COORDINATING PHENOMENOLOGICALLY IMMEDIATE AND SEMIOTICALLY MEDIATED CONSTRUCTIONS OF STATISTICAL DISTRIBUTION

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ABSTRACT

I locate the drama of mathematical learning in students' creative attempts to coordinate two different mental constructions of a situation designed to embody a targeted mathematical notion: A phenomenologically immediate construction of the source situation, and a construction that is semiotically mediated by a mathematical model of the same situation. To successfully synthesize these constructions, students must assimilate mathematical units of analysis into their tacit schematic structures. One strategy for accomplishing this assimilation is to invent realistic, hypothetical, or fantastic images that lend plausible coherence to the blending of these tacit and formal orientations. We focus on Razi, a 11.5 year-old student learning the binomial distribution. She conjures an imaginary mechanical system to concretize her intuitive sense of distribution.

Keywords: *Statistics; probability; education; learning; embodied cognition; binomial; design-based research; Seeing Chance; ProbLab*

1. OVERVIEW AND POSITIONING VIS-À-VIS THE SRTL-6 THEMATIC FOCUS ON INFORMAL INFERENTIAL REASONING[1]

The thematic focus of the Sixth International Research Forum on Statistical Reasoning, Thinking, and Literacy is *informal inferential reasoning* (IIR). The particular aspect of IIR I shall address is 9–12 year-old students' informal constructions of explanatory models in the context of tutorial clinical interviews. These interviews, enacted as part of an ongoing design-based research project ("Seeing Chance"), sought to elicit the cognitive resources that students bring to bear in attempting to solve a problem-based task involving experimental instructional materials, including a random generator, combinatorial-analysis tools, and computer-based simulations. Initially focused on participants' coordination of "theoretical" and "empirical" activities, our data analysis has gravitated to the "theoretical" phase of the interview, even before any actual sampling has been enacted. Therein, we have come to surmise, lies some previously overlooked critical coordination that students must accomplish regardless of, or perhaps prior to, the subsequent "empirical" work. This coordination is between naïve and mathematical constructions of the experiment, which involved drawing out samples of size 4 from an urn containing marbles of two colors, green and blue (equal numbers of each color—p=.5—so that the actual experimental outcomes are anticipated to converge toward a 1:4:6:4:1 distribution).[2] Participants' naive perceptual construction of anticipated outcomes, it turned out, was qualitatively sound—by and large they expected a plurality of outcomes with two marbles of each color, a rarity of color-uniform outcomes, etc. And yet their articulation of these expectations was exclusively in the form of five heteroprobable events tacitly aggregated over permutations, i.e., no-green, 1-green, 2-green, 3-green, and 4-green—whereas combinatorial analysis of the experiment constructs the distribution as five *sets* of equiprobable elemental events, i.e., one no-green, four 1-green, six 2-green, four 3-green, and one 4-green.

We have been intrigued by the types of reasoning that students manifested as they were guided to juxtapose and ultimately reconcile the two competing constructions of the expected distribution—the tacit aggregate-event-based phenomenology of the source situation, and the analytic elemental-event-based model of the same situation. These reasoning processes were particularly interesting, because typically the participants would initially reject our attention to elemental events (the permutations), as though these were irrelevant to the task of determining properties of the aggregate events (the combinations). Our working hypothesis is that students' eventual understanding can be explained through the lens of semiotics—the study of signs and how people invent, learn, and use them. Namely, we conjecture that despite the initially unintuitive nature of the elemental events as units of analysis, the sample space as a whole—which we intentionally organized so as to make salient its five event sets as well as the number of elements in each set—attracted the students as potential *means of objectifying* (Radford, 2003) their tacit judgments, thus functioning as a *material anchor for a conceptual blend* (Hutchins, 2005) between the tacit and mathematical constructions (see also Stavy & Tirosh, 1996). That is, the students could not in principle use the sample space to *articulate* their tacit perceptual judgment, because that judgment mechanism is cognitively impenetrable (Pylyshyn, 1973). Rather, they performed a "semiotic leap" that bridged epistemologically disparate resources—from tacit cognitive material to discursive formulation. Moreover, we conjecture, it was only once participants had appropriated the sample space globally, that they *ex post facto* accepted its inherent expansion of the aggregate events into sets of elemental event, so that students made sense of the combinatorial-analysis solution algorithm only after its product—the sample space—was meaningful to them.

To date, we have examined students' reasoning in interviews with 28 students in Grades 4–6 (9–12 years old) (Abrahamson & Cendak, 2006), 23 students in Grade 7/8 (12-14 yo) (Mauks–Koepke, Buchanan, Relaford–Doyle, Souchkova, & Abrahamson, 2009), and 25 senior undergraduate college students in mathematically-oriented programs (Abrahamson, 2007). We have analyzed students' coming to understand the targeted notions in terms of semiotic processes (Abrahamson, in press-b), cognitive conflict (Abrahamson, in press-a), creativity (Abrahamson, 2008), and the work of C. S. Peirce on "abductive" logical operations (Abrahamson, under review), and we have drawn conclusions for a heuristic framework for the design of cognitively ergonomic bridging tools—inscriptions that supports students' coordination of tacit and formal constructions of realistic situations (Abrahamson, 2006b, 2009, in press-c; Abrahamson & White, 2008; Abrahamson & Wilensky, 2007). Currently, we are returning to the data, seeking to understand the role that students' "wilder" creative inventions—ad hoc, explorative, sometimes fanciful and elaborate imagistic notions—play in their attempts to coordinate the phenomenologically immediate and semiotically mediated constructions of problematized situations. As a case study, we will examine herein the images that Razi, a 6th-grade female student, conjures as she attempts to make sense of the inclusion of elemental events in the formal analysis: Razi likens the five aggregate events to five sieves through which pasta-like elemental events [sic] are sampled—the more likely an aggregate event, the more porous its sieve.[3] We ask:

- What roles precisely do students' self-initiated images play in their reasoning?
- Do images act as scaffolds toward the coordination of tacit and formal constructions of the random experiment?

- How might an ostensibly irrelevant fanciful construction support Razi's mathematical sense making?

We begin below with a discussion of the theoretical background of our general research problem. Next, we present an "apologia" for the inclusion of an essentially "probability" paper in an explicitly "statistics" conference. (The pre-convinced reader may wish to skip this section.) We then turn to further descriptions of the instructional design employed in the interviews and furnish some relevant methodological points. Following an annotated transcription of the conversation between the interviewer and Razi, we end with some directions that the SRTL-6 interactive session might take.

2. BACKGROUND AND GENERAL RESEARCH PROBLEM

Commenting on *Intuition in Science and Mathematics* (Fischbein, 1987), Cobb (1989) discusses intuition as a "double-edged sword": Intuition is the root of all meaning, and yet advanced mathematical notions are liable to defy the "terrestrial" origin and scope of intuition, so that constructing these challenging notions demands cognitive capacity alternative to "primary intuition." Fischbein's conclusion, which Cobb appears to endorse, is that students' guided acculturation into mathematical practice is one of developing "secondary intuition" geared to operate meaningfully within non-intuitive disciplinary terrain. And yet, in *Where Mathematics Comes From,* Lakoff and Núñez (2000) argue that even Euler's Identity—a challenging "abstract" proposition—can be "embodied," i.e., constructed as iterative projections of successively elaborated conceptual metaphors. Notwithstanding Lakoff and Núñez's tour-de-force, which extends the work of Johnson's (1987) thesis on the necessarily embodied nature of reasoning, I concur with Fischbein and Cobb that some mathematical ideas are very difficult to embody, and in particular those ideas that resulted from historical need for closure, e.g., the intriguing expression $a^0 = 1$ (Rotman, 1993, 2000). Creating a list of officially unintuitive mathematical notions, however, may give mathematics educators—designers, teachers—too much license to file as "unintuitive" mathematical notions that students struggle to learn, whereas students could potentially ground a subset of these unintuitive notions in their primary intuitions. Characteristic of these mislabeled notions is that their canonical cultural formulations do not afford the tacit application of natural cognitive capacity, e.g., an expression of probability as ".13" does not readily afford part-to-whole comparison (Gigerenzer, 1998; Gigerenzer & Brighton, 2009; Gigerenzer, Hell, & Blank, 1988; Zhu & Gigerenzer, 2006)—the formulations are not "cognitively ergonomic" (Abrahamson, 2009) and are liable to subject learners to the *ontological imperialism* often inherent to canonical notations (Bamberger & diSessa, 2003). At the same time, creating cognitively ergonomic mathematical forms that do enable intuitive entry points might still pose the problem of fording a remaining gap from diagrammatic artifacts to normative symbolic inscription. How teachers and students in fact ford such gaps, and how some students fall short of crossing this cognitive divide, has been articulated through the lens of semiotics (Bartolini Bussi & Boni, 2003; Duval, 2006; Lemke, 1998; Presmeg, 2001; Radford, 2003). And, yet, where much work still lies, I believe, is in understanding, at a theoretical level, how human intuition comes to inhabit semiotics space (cf. Sfard, 2002; Sfard, 2007).

Taking on this general research problem of how humans coordinate intuitive and semiotic behavior in the context of learning mathematical notions, I strive to examine the potential compatibility of two compelling intellectual strands, which prima facie may appear at odds and yet are arguably complementary: On the one hand, phenomenology

(Heidegger, 1962; Husserl, 2000; Merleau-Ponty, 1992) and genetic epistemology (Piaget, 1968) view learning as contingent on reflexive abstraction over immersive embodied action; and on the other hand, cultural–historical psychology and its scions (Engeström, 1999; Lave & Wenger, 1991; Vygotsky, 1930/1978, 1934/1962) view learning as the internalization of activity-based cultural heritage through guided, legitimate peripheral participation in situated social practice. Indeed, diSessa (2008) has called for closer examination of a potentially "dialectical" relation between cognitivist and sociocultural theory. Operating within this dialectical program, I view artifacts—substantive or virtual embodiments of mathematical notions and tools for operating thereupon—as generative foci of mathematics-education research in general (cf. Sfard & McClain, 2002) and for the project of articulating a cognitivist–sociocultural theory of mathematical learning, in particular. Thus, the design-based research approach is specifically suitable for my research objectives, because its praxis of developing and field-testing instructional materials enables me to investigate student coordination of these complementary resources in constructing mathematical notions, even as I attempt to support this learning process.

3. APOLOGIA: WHY RESEARCH ON PROBABILISTIC COGNITION MAY BE RELEVANT TO RESEARCH ON STATISTICAL COGNTION

diSessa (1993; diSessa & Wagner, in press) maintains that individuals' conceptual understanding in science and mathematics is in the form of "knowledge in pieces." That is, students' construal of situations pertaining to a concept is mediated not by a single coherent structure (a 'theory') but, rather, recruits multiple phenomenological primitives. These p-prims are relatively small, domain-neutral regulatory schema, i.e., tiny bits of intuitive knowledge about the behavior of the world, such as the "Ohm's law p-prim," whereby "more effort yields more result." Making sense of a situation is tantamount to invoking a cluster of p-prims and coordinating them to each other as well as other cognitive functions active in the execution of analysis, description, inscription, etc. Learning, or 'conceptual change,' is the process of "tuning toward expertise" by routinizing the coordination, constraining, and calibrating, and application of the p-prims.

One immediate implication of the knowledge-in-pieces theory, and in particular the explanation of conceptual understanding as the aggregate activity of small-grain-size, "sub-disciplinary" actions, is that concepts traditionally classified as pertaining to different disciplines may in fact share much cognitive material. Specifically, cognitive activity that research has implicated in students' understanding of statistical constructs may play important roles in their making sense of probability concepts, and vice versa, albeit the specific constellation of the underlying pieces of knowledge may be differently configured and weighted for each domain. For example, notions of sampling, randomness, and distribution that are shared by, *yet have different meanings* in these two disciplines would be expressions of the same underlying p-prims that have been coordinated, constrained, and calibrated to meet the demands of each disciplinary practice. Consequently, students' guided conceptual development need not be a priori differentiated, e.g., into 'sampling-for-statistics' and 'sampling-for-probability,' but may, instead, begin by dwelling on the shared ideas and then proceeding as a nuanced differentiation into discipline-specific practice. Also—and more pertinent to the SRTL enterprise—insights from research on the cognition of probability may bear upon analogous research on statistics, especially in the case of early probability and statistics.

In closing his recent monolithic survey chapter, *Research on Statistics Learning and Reasoning*, Shaughnessy (2007, p. 1002) lists and elaborates on five implications of this research for the teaching of statistics. I shall focus on the first of these implications as a context for speaking further about the relevance of educational research on the teaching and learning of probability to analogous research on statistics.

Namely, Shaughnessy warns us to, *"Remember that there is a difference between statistics and mathematics!"* Indeed, statistics and mathematics—and specifically statistics and probability—are different. But they are similar, too, and for the young learner, who is constructing knowledge bottom-up from learning activities, the differences between statistics and probability may initially be very nuanced or even completely obscure. A fortiori, Abrahamson (2006a) demonstrated specific instructional activities involving uncertainty, sampling, and distributions that could be interpreted as pertaining to either probabilistic or statistical content contingent only on how these activities are framed and in particular on whether or not the student knows the structural properties of the objects used in these activities. Specifically, a student drawing a sample from an urn full of mixed marbles of two colors can be said to be engaging in a statistical activity if she is attempting to determine the color-ratio by studying the distribution of sample means (thus creating an opportunity to discuss the Central Limit Theorem), but she can equally be said to be engaged in a probability experiment if she already knows the color distribution (thus creating an opportunity to discuss binomial approximation of hypergeometric distributions and the Law of Large Numbers).

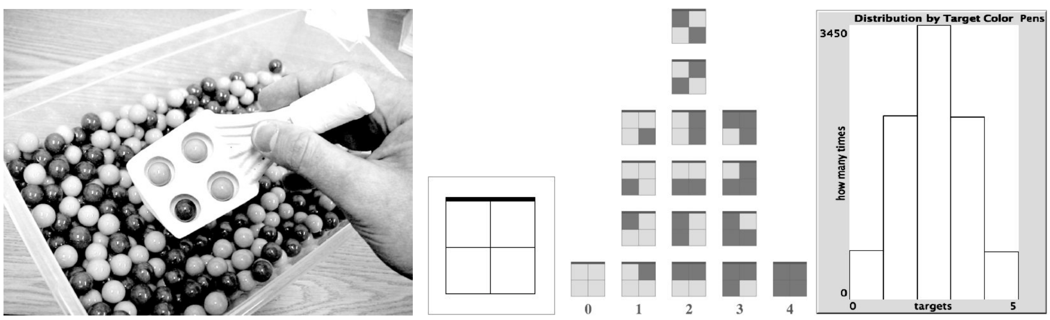
Media, such as urns, play a role in "fuzzifying" these boundaries of probability and statistics. Consider computer-based interactive modules that enable rapid dynamic sampling linked to "growing" distributions, iconic representations, and numerical displays. The computer environment, with its common interface tools for conducting either probability or statistics activities, underscores the common conceptual grounds of these disciplines and facilitates smooth transitions between their respective reasoning patterns (see in Konold, Harradine, & Kazak, 2007, the apparent pedagogical utility of having students engineer distributions; see in Pratt, 2000, how students either guessed the state of the 'workings box' or set this box and investigated the resulting actual experimental outcomes). Finally, the manifestation of uncertainty in statistics and probability is related, albeit reciprocal: In statistics, we know what our samples are, yet we are uncertain about the population measures that we conjecture on the basis of these samples; In probability, we know what our "population" is (e.g., the contents of an urn), but we are uncertain as to the values of each and every sample we are about to draw from this population (but the more samples we take, the more certain we are that our actual outcome distribution will converge on the anticipated distribution). Abrahamson concluded that we should revisit the traditional disciplinary boundaries between statistics and probability which—critically to our discussions—often play out in compartmentalized curriculum, too.

Implications of this line of reasoning are that: (a) it could be that the teaching and learning of statistics and probability should initially be completely undifferentiated, with the goal of building a strong common ground and only subsequently letting these disciplines drift apart on the basis of their critical differences that should be revealed and explained; and (b) research on the early cognition of probability is relevant to research on the early cognition of statistics.

4. METHOD AND DESIGN

The data excerpt cited below is from a clinical interview with Razi, an 11.5 year-old student ranked by her mathematics teachers as high achieving. Whereas her vocabulary and confidence were atypical of her 27 school-mates in this study, the nature of her creative insights, demonstrated below, were typical of this sample. The author conducted a semi-clinical interview using a flexible protocol (Ginsburg, 1997), the session was videotaped for subsequent analysis, and selected episodes were transcribed. The research team employed collaborative qualitative microgenetic analysis techniques (Schoenfeld, Smith, & Arcavi, 1991; Siegler & Crowley, 1991) as well as grounded theory (Glaser & Strauss, 1967) through which new constructs were articulated (diSessa & Cobb, 2004) and iteratively crosschecked against the entire data corpus.

Figure 1 overviews four central objects and target artifacts constructed, automatically generated, and/or used in the interview. The marbles scooper draws a sample of exactly four ordered marbles from an urn of mixed green and blue marbles (see Figure 1a). This hypergeometric experiment (without replacement) approximates the binomial due to the ratio of *n* (4 marbles) to the content of the urn (hundreds of marbles). We ask the student to guess the expected outcome distribution. Next, the student is guided to use crayons as well as a pile of stock-paper cards each bearing a blank 2-by-2 matrix (Figure 1b), so as to create the sample space of the experiment and assemble it in the form of the *combinations tower* (see Figure 1c). Later, the interview moves to computer-based simulations of the same experiment (e.g., see Figure 1d).



a. b. c. d.

Figure 1. Materials used in the study—theoretical and empirical embodiments of the 2-by-2 mathematical object: (a) The marbles scooper; (b) a template for performing combinatorial analysis; (c) the combinations tower—a distributed sample space of the marbles-scooping experiment; and (d) an actual experimental outcome distribution produced by a computer-based simulation of this probability experiment.

5. EMERGING EMPIRICAL FINDINGS: STUDENTS' METAPHORS AS WINDOWS ONTO THEIR REASONING

The following exchange took place 35 minutes into the interview, when Razi had completed the guided construction of the combinations tower. Like other participants, Razi anticipated that the experimental outcome distribution would have a plurality of two green and two blue marbles (hence, "2g2b"). Also similarly to her fellow participants, Razi experiences difficulty in reconciling two observations that are both valid yet appear to her as though they are contradictory: (a) The sixteen elemental events are equiprobable; and (b) The five aggregate events are heteroprobable. That is, Razi appears comfortable entertaining each of these views but is not sure whether or how they might be compatible.

The interviewer (the author) invites Razi to think deeply about the relation between the sample space and the experiment. Specifically, he asks her why a plurality of elemental events in the 2g2g subset of the sample space implies that we should get a plurality of actual experimental outcomes of that particular aggregate event. Where the interviewer means to lead Razi is toward an idea that would become clearer in the "empirical" phase of the interview, when the dyad would work with the computer-based simulations of the marbles-scooping experiment. Namely, that all sixteen elemental events in the sample space are equiprobable necessarily means that we expect to receive roughly equal amounts of samples of each type; therefore the actual experimental outcome distribution is expected to comprise sixteen subsets that, in turn, are mixed within their respective columns. For example, an experiment that drew 16,000 samples is expected to have roughly 1000, 4000, 6000, 4000, and 1000 outcomes in its five respective bins. Razi, however, does not or cannot take the interviewer's lead to engage in logical reasoning and calculation. Instead, she invents the following metaphor.

Dor: Can we be more rigorous about why having more here [more 2g2b elemental events in the combinations, as compared to the number of elemental events in other columns] means that we'll get more [in actual experiments]?

Razi: Well, since there's the same amount of blues and greens [in the box], and since it's the same amount of these…of this [lifts the marbles]… And there's the most…The most of it… There's… The most different patterns are in this [2g2b column in combinations tower], so it allows for the most…I guess you could say "wiggling room" almost.

Dor: In what sense, "wiggling room?"

Razi: Like, it allows for the most combinations. It lets the most combinations kind of slip into this category. [Razi's 'combination' = our 'permutation']…. Like through a sieve. Let's say, all of these [other columns]—they… they have, like, three green or only one green, or no greens, or all greens, uhhm… But they have too many greens, so they would be thrown aside. But this one [2g2b] has the most… like in a sieve of just two greens, it would have the most things slip through.

Dor: Yeah, I kind of get this—I'm not completely sure I got this, uhhh, this metaphor of the sieve. Can you tell me more about it? Or can you draw it? What do you mean by that? […]

Razi: Ok, let's say I had two green… I had… Pretend these [grabs some marbles from the box, holds in hand] are all pasta—every one of these is pasta. [returns marbles to the box] And so I had green pasta and blue pasta. And I had a sieve that would only let two-green pasta, like... And then each one of these [lifts scooper]… And then, like, one of these squares [scooper] was like a clump of pasta that's stuck together. And the sieve only lets through clumps of pasta that have two greens and two blues. And then there's another sieve that only lets through things with only… with *three* greens. And so… And let's say and I say… and I did… I sieved both things—the same amount in both containers—in both sieves. And, uhmm… But the one that only lets two greens and two blues through would let the most clumps fall through, since there're the most of these.

Dor: Hm'hmmm. So there's like more holes in that sieve?

Razi Almost… (see Figure 2, below)

Dor: It's kind of a magical sieve that kind of identifies what's coming and so it allows only them through?

Razi: Yeah, basically.

Dor: Got it. Got it.

Razi: That's a really bad metaphor.

Dor: No, I love it—it's great. So like you can pour in, and the holes know… There's like more holes for *that*…more, like opportunities. More… It grabs more of the stuff to that kind of place.

Razi: Yeah.

Dor: Ok. I think I have such a sieve here…Not with pasta, though, but kind of like… You wanna see?

Razi: Yes… Ok… [the dyad turns to work with computer simulations]

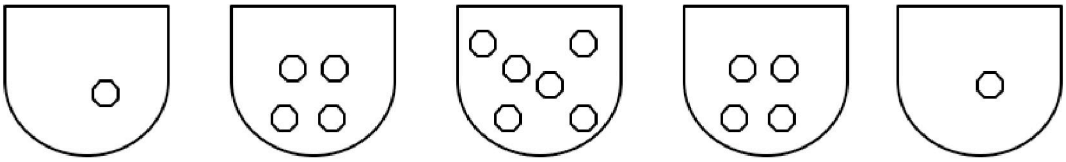


Figure 2. Schematic rendering of the interviewer's partial understanding of Razi's sieve metaphor. The interviewer thought Razi was saying that the number of holes in each of these sieves corresponds to the number of elemental events in its sample-space subset. But Razi was not quite there yet.

The sieve analogy appears to enable Razi to objectify the relative chances of the event classes as determined by directly examinable properties of an invented random generator. With this invented mechanism, Razi may reconcile two intuitions that are mathematically viable yet had appeared contradictory. Namely, by assigning aggregate events to sieves and elemental events to holes, Razi appears to have created ontologically distinct categories (sieves, holes) and thus differentiated these constructs cognitively.

I am currently unsure of the above interpretation. In fact, I am quite puzzled by several aspects of this episode:

- Why did Razi's invent the sieves metaphor?
- Why did the interviewer misinterpret Razi?
- Why did Razi appear not to understand the interviewer, or at least not to appreciate the utility of his embellishment on her own metaphor?
- Why was Razi apparently dissatisfied with her own metaphor?
- What function might such a metaphor play in Razi's reasoning?
- What pragmatic aspects of the social situation may have suggested to Razi the utility of invoking such a metaphor? Namely, was the metaphor primarily a discursive vehicle? (That is not to mitigate the potential capacity of the methapor to push Razi's thinking and create focused teaching/learning opportunities.)
- Where might a more skilled tutor–interviewer have taken this conversation? That is, despite the constraints of the interview protocol, might it have been useful to pursue this imagery further?
- What might we "take home" from analyzing such episodes with respect to broader theoretical issues pertaining to student learning—of mathematics, of statistics, of any STEM topic? In particular, does the episode promote the emerging model of student learning as a guided struggle to align phenomenologically immediate and semiotically mediated constructions of problematized situations?

6. INTERIM SUMMARY AND POTENTIAL IMPLICATIONS

I have suggested the relevance, to research on IIR, of research on students' spontaneous sense-making of phenomena embodying constructs fundamental to the mathematical theory of probability. To these ends, following a theoretical argument that builds on diSessa's epistemological framework, I furnished an empirical example of instances wherein students recruited metaphor and logical reasoning to coordinate between, on the one hand, their intuitive sense of a random generator's outcome distribution and, on the other hand, the sample space of this experiment. The notions at the core of this student's reasoning as well as the forms that this reasoning took are all germane to parallel research endeavors on students' IIR in the context of statistical activities. I conclude that research on probability cognition is well poised to contribute to the statistics education, in general, and to the SRTL on-going project specifically.

Furthermore, as our study participants' fanciful inventions should demonstrate, an educator might never be prepared for all specific ideas that students may bring to bear in their attempts to make sense of mathematical notions. However, my objective is to support teachers in developing a *disposition* toward such inventions. Namely, I believe that teachers should treat these inventions as authentic struggles to align and/or extend intuition so as to accord with disciplinary knowledge.

I am very excited at the prospects of interacting with my esteemed colleagues so as to keep critiquing and pushing forward both the emerging theoretical models, both of learning and instructional design, as well as the instructional materials, which may eventually prove of some use to students in institutional settings.

NOTES

[1] This draft is intended to furnish SRTL-6 attendees with relevant background of this project. For an MTL submission, I would modify the structure and tone of some sections.

[1] A note on terminology: For reasons elaborated elsewhere, I shall use the following vocabulary throughout this text. In each compound event there are four "singleton events." The compound, viewed with attention to the particular order of its four singleton events, constitutes an "elemental event"—one of 16 listed elemental events that make up the expanded sample space. Subsets of the sample space, each comprised of elemental events that have the same number of singleton events of uniform color, are called "aggregate events"—there are five aggregate events comprised respectively of 1, 4, 6, 4, and 1 elemental events. So aggregate events are the "combinations" that are made up of "permutations." Students are not initially aware of the aggregate–elemental or combination–permutation distinctions, as so if I write that students are seeing the situation as a set of aggregate events or as combinations, I do not mean to ascribe to them that differentiation but rather to re-formulate their utterances from a mathematical perspective, so as to highlight and hone the learning challenges. Using a possibly clearer noun that is perhaps colloquial or confusing, I would say that students initially see five "things" or five "objects." Note also that "aggregate" and "elemental" are not inherent properties of a compound event but may be in the eye of the beholder. That is, a student and a teacher who are both looking at the outcome sequence "green, blue, blue, green" may see it as an elemental event (teacher) or an aggregate event (student). The student herself may shift between aggregate and elemental constructions of the outcome.

[1] A 3'22'' video clip (18MB) of the focal excerpt can be viewed or downloaded at http://edrl.berkeley.edu/publications/journals/MTL/Abrahamson-MTL

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Video clip:

- File name: 2005-12-13_G6-Razi-sieve.mov
- Duration: 3’22 min.
- Recorded: Dec. 12, 2005. Begins at 35 min. into a 85 min. long interview
- Available: http://edrl.berkeley.edu/publications/journals/MTL/Abrahamson-MTL
- Content: The author interviews an Razi (pseudonym), an 11.5 yo consented minor participant, about her mathematical reasoning.