very processes that we wish to demonstrate and understand.

Eden and Uri, two Grade 6 male participants, were selected for a paired interview on the basis of compatible mathematical achievement (both were identified by their teachers as “high achievers”). Their interview was conducted by an apprentice researcher (DT), with the lead researcher (DA) occasionally intervening. The students sat side by side in front of the remote-action sensor system and computer display and each operated one of the two tracker devices (right-tracker device [RT] and left-tracker device [LT]). The students were presented with the task of making the screen green under an unknown 1:2 ratio setting (i.e., the screen would be green only when the right-crosshair [Rc] were double as high as the left-crosshair [Lc]).

*Hooking to the grid.* Prior to the introduction of the grid, Eden and Uri had been working together for nearly 11 minutes in the no-crosshairs (blank screen) condition and then another 7 minutes in the crosshairs condition. So doing, they identified two spatial dimensions—height and distance—as relevant to making the screen green and had articulated two theorems with regard to each of these dimensions: (a) Rc should be higher than Lc; and (b) the vertical distance between Rc and Lc is non-arbitrary. However, Eden and Uri disagreed as to whether this vertical distance should change or remain constant as the crosshairs move. Whereas both Uri and Eden observed different distances between the Rc and Lc at certain green locations, Uri interpreted this difference as a systemic principle for making green, while Eden attributed it to an HCI issue, as

---

3 Yet another apprentice researcher (JFG, the author) was present during this particular interview but intervened only once over the course of the 72-minute interview.
though the physical manipulation were inaccurate (Eden, apparently an avid video-game designer, referred to this error as the “human factor”). Uri articulated a covariant principle relating distance and height, explaining that “it has to get, like, farther away, the higher up we are” and that “the lower you are, the less distance apart it has to be”—a changing-distance theorem-in-action. Eden, however, courteously responded with, “Well I’m not sure if it matters if you’re lower or higher, but I think it’s just, like, you stay the same distance apart”—a fixed-distance theorem-in-action.

Thus, Uri and Eden’s collaborative hands-on problem solving enabled them each to notice and explicitly articulate a relation between the crosshairs’ height and distance, yet whereas Uri concluded from their empirical data that the distance should vary, Eden concluded from the same data that it should not. This disagreement bore practical implications, because the dyad were co-operating the two devices—they each depended on the other to enact a green-making theorem-in-action, yet their respective theorems were mutually exclusive. Consequently, the students’ success within this collaboration became contingent on whether or not they could rule between their incompatible fixed-distance and changing-distance theorems-in-action. At the same time, they apparently felt under-equipped to arbitrate in the continuous space. Namely, when the grid was subsequently introduced (see below), they recognized its potential for ruling between the theorems—they “hooked” to the grid largely for its discursive, argumentation, and arbitration affordances. Specifically, the grid served the boys to quantify the distance between the crosshairs and ultimately determine that this distance should in fact change between green
spots, as Uri had believed. Eden soon concurred. The excerpt below begins immediately after DT had layered the grid onto the screen.4

Eden: <19:36> **Grid.**

Uri: **Yeah.** [Grabs RT, lifts it, and remote-places Rc on the 1st-from-the-base-line gridline (hence “Rc up to 1-line”). Simultaneously, Eden, too, brings Lc up to 1-line. On the way up, between 0-line and 1-line, the screen flashes green for a moment but then turns red. Eden lowers Lc back down, holds it at .5 units. The screen turns green.] **Oh so you can like show where… Let’s see, so** [Rc up from 1-line to 2-line] // **if you’re on here…**

Eden: //**maybe it has to be two…** [Lc up to 1-line (see Figure 3a, next page)] **an entire box apart.**

Uri: [Rc up to 3-line] **If I go here…**

Eden: [Lc up to 2-line; screen goes red (see Figure 3b, next page)] **Then maybe you should raise it** [Uri raises Rc to just below 4-line; screen flashes green]. **So maybe the higher you go, the more boxes it is apart.**5

Uri: **Let’s just say like I’m here** [Rc down to 2-line], **then he has to be one box under me…**

Eden: [Lc down to 1-line; screen goes green] **And then the higher he goes//**

Uri: **//and when I go here** [Rc up to 3-line], **he has to be like in the middle** [Eden moves Lc up to 1.5 units; screen goes green]

Eden: **So the higher//**

Uri: **//And here** [Rc up to 4-line, while Eden moves Lc up to 2-line] **he has to be like two boxes under me.**

Eden: **So like the higher it goes, the more space there has to be between each.** [Both Eden and Uri place their tracker devices on the table]

---

4 RT = Right-Tracker device; LT = Left-Tracker device; Rc = Right crosshair; Lc = Left crosshair; // = overlap by next speaker. We mark spoken utterance with bold characters for readability.

5 In passing, note the different types of pronouns employed to designate action and measurement. Action is attributed to individuals (I and you), whereas measurement is about absolute magnitudes (it). This is a mere passing observation, but this linguistic approach may be useful in tracking subtle shifts between physical and reflective agency (see Rowland, 1999; Shreyar, Zolkower, & Pérez, 2010).
Thus, it appears that both Eden and Uri immediately appropriated the grid in view of its affordances to arbitrate among their conflicted theorems, however they differed with respect to the nature of their discovery, and this difference can be related to their idiosyncratic beliefs prior to the introduction of the grid. Namely, Uri had articulated a changing covariant relation between height and distance, so for him the grid afforded reiterating and quantifying this qualitative principle. Specifically, the grid enabled Uri to reformulate his continuous qualifier “get farther away” as the discrete quantifiers “one box” and then “two boxes.” Eden, who had acknowledged the in-principle possibility of a changing-distance rule yet nevertheless maintained a fixed-distance rule, soon changed his mind and articulated a changing-distance hypothesis (“the higher you go, the more boxes it is apart”). However, Eden ends with a qualitative statement about “space,” which suggests that he construed the grid as a means not of quantifying the “higher—bigger” conjecture but of evaluating whether or not this conjecture even obtained. Thus, Eden and Uri both hooked to the same artifact, yet they utilized it for different purposes. This episode demonstrates the plausible view that an artifact’s subjective utility is contingent on the
individual’s goals. The episode also suggests that a dyad can engage in collaborative hands-on problem solving even as they hold different theorems-in-action (cf. Sebanz & Knoblich, 2009).

In the following excerpt, below, we will continue at a point where the dyad initiated further inquiry. As we shall see, the dyad’s exploration will shift them from the now-consensual “higher–bigger” strategy toward a proto-ratio $a$-per-$b$ strategy. Both strategies can be viewed as expressing covariation—“the more $x$, the more $y$.” However, the former strategy is continuous–qualitative, whereas the latter is discrete–quantitative, so that student adoption of the latter strategy is a pedagogically desirable outcome of the interaction.

*Shifting with the grid.* Having reached consensus, Eden and Uri elaborate their explanation. They are now instrumented with the grid’s quantification affordances, and this new conceptualization of space will engender the semi-spontaneous emergence of a new mathematical form. In particular, in the transcription that follows we will observe that the students shift with the grid from the continuous–qualitative strategy to a discrete–quantitative strategy.

Uri:  

`<20:19> I think, like, uhm, when I go up to here [points to 2-line], he has to be one. Then when I go up/`

Eden:  

`Like for every… for every box he goes up, I have to move, go down$^6$/`

Uri:  

`//You have to go up half/`  

`//a box.`

Eden:  

`//Yeah/`

The dyad’s coordinated production of green tacitly modulated from simultaneous motions, in which the distance constantly increases and green coloration is maintained throughout, to sequential motions, in which each hand separately ratchets up to its respective designated destination and green is effected after a brief red interim, once the second ratcheted

$^6$ On several occasions throughout the interview, Eden used “up” to refer to the direction we would call “down” and vice versa. As it turned out, Eden is an expert video gamer. He thus may have been using schemes that are natural to video-gaming culture. Therein, and counter to non-gamer discourse, joystick actions are often mapped onto virtual actions such that swinging the joystick forward results in the first-person avatar looking or moving down.
motion is completed. Imperceptibly, the linguistic constraints of the speech modality as well as the linearity of discursive turn-taking and ownership over actions thus shifted the dyad from their higher–bigger continuous–qualitative strategy to an \( a\)-per-\( b\) discrete–quantitative proto-ratio strategy.

Eden and Uri thus co-discovered that in order to maintain green, they should progress at intervals of \( \frac{1}{2} \) (Eden) and 1 (Uri) coordinated vertical intervals, either both going up the screen or both going down. It is through this serendipitous discovery that their earlier observation, “The higher you go, the more boxes it is apart,” a covariation between height and distance, transformed (shifted) into a covariation that foregrounds the independent actions of the left and right crosshairs, “For every box he goes up—you have to go up half,” a new strategy that is closer to normative forms for ratio (i.e., \( a\)-per-\( b\)). We wish to underscore that whereas the general \( x\)-per-\( y\) covariation form was maintained, its semantic–mathematical content was replaced (see Figure 4, below).

<table>
<thead>
<tr>
<th>Mathematical Properties</th>
<th>“The more ( x)...”</th>
<th>“the more ( y)”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous–qualitative:</td>
<td>“The higher you go...”</td>
<td>“the bigger the distance”</td>
</tr>
<tr>
<td>Discrete–qualitative:</td>
<td>“The higher you go...”</td>
<td>“the more boxes it is apart”</td>
</tr>
<tr>
<td>Discrete–quantitative:</td>
<td>“For every box he goes up...”</td>
<td>“you have to go up half”</td>
</tr>
</tbody>
</table>

*Figure 4.* “Covariation” linguistic structure across strategy shifts.

In addition to explicating our hook-and-shift construct, our analysis of the case has demonstrated that collaborative mathematical learning processes are impacted by nuances of personal/interpersonal framing to the extent of dissociation between a dyad’s physical and epistemic actions. Namely, whereas the two dyad members collaborated on using a single symbolic artifact (the grid), their joint experiment simultaneously enacted an exploration of two
different hypotheses (“same distance” and “different distance”). Uri was quite comfortable from
the very onset with the higher–bigger principle, so he did not need arbitration but refinement,
whereas Eden, who challenged Uri with a fixed-distance theorem, needed resolution. Once a
common ground was established, the dyad was able to continue mathematizing the mystery
artifact–phenomenon and jointly articulate a new mathematical form, which we recognized as
pedagogically desirable.

4.1.2. Bootstrapping Mathematical Forms Through Appropriating Arithmetic
Affordances of Symbolic Artifacts: From a Recursive to an Explicit Strategy for Enacting Green.
Following the crosshairs and the Cartesian grid, the y-axis numerals are the third symbolic
artifact layered by the interviewer onto the display monitor, in accord with the protocol. To
further illustrate the dual construct of hook and shift, this section presents a case-analysis of
Shilav, a 5th-grade female student identified by her teachers as middle-achieving, as a
paradigmatic instance of a participant who hooks to and shifts by using the numerals. Shilav, too,
was interviewed by an apprentice researcher (DT), with the lead researcher (DA) intervening.

Hooking to the Grid+Numerals. To prepare the reader for the hook-and-shift analysis of
Shilav’s behaviors surrounding the y-axis numerals, a brief overview of her behaviors in the
crosshairs and crosshairs+grid (henceforth “grid”) conditions is necessary. Prior to the
introduction of the numerals, Shilav had worked for approximately 20 minutes in the crosshairs
condition and then another 7 minutes in the grid condition. While in the crosshairs condition,
Shilav identified two spatial dimensions governing the operation of green: (a) the distance
between the left and right trackers; and (b) the hand-pair elevation (i.e., height from the
baseline). In particular, Shilav and DA worked together to find various green locations, with DA
taking instructions from Shilav to position either one or both tracking devices. Asked if there
were anything she had noticed as they worked to find “green spots,” Shilav responded with, “Uhm the distance, like how far away are they” and that “wherever spot it is [i.e., particular hand-pair locations], if it’s like the same distance apart then it’s green.” Thus she articulates a fixed-distance theory, namely that the distance remains constant as the hand-pair moves.

Minutes later when the grid was subsequently introduced into the problem space, Shilav hooked to it as a means of quantifying the distance between the Lc and Rc and objectifying an emergent changing-distance strategy. Once the grid was overlain on the display monitor, DA—again operating both tracker devices—produced green in various locations by remote-placing the crosshairs on precisely the horizontal gridlines that would satisfy the 1:2 ratio. While DA was manipulating both the Lc and Rc to produce instances of green, Shilav, gazing at the display monitor, apparently noticed that the vertical distance between the hands was not constant, that the distance was growing. To wit, she articulated a changing-distance principle relating distance and hand-pair location: “If you go up by uhm one more square than you were before...”

Similar to the case of Uri and Eden, Shilav, too, had instrumented herself with the grid’s quantification affordances, using the grid as a means of measure the vertical distance between the Lc and Rc at certain green locations; she thus noticed that a fixed-distance theory did not obtain. At this point, DA returned both tracker devices to Shilav, and she began to move the crosshairs, remote-placing them on certain gridlines, attempting to produce a green screen via a growing-distance theory. So doing, she found herself enacting a new strategy for effecting

---

7 Within the 15 analyzed interviews, the interviewers occasionally intervened and served as students’ “trusty robot” for both technical and conceptual reasons. For instance, Shilav’s interview experienced constant “glitches” with the technology, which we were still developing at the time. Whereas Shilav was very patient throughout the entire interview, she often found it difficult to maintain remote traction on the crosshairs. Thus in Shilav’s interview DA intervened more than usual for the sake of mitigating some of these technical issues. Furthermore, DA took control of one or both tracker devices also to probe Shilav’s fixed-distance theorem-in-action. So doing, DA demonstrated various green locations, both down low and up higher along the display monitor, to highlight and thus create opportunities for Shilav to notice and appreciate a changing spatial interval between the hands/crosshairs.
green—that is, Shilav *shifted* with the grid, from using it as evaluative and enactive means (i.e., first determining whether or not a fixed-distance theory would obtain, followed by enacting a changing-distance solution strategy) to using it as a means for enacting a novel recursive strategy. Namely, as she moved the crosshairs, Shilav stumbled upon a quantitative strategy whereby the production of the next green spot was dependent on the *current* hand-pair location. When asked to explain her strategy, Shilav verbally utters, while at the same time physically manipulating the tracker devices, that one needs to begin at a specific green location, namely, with Lc and Rc at the 1-line and 2-line, respectively [henceforth “(1, 2)”]; the screen goes green. This move suggests that Shilav’s new strategy depended on an initial condition. Once she placed the crosshairs at (1, 2), she then raised Lc to 2-line, so that for just a brief moment, the Lc and the Rc were on the same horizontal gridline; the screen went red. She then raised the Rc to 4-line, and the screen became green. Thus her new strategy was recursive.

It is important to note that up to this point, Shilav’s attempts to produce green had been informed by a growing-distance theory, that is, at every move, she had increased the distance between the hands, from one “square” to two “squares.” Yet, her movements involved raising Lc to the height of Rc as a necessary interim step toward producing the next green, which, by chance, happened to work for the first two cases. More precisely, her production strategy can be expressed mathematically using function notation as follows: \( F(a, b) = (b, b + (b-a) + 1) \). As Shilav experienced, her initial condition of (1, 2) produced green, and the first iteration of (2,4) worked as well. However, her strategy breaks down because, at the next iteration, her strategy located Lc and Rc at (4, 7) which does not satisfy the 1:2 ratio. Shilav indeed recognized that her production strategy would not effect green for instances higher along the computer display monitor, even though she was faithful to her growing-distance theory and increased the distance
by one “square” each time. In need of repair, Shilav and DA worked together for nearly 4
minutes—DA operated LT, Shilav operated RT—trying to reconcile the growing-distance
theory with a faulty enactment interpretation, but with minor success. Faced with an apparent
“dead end,” DA then prompted for the y-axis numerals to be introduced into the problem space.

As we hope to illustrate with the transcription that follows, Shilav immediately perceived
the numerals’ indexing affordance. That is, she hooks to the numerals as a means of indexing
specific crosshair locations so as to try to repair her recursive strategy for making green. The
following transcription begins immediately after the numerals appeared on the display monitor
(see Figure 2e, in the Methods section, above). DA operates Lc, while Shilav operates Rc.

DA: <39:26> So you see those numbers there?
Shv.: Mmmh.
DA: Ok. Let’s see if that ahh [Lc to 1-line] does anything for us. So
there’s a place we know that’s green, right? [Shilav raises Rc to 2-line; screen
goes green (see Figure 5a, below)] Should I go up or... So now what are your
instructions?
Shv. Go up one. [DA raises Lc to 2-line, while at the same time, Shilav raises Rc to 4-
line; screen goes green (Figure 5b)] Go up to four.
DA: [DA raises Lc to 4-line; screen goes red (Figure 5c, on the next page). And then
Shilav raises Rc to 8-line (Figure 5d); screen goes green]. Huh. That was cool.
Shilav used the numerals as a means of coordinating the production of green and repairing her faulty recursive strategy. Specifically, recall that Shilav had presented the strategy of raising the Lc to the same height as the Rc and then incrementally increasing the difference between the two hands by one “square.” Shilav used the numerals to indicate to DA specific gridline locations for the Lc, so as to effect green—that is, she hooked to the numerals for discursive purposes of joint physical production.

*Shifting with the Grid+Numerals.* While engaged in the collaborative production of green, Shilav’s use of the numerals tacitly changes in terms of their function, from indexical to arithmetic. Namely, as we shall see in the transcription below, Shilav shifts from a recursive to a closed/explicit strategy.

It is important to recall that during a previous production of green, under the grid condition, Shilav had initially asked DA to reposition the crosshairs according to a recursive strategy for producing green (see Figure 4a-4d, above). But Shilav will now use the numerals to explain in mathematical terms her choices for positioning the trackers. We rejoin the interview immediately where the previous segment left off; DA still operates Lc, Shilav operates Rc.
If you... so at one and two [DA lowers Lc to 1-line; Shilav lowers Rc to 2-line; screen goes green]... umm if you... one plus one is two.

Ok.

And if you go to the two [Lc to 2-line] and go to four [Rc to 4-line], two plus two is four.

Cool!

And if you go to four [Lc to 4-line]—

Oh not three?

And go to the eight [Rc to 8-line]... four plus two—four plus four is eight.

What about if I’m down [Lc to 3-line] to the three? What do you want to do?

Go to [Rc to 6-line] six.

Cool.

Because three plus three is six.

Shilav has discovered an additive numerical strategy that directly expresses the height of Rc in terms of the height of Lc. This strategy can be expressed mathematically using function notation as follows: \( F(a, b) = (a, a+a) \). DA then encourages Shilav to operate both Rc and Lc by herself to make the screen green, using her new strategy. So doing, Shilav retraces their journey-toward-green by revisiting the hand-pair locations that had previously yielded green, beginning at (1, 2), moving up to (2, 4), (3, 6), and then (4, 8).

DA then attempts to probe her emerging rule. In response, Shilav shifts yet again with the numerals to a closed–multiplicative strategy for enacting green.

Ok, so the rule is, if I understood correctly, wherever the left one [Lc] is...

Then you double the number that the left one is on and you put the right one [Rc] on the number... on that number.

Yeah. Cool. So you’ve cracked the code.

4.2 Interaction Dimensions Predicting Hooks and Shifts

Our objectives thus far have been to illustrate the construct of hook and shift with empirical episodes from our study. We believe that the case studies presented above are representative of students’ successes with hooks and shifts. In sum, our qualitative analyses have found that students hook to symbolic artifacts for three main reasons, namely, so as to:

(a) enact – to master the skillful control of the system, by progressively increasing the
precision, efficiency, collaboration affordances, and predictive capacity of their strategy;

(b) *explain* – to support discursive representation of personal solution strategies in accord with the pragmatic framing of the interaction as proto-mathematical (i.e., as demanding unequivocal, generalizable, quantitative indexing of procedures); or

(c) *evaluate* – to arbitrate among conflicted theorems-in-actions held among students and instructors participating in the collaborative inquiry.

Yet students may not hook. Comparisons across participants who did and who did not hook to, and shift with the artifacts suggest a set of five interdependent interaction dimensions (see below) as predictive of students’ prospects of hooking to, and shifting with the symbolic artifacts layered onto their problem-solving space.\(^8\) Each and all of these five parameters, we maintain, is necessary for students’ to hook a new symbolic artifact:

1. *content* – students must have conjectured an effective operational strategy;
2. *validation* – students must have confirmed the plausibility of the strategy;
3. *priming* – the strategy should be cognitively available when the artifact is introduced;
4. *fluency* – the students should be minimally familiar with the symbolic artifact introduced by the instructor; and
5. *pragmatic* – the instructor should frame the artifact as potentially instrumental in solving the interaction problem.

Shifting, too, is contingent on students’ preparedness. In particular, the following three additional interaction parameters partially predict the likeliness for a shift to occur:

6. *correct dimension, incorrect value* – students may have hooked an artifact whose affordance dimensions cohere with their strategy, only that their strategy had assumed particular values along this dimension that turn out to differ from the values the designer had embedded into the system’s interactivity; these students need still to experience and resolve cognitive conflict along this dimension; and

---

\(^8\) Whereas study participants who generated and connected more strategies and meanings were typically the students who had been characterized by their teachers as higher achieving, the other participants were impeded as much by our experimental methodology as by their own knowledge. Namely, at times we marched on through the interview just to ensure that we “cover” all protocol items for subsequent analysis, regardless of whether or not participants were optimally prepared to work with the new symbolic artifacts. Only during analysis did we fully appreciate the potentially detrimental cumulative effect of our facilitation practice on the quality of the experimental implementation as a *learning* experience for some participants. Serendipitously, it is these children’s failure to learn under the inadvertently compromised methodology that alerted us to implicit dimensions of normative instruction.
7. **perceptual compatibility** – students may have determined, performed, and articulated an effective rule in accord with the design’s interaction parameters and values, yet their perceptual construction of one of the phenomenal properties differs from the disciplinary view implicit to the design, a mismatch that surfaces only with the introduction of the artifact and may be aligned through discourse.

8. **implementation** – to experience a shift, students who have hooked a symbolic artifact need sufficient time to further explore it; indeed, students productive engagement with an artifact is greatly impacted by the quality of the interviewer’s facilitation.

In the sections that follow, I elaborate each of these interaction parameters and present empirical data demonstrating each.

4.2.1 **Content.** Before the new artifact is introduced, students need to have determined an effective theorem-in-action regarding the production of green, which may be articulated as either a qualitative relation or a quantitative principle. For example, some students observe that the distance changes, in fact that “the higher you go, the bigger the distance” (hence “higher–bigger”); while others observe that, “the right hand is double the left one” (hence “doubling”); or that the “right one goes faster than the left” (hence “faster”). This relation, pattern, principle, or rule constitutes the cognitive content that the student is attempting to enact, express, or evaluate.

When the artifact is introduced, students who have established such content recognize the artifact as a means of better enacting, explaining, or evaluating the principle. For example, when the grid is superimposed on the display monitor, students need to have articulated the relation between the hands’ height and distance. A student who does not bear a theorem-in-action would not be attracted to the newly introduced symbolic artifact as a means of anchoring and elaborating this theorem and consequently would be unlikely to assign new meanings to the artifact. Instead, the artifact would likely appear as a redundant utensil for an unknown task. Such is the case for Liat, a middle-achieving 6th-grade female student, who does not bear the appropriate “content” and therefore does not hook to the grid.
Prior to the introduction of the grid, Liat worked for nearly 10 minutes in the crosshairs mode. Initially, she had set out to make green by moving the two tracker devices in parallel. Her first green appeared at a very low height, where the trackers were essentially at the same height. Asked to find green some place else, she kept moving her hands in parallel. Met with little success, she then explores moving her hands at different heights, until she arrived at her first green in which the height of the LT and RT were clearly different. She then continued to work on the problem in silence for nearly 3 minutes, finding other instances of green, both up high and down low on the display monitor. When asked by the interviewer if she had any guesses about where green can be found, she did not know. Liat returns her attention to the screen and explores for nearly 2 more minutes, finding other instances of green. Prompted by the interviewer, Liat states that the RT should be higher than the LT, and even as she spoke, she held the two tracker devices at a green position and then moved the hand-pair up and down holding a fixed distance; the screen changed repeatedly from green to red to green. Liat appeared confused and returned to exploring different distances between the two trackers but with little success. Just moments before the introduction of the grid, Liat concluded that both hands should be involved in the production of green, but she did not articulate how so. When the grid was introduced, she initially moved her hands in an explorative fashion as she had done before. However, a close scrutiny of the video data suggests that Liat ignored the grid; at least, she did not use it in a way that indicates that it regulated or assisted her search for a principle.

Liat: **There’s squares.**
DA: **Ok, so maybe they can help us figure it out.**
Liat: [picks up Rc and Lc and initially moves them in parallel, slowly, up from 0-line, then flip-flops them up and down, the screen flashes from red to green, red to green, etc., until Rc finally lands at a height just above the 3-line and Lc at about half that height; screen goes green] **I know that before it turns green it turns yellow. A little bit like… orange. So maybe it’s like—it’s kind of reminding me a rainbow when it goes from red to orange to yellow to green.**
When the grid was introduced, Liat did not assign any apparent utility to it in her search for green and, interestingly, she instead focused her attention on the coloration gradient that occurs as the crosshairs are repositioned. In particular, as Liat manipulated the positions of $L_c$ and $R_c$ to make green, neither of the crosshairs ever landed on any of the gridlines. That is, all the green spots that she found in the presence of the grid were not the result of her intentionally placing the crosshairs on an ordered pair, for example, at (1, 2) or (3, 6), which is typical of other students who successfully hooked to the grid. Therefore, without having explicitly articulated or enacted a principled relation between the distance and hand-pair locations, the grid did not appear to her as a means for enhancing the production of green. Liat did not hook, we maintain, due to this lack of enactive or semiotic content.

4.2.2 Validation. Before the new symbolic artifact is introduced, students should not only have conjectured an interactive principle (the “content”) but also have established the principle as appropriate and useful. For example, students should not only have suggested the higher–bigger theorem-in-action but also have confirmed its accuracy and utility through empirical trials and/or the instructor’s or peer’s sanctioning. A student whose insight is at best tentative may not be predisposed to appropriate a new symbolic artifact as a means of implementing, articulating, and defending the insight. Consider for example the case of Boaz, a low-achieving 5th-grade male student, whose emergent principle was not validated and therefore did not hook.

Similar to Liat, Boaz inferred that the right hand needs to be higher than the left in order to produce a green screen. In particular, while working in the blank-screen mode, the interviewer asked Boaz if he was noticing anything about the production of a green screen, to which Boaz responds, “All I know right now is right is higher than left.” As Boaz continues to work on the task, he explicitly articulates a relation between the hands’ collective height and the distance
between them (albeit an indirect relation). For example, still operating in the blank-screen mode, Boaz analogizes his hands to moving vehicles.

Boaz: You see, if you keep them in the same, like, pace, like, same, like for a car... If you wanted to do this with a car, it would sort of be the same speed limit. [Gestures with both hands moving back and forth horizontally] Like this one’s going 20, and this one’s going 50. And they have to keep on going, so... But you always have to keep the left hand... and then the right hand, so if you’re going back down, the left hand needs to go down with you [gestures in vertically descending, diminishing distances].

Boaz’s utterance reflects a theorem-in-action of effecting a green screen by moving the hands at different constant speeds, where LT moves up at a speed of “20,” while RT rises at “50.” However, perhaps puzzled by his analogy, the interviewers did not probe his “faster” theorem-in-action in a substantive way, thus the plausibility of the strategy was not ever confirmed. Possibly, one way of validating Boaz’s “faster” strategy would have to been to work toward a re-description of the individual hand speeds as discrete distance/time units, thus rendering apparent for Boaz the co-variant relation inherent to his strategy. Yet, not having thus re-described his own theorem, when the grid was presented Boaz could not recognize it as an appropriate means of objectifying/enhancing his speed-limit theorem. That is, the grid did not intimate itself as a means of elaborating or improving a principle that had remained un-validated. In Boaz’s words, “I don’t see what the squares are for.”

4.2.3 Priming. For students to appropriate a symbolic artifact as a means of articulating and enhancing a theorem-in-action, they should retain the particular theorem on the fore of their minds when the new artifact is introduced. If the principle is not cognitively available just when the new symbolic artifact is introduced, students may fail to recognize the artifact’s potential to further their objectives. We demonstrate this dimension with the case of Siena, a low-achieving 6th-grade female student.
Prior to introduction of the grid, Siena had verbally expressed that RT has to move “faster” than LT, while each tracker moves at “their own continuous pace.” Moreover, she empirically demonstrated the veracity of her “faster” strategy, and thus noticed that the distance between crosshairs changes as the hand-pair height increases. However, nearly 3 minutes had passed between the time her pacing/faster strategy had been mentioned and the introduction of the grid. This lag time, we argue, negatively impacted the prospects of her hooking to the grid.

When the grid was overlain onto the display monitor, Siena immediately responded that the grid would “make it easier to say where it is,” gesturing “it” to mean the hand locations effecting green. However, she did not in fact use the grid so as to “say” or otherwise demonstrate green locations. Even after extended probing by the interviewers, Siena did not ascribe any meaning to the grid that would result in effectively producing green. In sum, although Siena bore the necessary semiotic content (faster/pacing strategy), and although that content was indeed sanctioned by the interviewers as a plausible avenue toward accomplishing the goals of the activity, her strategy was not retained and therefore cognitively unavailable when the grid was brought in.\(^9\)

4.2.4 Fluency. Students who have established, validated, and retained a new principle still may not adopt the new symbolic artifact if they do not recognize its function. For example, students who have had limited experience with coordinate systems (e.g., a Cartesian plane) may not orient to the grid as parsing the working space into enumerable quotas of spatial extension; they may thus interpret the gridline’s function only as marking specific locations, not as equipping the viewer to assign these locations height values. Consider the case of Lilah (female)

\(^9\) While this is our main hypothesis for why Siena did not hook to the grid, we are also entertaining a supplementary explanation. Namely, we speculate that the grid’s “discretization” of the problem space was at odds with Siena’s notions of “continuous” speed. But the data are inconclusive specifically in this regard.
and Gideon (male), two high-achieving 4th-grade students, who had been paired for an interview and each experienced the grid differently. That is, Gideon but not Lilah hooked to the grid.

Before the grid, Gideon and Lilah had co-constructed, expressed, and enacted a changing-distance strategy. While working together under the crosshairs condition, Lilah first noticed that her RT should be higher than Gideon’s LT. DA suggested that one of them try to make green by themselves, so Gideon tried it and found green. When asked to find another green, he moved his hands up at a fixed distance. Seeing that the screen had become red, Gideon lowered his left hand; the screen became green. He moved up again holding a fixed distance between the two trackers; the screen went red. Again, he adjusted his left hand, and the screen turned green. He continued in this way, working the crosshairs up the display monitor, and all the while his adjustments become finer and finer, so that finally, he moved the crosshairs using what would appear to an onlooker to be a changing-distance strategy.

Lilah tried it next, but she initially employed a fixed-distance strategy, moving her hands up at the same rate. When asked if they were noticing anything about green, Gideon responded with, “My right hand goes up a little faster than my left hand and that way it stays green almost all the time.” Prompted by the interviewer, Lilah tried Gideon strategy, finding an initial green spot, and then moved both trackers up, with the right hand moving faster. However, she found it difficult to execute this particular strategy well, so she gave up after only a few attempts.

Up to this point in the interview, and just before the grid was introduced, both Gideon and Lilah had acknowledged the plausibility of a changing-distance strategy and had both attempted to enact it. Yet upon introducing the grid, Gideon but not Lilah hooks to the grid. If both students had appreciated the utility of the changing-distance strategy and had both enacted it
on the blank-with-crosshairs display, why then do we witness a differential uptake of the grid to enact, explain, and evaluate this strategy?

An argument could be made that Lilah was missing the necessary “content” (as described above) because, after all, it was Gideon who had initially expressed the rule, and even after the rule had been established as a viable strategy, Lilah still had difficulties executing the rule manually. However, we argue that the most critical factor relevant to their differential hooking experiences pertains to their prior knowledge and experience working with mathematical-diagrammatic representations—in this case, a Cartesian plane. Specifically, whereas a graph-fluent person might immediately recognize the contextual systemic affordances of the grid’s horizontal and vertical lines (e.g., for counting the gridlines from the bottom and up toward a particular line in question and potentially determining quantitative relations), to Lilah the gridlines per se afford narrower pragmatic utility. Namely, she valued the discursive utility of the grid as indexing the crosshairs’ particular locations relative to the nearest “box.” In fact, Lilah immediately refers to the grid as “boxes,” whereas Gideon identifies it as a “grid.” Although these appellations might be interpreted as trivial, we propose that they bespeak the students’ differential prior experiences with the Cartesian plane as a mathematical instrument. Again, as in the Uri/Eden episode, we witness how individuals who are “yoked” together in a joint perceptuomotor production may see a situation differently. But whereas Uri and Eden differed in their theorems, Lilah and Gideon appear to differ in their mathematical fluency.

Upon introducing the grid, the interviewer, JFG, prompts Lilah and Gideon to collaborate on finding green. As they manipulate the crosshairs on the screen, they find various green locations. After a few minutes of exploration, DA poses a question to the dyad, “Are you noticing anything?,” to which Lilah responds:
Lilah: **Well sometimes it’s on the top and the bottom, like [Gideon’s] is on the bottom and mine is on the top, and sometimes it’s when he’s on the middle [of a row of boxes] and mine is on the middle [of a different row of boxes].**

Lilah, too, notices that the relation between the two hands’ locations is relevant to achieving green, yet she does not interpret these locations as magnitudes (i.e., the hands’ height above the baseline) but only as within-box locations. That is, Lilah parses the gridded space into three types of possible crosshair locations relative to box yet regardless to the box’s height above the baseline: below a box, in the middle of a box, and on top of a box. Thus it appears that Lilah did not bring to bear previous experiences (with mathematical coordinate systems) that could have equipped her to assign height values to the relative locations she had identified.¹⁰

In contrast, Gideon, apparently a more graph-fluent student, orients to the grid as parsing the working space into enumerable quotas of spatial extension. In particular, he hooks to the grid as a means of regulating the execution of his “faster” strategy, only to find himself quantifying the vertical distance that each hand sequentially travels from one gridline to another in order to maintain a green screen. In his own words, “Every time the one on the left goes up one, that one [RT] goes up two.” Thus, Gideon’s prior experience with “grids” enables him to hook to the grid as a means of enacting a “faster” strategy, and then shift to an emergent *a*-per-*b* proto-ratio strategy; Lilah, on the other hand, was unable to hook to the grid so as to because she was not trained to recognize its normative mathematical function.

4.2.5 Pragmatic. How an instructor frames the introduction of the new symbolic artifact is likely to affect students’ mode of inquiry—as continuous or as ruptured. That is, students need to appreciate that the pragmatic intention of interpolating a new object into the problem space is to afford new means of engaging in the same activity as before rather than to begin a new

¹⁰ Lilah’s perceptual privileging of the “box” as compared to the gridlines is typical of early interpretations of reference frames. In the context of number lines, this phenomenon is often referred to as “fence vs. fencepost.”
activity. Otherwise, even a student who has discovered, confirmed, and sustained a naïve theorem-in-action and who is fluent with the new artifact might nevertheless interpret the introduction of this artifact as a cue for cognitive reoration—as though the symbolic artifact is in fact a new object of inquiry onto its own or even a distracter that should be actively ignored.

Consider Irit, a high achieving 5th-grade female student, who does not hook to the grid despite having developed and explicitly articulated a changing-distance theorem-in-action. While working in the crosshairs condition, several minutes prior to the introduction of the grid, Irit first asserts a fixed-distance theory, namely, that “they always have to be the same distance apart to get this green.” However, after testing her proposition empirically, she soon discovered that her strategy does not apply to green locations lower down along the display monitor. In response, Irit articulated a higher–bigger theory. Specifically, while comparing two green spots—one down low and the other a bit higher but just below the middle of the monitor—Irit noticed that a fixed-distance theory does not obtain. She exclaims, “Well—oh! No, it gets shorter if you go down more and then it gets tall… longer if you go up.” The interviewer, DA, then prompts her to find green in other locations, higher along the monitor, so as to test the veracity of her newly articulated theory. Irit confirms her changing-distance theory and, immediately, DA introduces the grid.

DA: **So I’m just going to add something here. Some ahh...** [operates computer console so to make the grid appear on the display monitor] **here we go... ok.**
Irit: [picks up both trackers and remote-places Lc and Rc near the middle of the display to make green; she then raises both trackers to make green higher up] **Yeah it’s the same thing, it still has to be higher—this one** [gestures with her right hand, points index finger straight up in the air so as to indicate that this hand is moving upward] **the lower this one** [LT] **has to go. The distance gets bigger up higher.**  
DA: **Ok, the distance gets bigger up higher.**
Irit: [Lowers trackers to make green down low] **And smaller down here.**
DA: **Ok, this stuff that I added, kind of grid thing, does that—**
Irit: **Doesn’t really do anything.**
DA: **Doesn’t do anything… Ok… Ok.**

Irit’s interpretation of the grid, we argue, was different from most students and apparently sprung from a pragmatic misunderstanding between the interviewer and her respecting the interviewer’s act of introducing the grid into the environment. The interviewer expected the student to appropriate the new tool as a means of enhancing her inquiry into the phenomenon *as is*. The student’s interpretation of the act, however, was that the phenomenon *itself* may have changed and consequently the rules might have changed. Consequently, rather than instrumentalize the grid as a means of enhancing her higher–bigger strategy, she examined whether her strategy still produced green given the ostensible change in the situation. She ascertained that her strategy was still effective and subsequently did not search for any new uses for the grid.

4.2.6 **Correct dimension, incorrect value.** A student might establish a theorem-in-action that articulates a relation among dimensions that indeed are embodied in the interactive device’s functionality yet differs with respect to the particular values assigned to these dimensions; the child’s theorem would ultimately be at odds with the targeted mathematical principle, yet the child would nevertheless recognize the new symbolic artifact as potentially conducive to enhancing their solution procedure and therefore hook to the artifact. This can occur when children have tested their theorem only within a narrow range of the interaction spectrum along the critical dimension, such that the feedback appears consistent with the theorem, or have employed confirmation bias of data that objectively refute their theorem.

For example, a child might believe that the distance between the crosshairs should remain fixed as they rise along the display—she has identified “distance” to be a phenomenal dimension shared with the instructor as a means of discussing the device yet he entertains “fixed” as a value
along this focal dimension. Consequently, when the grid is illuminated, the child recognizes its strategic and/or discursive potential—the child hooks to the grid. Moreover, this child might subsequently shift to a pedagogically desirable form even as she still maintains her non-normative theorem as the contents of that form—she might, for example, express the incorrect fixed-distance rule numerically. However, we found that symbolic artifacts introduced into the learning environment often create opportunities for these students to amend the incorrect values, probably because the new artifact’s features, such as the vertical span of gridlines, encourage exploration and articulation of a wider band of possibilities whose results spur cognitive conflict. To demonstrate this parameter, let us consider the case of Esau, a high-achieving 4th-grade male student, who hooks to but does not shift with the grid.

Early in the activity, Esau found “green spots” both near the bottom of the screen, where the distance was small, and higher up on the screen, where the distance was considerably larger. However, he never developed from these findings a verbalized hypothetical inference regarding the covariance of height and distance. Moreover, Esau explicitly stated that the distance between his hands is irrelevant for determining a rule governing the production of green spots: “So, it’s not if they’re far apart or not—it’s something else.” It appears that Esau rejects “distance” as a candidate dimension for articulating a rule precisely because the distance varies between the two spots—sometimes it is “far apart” and other times it is not. And yet, the rule of green, which most of our participants discovered, actually builds on this variance by tying it to the empirically corresponding dimension of height. A thorough analysis of Esau’s speech reveals that he never articulated the green spots in terms of their variability along the dimension of height. Rather, he refers to the low spot as “here” and to the high spot as “there.” However, although Esau does not explicitly state a changing-distance relationship between height and distance, when the grid is
subsequently introduced Easu nevertheless hooks to it as a means for regulating his search for an invariant quantity or property across green locations. Presumably, the sheer attention to the height as a candidate useful dimension appears to have sensitized Easu to this dimension.

Easu: [remotes-places Rc at about two-thirds of the way up the height of the display monitor and Lc at about half that height; screen goes green] Here it’s four boxes… so here’s four boxes… yeah. And then… [remote places Lc at about half way up the height of the monitor and Rc at about half of that height; screen goes red] No. I don’t see why it makes green down here [Lowers Rc to about 1-line and Lc to about half that height] Oh! So it’s… it’s also four boxes [sic!]. So that’s weird. So it’s four boxes here [sic!; where the “here” again refers to the “low” green location] and… where is it? And it’s four boxes there [finds the same “high” green location from before, with Rc at about two-thirds up the height of the monitor].

DR¹¹: When you say “four boxes,” which four boxes do you mean?

Easu: One—uh diagonally. One two—

DR: Oh diagonally so// one two three four// [indicates the four boxes that subtend a “diagonal” from the Lc to the Rc]

Easu: //One two three four// and then down here… [lowers trackers to the same “low” green position as before, with Rc at about 1-line and Lc to about half that height] one two three four apart so I guess they can’t move sideways so of course they’ll always be four apart, unless you move them diagonally in a way. But yeah so I guess they’ll always be four apart, going this way [horizontally] or diagonally.

In sum, when the grid was illuminated, Easu immediately began to use it as a means for measuring the distance between the left and right crosshairs. Whereas Easu took advantage of the grid’s affordance to quantify magnitude along the dimension of distance—the dimension intended by the designer—Easu did so to validate his emerging, mathematically incorrect, fixed-distance theorem-in-action. Easu thus groped for any distance within the perceptual problem space that remained invariant across multiple green locations, and consequently the horizontal distance became salient to Easu. In his attempt to construct the horizontal distance as meaning a fixed vertical distance, Easu articulates the would-be fixed vertical distance as an invariant diagonal (because diagonal lines afford both a “rise” and a “run” construction). In sum, Easu hooked to the grid’s pedagogically targeted dimension (distance) but not to the targeted value within that dimension (horizontal instead of vertical) and therefore could not disprove his

¹¹ Easu’s interview was conducted by an apprentice researcher (DR), with the lead researcher (DA) occasionally intervening.
incorrect theorem (fixed distance) but only bolster it (always four units apart). In any case, Esau consequently could not shift to more sophisticated forms of mathematical reasoning.

4.2.7 Perceptual compatibility. Students who have discovered and confirmed the utility of a proposed theorem-in-action, have retained the theorem when the new symbolic artifact was introduced into the working space; are fluent with the artifact; and even appreciate its pragmatic framing… may still not appropriate the artifact, if their idiosyncratic orientation of view toward the perceptual field, which had served them well in the embodied interaction, proves incompatible with the normative view. That is, whereas a student may have identified a strategy that is mathematically valid and consistently effects the desired feedback, the student’s particular perceptual object being manipulated by said strategy may be at odds with the didactical goals, and therefore impede their ability to shift to other, pedagogically advanced strategies.

For example, consider the case of Shani, a low-achieving 5th-grade female student who hooked to the grid as a means of elaborating a changing-distance strategy but with little success. Similar to several participants in this study (e.g., Esau, above), Shani had been attending not to the vertical distance between the two crosshairs but to their diagonal distance. However, Shani refers to this diagonal distance verbally as “distance,” and this ambiguous label is consistently interpreted by the interviewers as referring to vertical distance. Thus, Shani and the interviewers work together in the crosshair mode for nearly 14 minutes, with Shani consistently referring to the “distance”—how she is increasing the “distance” as she raises her hands and decreasing the “distance” as she lowers her hands; she even articulates covariant principles associating height and “distance.” However, Shani’s orientation of view, her indexical gestures eventually revealed, was toward the diagonal distance between the two crosshairs—it was the diagonal “object,” not the vertical “object,” that Shani was remote-controlling. Once the interviewer, DT, became aware of Shani’s orientation of view, he approbated yet probed her diagonal-based strategies. DT
did not attempt to deter Shani from implementing what was clearly a working strategy.

Consequently, when the grid is illuminated on the screen, Shani immediately hooked to it as a means of objectifying her diagonal construction. As we shall see below, though, hooking a diagonal orientation of view to a patently perpendicular plane was not unproblematic.

In particular, Shani noticed that, “[I]f you make rows diagonally uhm and find green, these two [crosshairs] will never like be on the same diagonal row,” and goes on to explain that “there’s squares and they’re going diagonally down.”

```
DT: [indicates a “diagonal row” subtending Rc and Lc] Like this?
Shani: Yeah. So they’re never on the same row, like you see uhm…
DT: Here [hands over RT to Shani].
Shani: [finds green spot at a height that is a little lower than mid-screen] So there… so now it’s green but they’re not on the same diagonal row, and the rows are next to each other though.
DT: So we’re talking about these rows? They’re next to each other…
Shani: So there’s two of them… that make… like you know if… if we have one little x on each end [where “x” refers to the crosshairs].
DT: What about if you try putting this one [RT] a bit higher?
Shani: [Raises the Rc to about three-quarters of the height of the screen; screen goes red] Well it…with the distance… it’s not the same distance so it wouldn’t really work.
```

Only by attempting to use the vertical–horizontal grid as a means of articulating a diagonal-based covariant strategy did Shani begin to appreciate the limitation of her idiosyncratic construction vis-à-vis the spatial structure inherent to the imposed frame of reference. That is, although she was able to objectify her diagonal by using the grid, the diagonal did not help her to leverage the grid’s mathematical affordances by articulating a strategy that quantitatively relates the crosshairs’ elevation and diagonal distance. After some probing by the interviewer, Shani, too, came to realize the contextual limitations of her otherwise perfectly legitimate diagonal construction. When asked how the grid might help her express the observation that different green spots have different diagonal distances, Shani responded that the grid enabled her merely...
to further *validate* the observation: “In a way, it kind of helps to like *know* that it *is* about distance, but from like my point of view.” Indeed, Shani’s idiosyncratic orientation of view—her remote-control object of a diagonal distance—was incompatible with the interim learning objective, namely, to recognize a covariant relation that could be expressed using height and vertical distance. Thus, whereas Shani certainly hooked to the grid, the idiosyncratic-vs.-normative perceptual mismatch did not enable her to shift beyond to different strategies.

4.2.8. Implementation. Finally, students who have hooked to a symbolic artifact may not shift due to factors that are not cognitive or social-interactional per se but rather are related to implementation. The interview protocol used in this project structures the interaction into a series of episodes, each onset by the introduction of a symbolic artifact. Whereas the sequencing of the symbolic objects remained invariant across all the interviews, there was variation with regard to timing and prompting. For example, students who hooked the grid yet were provided with relatively shorter durations of time to engage and explore it, were less likely to shift.

Furthermore, we noted variation across interviews with respect to the types of prompts that were used immediately following the introduction of the grid, and this variation appears to be related to variation in the prospects of students to hook or shift. For example, consider the case of Asa and Kaylen, two low-achieving 5th-grade male students, who both hooked to the grid but did not experience a shift. Prior the introduction of the grid, they had both observed, articulated, and agreed upon a higher–bigger theorem-in-action. Upon introducing the grid into the problem space, they both immediately used it as a means of better coordinating the coproduction of green and, moreover, they both used it to explain their emerging strategy. However, the dyad did not experience a shift because, we argue, the interviewer’s prompting did not properly equip them for one. Namely, once the dyad had established that “distance definitely
does seem to grow” from “one box,” to “two boxes,” to “three boxes,” and so on, the series of prompting that ensued did not engage this strategy in a substantive way. What the apprentice researcher (JFG) focused on instead was to determine whether or not the grid could be useful in making the screen green continuously. That is, JFG invited them to make the screen green “continuously from the bottom to the top,” a prompt that evokes notions of continuity within the problem space, whereas the students had only just articulated a strategy that explicitly relies on the discretization of that same space. Not surprisingly, in their attempts to effect continuous green, the students temporarily regressed from quantitative grid-based strategies to qualitative pre-grid descriptions. Clearly, students’ scope of productive engagement with cultural artifacts is heavily determined by the quality of instructors’ facilitation, which is manifest in nuanced detail.

In sum, we tentatively submit that these emerging categories (sections 4.2.1 – 4.2.8 above) may bear implications beyond the specific case of the grid and y-axis numerals. That is, we view students’ efforts to instrumentalize the grid and numerals as paradigmatic of the pedagogical program writ large of having students appropriate symbolic artifacts meaningfully. Evidently, steering students to spontaneously appropriate disciplinary forms as means of expanding and signifying their embodied interactions requires of designers and instructors to listen very closely to the students. Though learners’ ways of using the tools may be unintended by the designer, these ways embody the students’ understanding and, hence, their subjective learning potential (see also Hornecker & Dünser, 2009; Olive, 2000; White, 2008).

I now present a quantitative overview of hooks and shifts across our data in light of the interaction parameters identified in our study.
4.3 Quantitative Findings – Hooks and Shifts Across 15 Interviews

The data excerpts presented above elaborated on our proposed list of interaction parameters determining students’ likelihood to hook-and-shift through their interactions with the Cartesian grid and $y$-axis numerals. Table 1, below, uses this parameter list as one of several lenses on the data corpus. The table captures individual participants’: (1) motivations to hook the symbolic artifact; (2) strategies employed prior to appropriating the focal artifact; and (3) strategies emerging as the result of appropriating the symbolic artifact (i.e., from shifting).
Table 1.

**Student Motivations for Hooking/Shifting & Interactive Strategies**

**Motivations**
- **En**: “Enact” – student used the artifact to enact strategy.
- **Ex**: “Explain” – student used the artifact to explain a strategy.
- **Ev**: “Evaluate” – student used the artifact to evaluate a strategy.

**Main Qualitative Strategies**
- **FD**: “Fixed-distance” – student’s strategy is to maintain a constant spatial interval between hands/crosshairs.
- **CD**: “Changing-distance” – student strategy is to modify the spatial interval between hands in relation to the height of the hands/crosshairs above the baseline.

**Changing-Distance Numerical Strategies**
- **RS**: “Recursive” – indirectly expresses the height of the hands/crosshairs as a function of their previous locations.

**Specific Recursive Strategies**
- “a-per-b” – proto-ratio strategy that reflects sequential hand motions, in which each hand separately moves up or down according to its respective quantity. For example, some students expressed that Rc goes up by 2 units for every 1 unit Lc goes up (see the case of Uri and Eden, Section 4.1.1 above, for further explication).
- “a-per-(b-a)” – proto-ratio strategy that attends to the interval between between Lc and Rc as it changes with respect to the height of the left hand/crosshair. For example, some students noticed that as Lc moves up by 1 unit, the difference between Rc and Lc also increases by 1 unit (see the case of Shilav, Section 4.1.2 above, for other examples and further explication).

- **CE**: “Closed/explicit” – directly expresses green hand-pair locations as a function of the height of only one of the crosshairs. This strategy need not attend to previous pair locations but can offer right-hand location for a given left-hand location and v.v.

**Specific Closed/Explicit Strategies**
- “Additive” – an addition-based strategy. For example, some students expressed the form Height(Rc) = Height(Lc) + Height(Lc).
- “Multiplicative” – a multiplication-based strategy. For example, some students expressed the form Height(Lc) = ½*Height(Rc)—a “halving” numerical action. Similarly, other students also expressed the form Height(Rc) = 2*Height(Lc)—a “doubling” numerical action.

Table 2, below, gives an overview of participants’ successes and failures to hook and/or shift across fifteen case interviews (18 students total) with respect to the eight interaction dimensions identified above. In particular, this table represents hooks and shifts with the MIT set.
at a 1:2 ratio (i.e., the right hand/crosshair needs to be twice as high along the monitor as compared to the left hand/crosshair to produce green). In addition to Table 1 classifications, the entries also include timestamps that suggest how far along the interview these interactions occurred. [“Hf” = high-achieving female student, “Lm” = low-achieving male student, etc.; see Table 1 for more abbreviations].

A methodological clarification is due here, that would have perhaps been unclear if put in the Methods section. In creating Tables 1 & 2 for this study, the inter-rater reliability is “built into” the process of collaborative microgenetic analysis (Schoenfeld, et al., 1991). During the 2010 Summer and Fall academic semesters, our research team systematically reviewed each and every case. Over the course of our weekly group meetings, at least two members of the research team focused on each student listed in Table 2. For example, the principle investigator, DA, and the author, JFG, focused individually on the paired-interview case of Eden and Uri. DA provided the initial analysis of hook-and-shift for instances involving the grid only and presented his findings to the rest of the group. A general consensus was researched, and then JFG subsequently extended the analysis to instances involving the grid and numerals. These additional findings were presented to the group, and any discrepancies were resolved for this particular case. This iterative, intensive analytic process was repeated for all 15 cases presented here. We thus maintain that the collaborative, cyclic, intensive analytic approach is perhaps even stronger than determining inter-rater reliability between two independent coders, because each coding in Table 2 is already the result of resolving any disagreement among the research team.
Table 2. *Progressive Mathematization as Hooks and Shifts*

<table>
<thead>
<tr>
<th>Student</th>
<th>Grade 4</th>
<th>Grade 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Crosshairs + Grid</td>
<td>Crosshairs + Grid + Numerals</td>
</tr>
<tr>
<td><strong>Hook</strong></td>
<td><strong>Shift</strong></td>
<td><strong>Hook</strong></td>
</tr>
<tr>
<td><strong>Grade 4</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lilah (Hf) &amp;</td>
<td>(No hook - <em>Fluency</em>)</td>
<td></td>
</tr>
<tr>
<td>Naama (Lf)</td>
<td>(No hook - <em>Content</em>)</td>
<td></td>
</tr>
<tr>
<td>Shani (Lf)</td>
<td>[21:02] Ex, Ev: <strong>FD</strong></td>
<td>(No shift – <em>Perceptual compatibility</em>)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:**
- **FD**: Find dimension, incorrect value
- **CD**: Correct dimension, valid value
- **RS_a-per-b**: Rounding shift $a$-per $b$
- **CE_mult.**: Content and Emotional Multidimensional
- All timestamps are in 24-hour format.
Hooks and Shifts

<table>
<thead>
<tr>
<th>Name</th>
<th>Hook Type</th>
<th>Shift Details</th>
<th>Time</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liat (Mf)</td>
<td>No hook - Content</td>
<td>-</td>
<td>[27:50]</td>
<td>En, Ex: CE&lt;sub&gt;mult.&lt;/sub&gt;</td>
</tr>
<tr>
<td>Shilav (Mf)</td>
<td>[33:49]</td>
<td>En, Ex: CD</td>
<td>[34:50]</td>
<td>En, Ex: RS&lt;sub&gt;a-per-(b-a)&lt;/sub&gt;</td>
</tr>
<tr>
<td>Irit (Hf)</td>
<td>No hook - Pragmatic</td>
<td>-</td>
<td>[22:10]</td>
<td>En, Ex, Ev: RS&lt;sub&gt;a-per-(b-a)&lt;/sub&gt;</td>
</tr>
<tr>
<td>Elsie (Hf)</td>
<td>[18:22]</td>
<td>Ev: CD</td>
<td>(No shift – Correct dimension, incorrect value)</td>
<td>[19:30]</td>
</tr>
</tbody>
</table>

**Grade 6**

<table>
<thead>
<tr>
<th>Name</th>
<th>Hook Type</th>
<th>Shift Details</th>
<th>Time</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Siena (Lf)</td>
<td>No hook - Priming</td>
<td>-</td>
<td>[23:15]</td>
<td>En, Ex, Ev: CE&lt;sub&gt;mult.&lt;/sub&gt;</td>
</tr>
<tr>
<td>Penuel (Mm)</td>
<td>[23:15]</td>
<td>En, Ex, Ev: CD</td>
<td>(No shift – Correct dimension, incorrect value)</td>
<td>[40:08]</td>
</tr>
<tr>
<td>Totals</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Shaded row represents paired-student interview.
Our analysis across 15 interviews (18 participants) yielded that 5 of the 18 students hooked to and shifted with the grid; 7 students hooked to but did not shift with the grid; and 6 students did not hook the grid (and therefore did not shift). All 18 students hooked to and shifted by using the numerals. From our analyses, what appears central for the students’ deep/grounded learning is whether and when they discovered a changing-distance interaction principle, namely, the “higher–bigger” strategy. The implications of these findings are explored next in-depth; however, let us first revisit the MIT’s design rationale in order to interpret these findings.¹²

4.3.1. Interpreting the Quantitative Findings vis-à-vis the Design Rationale. During the initial design phase of the Mathematical Imagery Trainer, we neither intended for nor anticipated the hook-and-shift behavioral pattern. Rather, the hypothetical constructs that we eventually named hooks and shifts emerged as a means of describing a stable, ubiquitous phenomenon that we discerned during our data analysis. The objective of this study has been to delve deeper into this pattern by attempting better to characterize, illustrate, and quantify its parameters. We thus identified a set of interaction factors that were apparently critical for the phenomenon to occur. Whereas we are only beginning to make sense of this phenomenon, we nevertheless assert that it is pedagogically desirable. In turn, then, we ask how better to promote this phenomenon, and we strongly believe that our list of interaction parameters greatly advances our understanding of how to promote hooks and shifts in instrumented mathematics learning.

The rationale for our MIT experimental design is grounded in cognitive science theory (Barsalou, 1999), cognitive linguistics (Lakoff & Núñez, 2000), and pedagogical philosophy (e.g., "Realistic Mathematics Education," Freudenthal, 1991). Specifically, our theoretical focus

¹² For this report, the purpose of Tables 1 and 2 is to overview which students exhibited specific motivations and strategies. Due to space constraints on this manuscript, we omit an existing analysis involving variability of student behaviors/strategies across all 18 participants. For example, Table 2 suggests that older students (Grades 5 & 6) are more likely to experience more than one shift with the grid+numerals as compared to younger students (Grade 4).
is embodied cognition, which posits that action, perception, and reasoning are inextricably bound. Accordingly, the MIT was designed so as to create opportunities for students to partake in coordinated physical action that builds multimodal imagery consistent with our targeted mathematical content, proportional transformation, and grounding its learning. So doing, we designed for our research a context in which to explore the role that body-based multi-modal imagery serves in mathematical learning and reasoning. We selected the content domain of proportionality (e.g., $2:3 = 4:6$) because the learning of rational number concepts has historically been fraught with conceptual impediments (Davis, 2003; Lamon, 2007). We focused specifically on proportional progression, because this content, we conjectured, might lend itself to challenging physical actions and, thus, to challenging mental simulation that is necessary for learning the content. In sum, the general experimental design rationale for the MIT was to systematically investigate a type of embodied resource that would support the learning of proportional progression as well as how this resource might become mathematically signified.

Built into our experiment’s protocol are specific pedagogical considerations. Namely, the intent of our protocol-as-instructional-design was: (a) to elicit students’ default sense of invariance (fixed distance) and then enable the students to discover proportional progression as a variant on this default sense (i.e., that the interval should not be constant but actually grow); and (2) to cultivate this sense of a changing distance as the basis of understanding. We are interested in exploring whether and how students enact physical solution procedures for the MIT that, unknown to them, inscribe the kinesthetic image schema of proportional progression. Moreover, we seek to understand whether and how students ground their mathematical analysis in this physical solution and, specifically, in their qualitative, embodied sense of a changing distance.
Having rearticulated our design rationale and pedagogical intent, we are now in a position to recast our analysis and interpret our main finding in terms of the hook and shift constructs.

What appears central to achieving a powerful, grounded learning experience with the MIT is whether or not a student hooks to the grid. Our analyses suggest that students who articulate the “higher–bigger” rule early in the activity sequence (i.e., before the grid is introduced) and progressively mathematize it using increasingly formal symbolic artifacts have greater opportunities for grounding those symbolic artifacts—as well as the conventional manipulation procedures—in their embodied–intuitive experiences.

Table 2 above indeed shows that some students’ individual learning trajectories manifest in the form of a series of hooks and shifts (e.g., students Uri and Shilav). In particular, some students hook to the grid as a means of enhancing a changing-distance strategy then shift with the grid to different mathematical forms; following that, they hook to the numerals and then shift with the numerals to other, more sophisticated mathematical forms. We argue that individual students whose manifest trajectories take on this particular form experience a series of learning moments that closely align with our overall design intent and are pedagogically desirable.

At the same time, Table 2 shows that some students nevertheless hooked to the grid even when they had articulated a fixed-distance principle prior to its introduction into the learning environment (e.g., students Shani and Eden). In these instances, what implications does a fixed-vs. changing-distance intuition bear on students’ learning experience? We have already argued that having a changing-distance strategy as the basis for understanding is ultimately a powerful way to go. However, we are entertaining an alternative view: initially articulating a fixed-distance strategy and hooking to grid to substantiate that strategy, only to discover that the strategy does not obtain, is, too, a powerful, if different, learning experience. Yet this child
should then be enabled to re-experience the initial embodied interaction with the MIT that may now serve their new “intellectual needs” (Harel, in press) and, in turn, provide greater opportunities to further ground new inferences in embodied experiences.\textsuperscript{13}

Finally, our design rationale has been first to induce embodied interactions and only then introduce mathematical instruments for signifying these interactions. Consequently, our data consist exclusively of trajectories from embodied to instrumented action. Still, perhaps alternative intervention sequences would be equally if not more powerful. Indeed, our data indicate great variability in the way students inhabit the space created between their initial contact with the device and the end-state of having all the symbolic artifacts layered onto the display monitor. That is, whereas by-and-large all students arrived at the same destination, namely, all 18 student-participants hooked to and shifted with the grid+numerals, the path that each individual student embarked on is different. Could it be that students nevertheless experience cognitive conflict even with alternative intervention sequences, for example, where embodied and instrumented action co-occur? To explore the space that is opened by this question, let us first briefly revisit our thematic conjecture that informs our design.

The MIT intervention was designed and implemented in accord with conjecture-driven design-based research methodology. Specifically, our conjecture implicates lack of body-based action schemes and rehearsal of these schemes, as underlying a certain class of conceptual learning difficulties. Whereas inducing physical action while simultaneously manipulating symbolic artifacts still may give rise to kinesthetic–visual schemes, we worry that students may be less prone to rehearse those schemes or may even ignore these schemes all together, because

\textsuperscript{13}At this point, it is important to note that our protocol indeed calls for the interviewer to move all students back and forth between working in continuous space mode (sans artifacts) and working with the grid and grid+numerals, so that students have opportunities to blend these activities in a coherent, systematic, meaningful way. Table 2 reflects roughly the first third or half of each interview. We leave the analysis of the rest of these data for later publications.
their focus may be drawn too powerfully to the symbolic artifacts, such as their arithmetic affordances. That is, if we implement our intervention sequence with the grid+numerals at the very onset of the activity, for example, some students are liable to go into “number land” and use number-based solutions that never let them construct the embodied sense of a variant distance, because number-based solutions are what students are comfortable and familiar with. Indeed, mathematics-education researchers studying conceptual learning have pointed to the cognitive-pedagogical virtues of fostering qualitative sense prior to manipulating symbols (see Arcavi, 1994; Papert, 1996; Resnick, 1992; Thompson, 1993).

5. Conclusions

This paper contributes to the study of pedagogical situations formed by embodied-interaction design. In particular, our study has shown that students can experience powerful learning trajectories whereby a new type of mathematical principle expresses an embodied inquiry experience. In particular, fundamental meanings and principles of proportion emerged in students’ interactions as features of the ‘thing’ they were trying to make sense of through attempting to control it (cf. Granott, Fischer, & Parziale, 2003). Drawing on both constructivist and socio-cultural theory, we have offered the “hooks and shifts” constructs to explain the process of ontological emergence and account for variation across individual student trajectories. We have characterized discovery-based learning as a two-step process whereby students bootstrap themselves into pedagogically targeted ways of thinking and speaking by virtue of appropriating available symbolic artifacts as enactive, explanatory, or evaluative means (hook) and then spontaneously reconfiguring a current or emerging strategy to capitalize on the artifacts’ newly discovered affordances (shift).
Furthermore, we assert that the hook-and-shift behavioral pattern is a pedagogically desirable phenomenon and have offered a set of interaction parameters that would likely promote it. Yet, much further substantiation, elaboration, and validation of this emerging framework are required—across pedagogical contexts, mathematical content, and instructional designs. Indeed, we hope to build, refine, scale up, and research improved embodied-interaction designs for mathematics learning. In a sense, we are hooking to, and shifting with our empirical data, just like students learning within our designs.

References


