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Learning Is Moving in New Ways: The Ecological Dynamics of Mathematics Education

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Whereas emerging technologies, such as touchscreen tablets, are bringing sensorimotor interaction back into mathematics learning activities, existing educational theory is not geared to inform or analyze passages from action to concept. We present case studies of tutor–student behaviors in an embodied-interaction learning environment, the Mathematical Imagery Trainer. Drawing on ecological dynamics—a blend of dynamical-systems theory and ecological psychology—we explain and demonstrate that: (a) students develop sensorimotor schemes as solutions to interaction problems; (b) each scheme is oriented on an *attentional anchor*—a real or imagined object, area, or other aspect or behavior of the perceptual manifold that emerges to facilitate motor-action coordination; and (c) when symbolic artifacts are introduced into the arena, they may both mediate new affordances for students' motor-action control and shift their discourse into explicit mathematical re-visualization of the environment. Symbolic artifacts are ontological hybrids evolving from things with which you act to things with which you think. Students engaged in embodied-interaction learning activities are first attracted to symbolic artifacts as prehensible environmental features optimizing their grip on the world, yet in the

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course of enacting the improved control routines, the artifacts become frames of reference for establishing and articulating quantitative systems known as mathematical reasoning.

Mathematics, like music, needs to be expressed in physical actions and human interactions before its symbols can evoke the silent patterns of mathematical ideas.—Skemp (1983, p. 288)

Rules, like birds, must live before they can be stuffed.—Ryle (1945, p. 11)

INTRODUCTION: IN SEARCH OF AN ACTION-ORIENTED THEORY OF MATHEMATICAL ONTOGENESIS

Background and Objective: Why Educational Theory and Practice Need an Action-Oriented Theory of Mathematics Learning

With the increasing public availability of advanced technological platforms, we are witnessing an efflorescence of commercial products designed for interactive learning of mathematics content. In this brave new world, users manipulate virtual objects to complete engaging tasks and, in so doing, per the vendors, develop conceptual understanding of target notions, such as arithmetic operations. Although these electronic devices are slow to enter mainstream education, they are literally at the fingertips of any child who has access to a tablet; a smartphone; or any other natural user interface platform, such as Wii, Xbox Kinect, or Leap Motion. It is understandable that this unprecedented outburst in downloadable, over-the-counter edutainment is slow to be evaluated, let alone guided by the educational research community (Abrahamson, 2015). It is problematic, though, that extant theory of learning is by and large a theory of learning with paper, informed neither by the interaction possibilities of emerging technologies nor by what these possibilities could imply for mathematical epistemology and pedagogy (Papert, 2004). In the short term, the scarcity of bold research on interactive mathematics learning impedes the formulation of empirically based progressive policies concerning the integration of technological environments into educational institutions. In the long term, this scarcity is accelerating misalignment between theory of learning and emerging practices to which it should apply. As children are learning to move in new ways, theory of learning should move in new ways, too.

A motivation behind this article is that the pedagogical quality and institutional acceptance of action-based learning environments largely depends on developing informed scholarly and public discourse concerning what it means to learn a mathematical concept and what an instructor's role might be in this process. Thus, we are echoing Papert's consistent call to leverage the technological revolution as an opportunity for deep discussion of the potentially radical changes educational systems should undergo (Papert, 1993, 1996). Similar to Papert, we are optimistic that technological advances in educational media bear the potential of fostering students' deep understanding of mathematical concepts. Complementarily, these technological advances bear the potential of fostering researchers' deep understanding of learning processes. The objective of this article is to contribute first steps toward developing a theory of action-based mathematics learning. We take these first steps by arguing for what we believe to be productive directions for investigating action-based learning, namely, adopting perspectives from scientific disciplines dedicated to the study of motor action.

We begin by introducing the empirical context and findings that have motivated us to seek, beyond seminal theories of mathematics learning, new approaches oriented on cognitive, physiological, material, and social factors at play in motor-action skills development.

Empirical Context: The Mathematical Imagery Trainer for Proportion (MIT-P)

Our argument for the added value of action-based disciplinary perspectives is situated in emerging findings from qualitative analyses of empirical data gathered in the context of implementing an experimental design for mathematics learning, the MIT-P. In this study, volunteering study participants manually operated an unfamiliar technological system with the task objective of bringing this system to a prescribed goal state, namely, moving their hands in space to make a screen green (Abrahamson & Trninic, 2011; Howison, Trninic, Reinholz, & Abrahamson, 2011). Figures 1 and 2 offer an overview of the design.

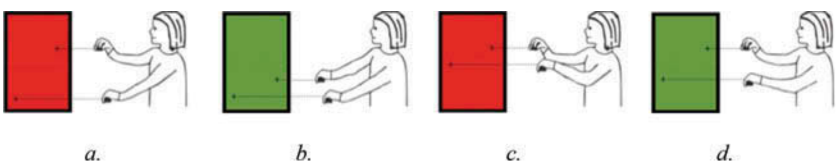


FIGURE 1 The Mathematical Imagery Trainer for Proportion set at a 1:2 ratio, so that the favorable sensory stimulus (a green background) is activated only when the right hand is twice as high along the monitor as the left hand. This figure sketches out our Grade 4–6 study participants' paradigmatic interaction sequence toward discovering an effective operatory scheme: while exploring, the student (a) first positions the hands incorrectly (red feedback); (b) stumbles on a correct position (green); (c) raises hands, maintaining a fixed interval between them (red); and (d) corrects position (green). Compare (b) and (d) to note the different vertical intervals between the virtual objects.

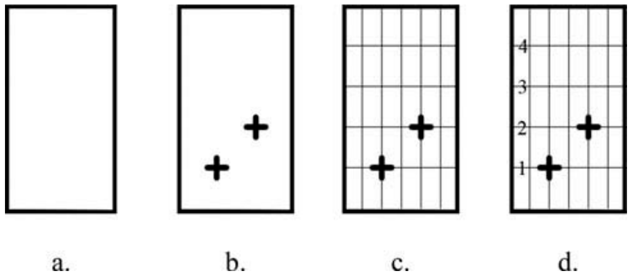


FIGURE 2 Mathematical Imagery Trainer for Proportion display configuration schematics, beginning with (a) a blank screen and then featuring a set of virtual objects overlaid incrementally by the facilitator onto the display: (b) cursors, (c) a grid, and (d) numerals along the y-axis of the grid. For the purposes of this figure, the schematics are not drawn to scale. Also, the actual device enables the tutor to flexibly calibrate the grid, numerals, and target ratio in between trials. The full protocol includes a range of ratios as well as a ratio table.

As they attempted to determine effective bimanual choreographies for manipulating the system, and still before the grid was introduced, participants discerned within the sensorimotor interaction field latent structures affording utilities for better satisfying the task objective. In particular, the negative space between their hands became foregrounded as a thing that they manipulated as a means of making the screen green—the higher they raised the interval, the bigger they made it. Moreover, when we then introduced into the interaction system certain symbolic artifacts—a grid and then numerals (see Figure 2)—the participants adopted these screen elements as frames of action and reference. In turn, using these artifacts shifted the participants' manipulation strategies into forms of engagement closer to mathematical visualization and reasoning. For example, for a 1:2 ratio they raised their hands sequentially, with the left hand going up 1 unit and the right hand going up 2 units (Abrahamson, Gutiérrez, Charoenying, Negrete, & Bumbacher, 2012; Abrahamson, Lee, Negrete, & Gutiérrez, 2014; Abrahamson & Trninic, 2015; Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011).[1]

[1]Readers are referred to earlier publications for more detail on the design rationale that led to the development of the MIT-P, including a critical reading of previous literature on the cognition of multiplicative concepts (Abrahamson, 2015; Abrahamson et al., 2014; Reinholz, Trninic, Howison, & Abrahamson, 2010). The didactical principle is to support classroom teachers in implementing their own intuitions for proportionality. Teachers (and textbooks) often introduce the concept of proportional equivalence by way of presenting a situated recipe notion. Per the recipe notion of proportional equivalence, some sensory perception of a phenomenon, such as its color, is maintained amid supplementing substance into the situation. Thus, the idea of equivalence in 1:2 = 2:4 might be presented as receiving the same color of green whether one mixes 1 cup of blue paint and 2 cups of yellow paint or 2 cups of blue paint and 4 cups of yellow paint. Whether we compare ratios of paint components (color perception), geometrically similar rectangles (aspect ratio), or food ingredients (flavor), this notion of

We conceptualize the emergence of these embodied structures and interaction strategies as pivotal to the guided cognitive process of first controlling the device and then mathematizing this skill. Therefore, and putting aside the question of what these students ultimately learned through these activities, we view their documented behaviors as relevant and perhaps paradigmatic for a discussion of action-based learning.[2] Still, what theory of learning might best inform our understanding of these behaviors?

Open Questions and General Arguments: A Rationale for Action-Based Theories

In making sense of these data, we were inspired by seminal theories of cognitive development. Per genetic epistemology (Piaget, 1968), spontaneous coordination of sensorimotor activity around an interval between the hands can be viewed as reflective abstraction of a correlational invariant—the higher, the bigger—which then becomes encapsulated and generalized via discourse (Abrahamson

the equal sign as meaning equivalent sensory perception between two phenomena is pedagogically sensible, because it paves conceptual continuity from earlier arithmetic notions of parity associated with the equal sign—it makes the two different mathematical expressions the same (Abrahamson, 2002, 2012a). At the same time, our argument goes, teachers are technically challenged in creating within their classrooms effective opportunities for their students to experience the perceptual equivalence of situated proportionality, let alone to rigorously investigate, predict, enact, measure, calculate, challenge, and reinvent proportionality. Thus, we view the pragmatic outcomes of this line of work ultimately as contributing to forms of mathematical instruction that encourage students to inquire into curricular content by engaging in carefully designed sensorimotor experiences (Abrahamson, 2012a; Kalchman et al., 2000; Nemirovsky, 2003).

[2]We acknowledge that our focus, in this article, on the 1:2 ratio might appear as setting too low a bar even for a proof-of-existence argument for the utility of our approach. After all, children demonstrate particular sensitivity to visual displays of half by using this perceived symmetry as a benchmark for inference making (Nunes & Bryant, 1996; Sophian, 1995; Spinillo & Bryant, 1991, 1999), and these sensitivities may be related to primitive actions of equipartitioning (Confrey & Scarano, 1995). We thus refer readers to our earlier publications in which we report results from using other ratios beyond 1:2 (Abrahamson & Howison, 2010; Reinholz et al., 2010). Also, we wish to highlight that our design explicitly took on students' endemic confusion around additive versus multiplicative reasoning (Post, Cramer, Behr, Lesh, & Harel, 1993). We created interactions wherein sensorial constancy (i.e., keeping something the same about a situation) could be achieved by *changing* the difference between two quantities and figuring out how this change covaried with the quantities. Our learning materials and activities (see Figures 1 and 2) were thus explicitly focused on creating opportunities for students to experience cognitive conflict when they initially attempted to keep the difference constant amid changes in the quantities; come to realize that their tacit theory of action had failed them; and then reconcile this conflict by way of reflective abstraction to coordinate, encapsulate, generalize, and progressively mathematize new rules. Thus, we build on a generation of constructivist design-based research on mathematics learning. We offer that the efficacy of those designs might be attributed to the tacit sensorimotor struggles and coordinations inherent in successfully enacting those tasks, and our work might be viewed as making visible those invisible actions.

& Sánchez-García, 2015; Abrahamson, Shayan, Bakker, & Van der Schaaf, in press). Per cultural–historical psychology (Vygotsky, 1930/1978), we conceptualized the interval as a self-generated artifact. Wedged between the subject and the object (Sfard, 2007), the interval served as an emergent auxiliary construction mediating the students' participation in the social enactment of a novel cultural practice. The respective work of Saxe (2002) and Radford (2005) clarified the transitional function of cultural forms in the MIT-P activity. In particular, we realized that changing the task from enactment to representation changed the function of the gridlines from handles anchoring visual attention to semiotic means of objectifying new meanings. According to distributed cognition theory (Chandrasekharan & Nersessian, 2015; Schwartz & Martin, 2006; Zhang & Patel, 2006), the students adjusted to the new external feature by offloading aspects of their action scheme onto its structure so that the grid, an external feature, became an integral component of situated action. We further used instrumental genesis (Verillon & Rabardel, 1995) to identify and articulate how the students plied and applied the interval, once it was constructed, as a means of accomplishing a situated task. Situated cognition (Greeno, 1994, 1998, 2015) inspired us to consider this development of a new motor-action routine as the emergence of new perceived affordances in the environment. A best fit was found in assertions from the philosophy of enactivism (Varela, Thompson, & Rosch, 1991), such as "(1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (pp. 172–173). The cognitive structure *interval* and its sensorimotor pattern *higher–bigger* coemerged dialectically from exploring the MIT-P. Enactivism, however, was not readily enabling us to insert the teacher, a guiding cultural agent, into the picture of individual learning.

The elegance and power of these theories notwithstanding, we thus sensed that they were not optimally geared to explain some intriguing aspects of our empirical data. In particular, the theories did not illuminate the (a) *spontaneous microgenesis of imaginary objects mediating sensorimotor control of interactive systems* (the interval); (b) *adoption of symbolic artifacts as regulatory enactive, epistemic, and discursive devices* (the grid); and (c) *educators' active role in facilitating effective sensorimotor engagement.* And yet these three observed phenomena could be typical of learning processes in action-based design and therefore important for building a theory of action-based learning. We were thus inspired to consider a theory coming from a discipline focused specifically on individuals' guided development of situated motor-action skills. Namely, we were favorably inclined to adopt from sports sciences the theory of ecological dynamics. This article spells out how ecological dynamics could illuminate these three phenomena.

We hasten to note up front that our focus in this article on fostering motor-action skill should not for a moment suggest that we are disregarding or mitigating the formative role of symbols in the development of mathematical knowledge or disavowing the rich theoretical and practical challenges that the symbolic register introduces (Duval, 2006; Kirsh, 2010a; Lesh, Post, & Behr, 1987; Radford, 2013). Rather, we believe that there has not been sufficient focus in the literature on the initial development of action schemes via direct or vicarious interaction with instructional media (but see de Freitas & Sinclair, 2012; Kim, Roth, & Thom, 2011). At the same time, we view embodied-interaction technologies as powerful yet underresearched means of fostering those action schemes (Abrahamson, 2012b). Accordingly, this article treats the initial construction of operatory schemes more so than the subsequent signification of these schemes in semiotic systems of the mathematics discipline. Per the embodied-design framework (Abrahamson, 2009, 2014, 2015), we thus discuss Step 1 more than Step 2.

The next section introduces the theory of ecological dynamics as it relates to our research. The section called "The Ecological Dynamics of the MIT: Paradigmatic Examples From Implementing an Action-Based, Concept-Oriented Learning Activity" presents paradigmatic vignettes from the MIT project so as to address the three problematic phenomena (see above) from an ecological dynamics view. The final section offers concluding comments on the scope of our approach vis-à-vis the manifold of mathematics pedagogy.

ECOLOGICAL DYNAMICS

What Is Ecological Dynamics?

Ecological dynamics (Vilar, Araújo, Davids, & Renshaw, 2012) is a theoretical approach used in sports sciences to study skill acquisition in natural settings or naturally occurring activities. This framework blends *dynamical systems theory* (Edelman, 1987; Thelen & Smith, 1994) and Gibson's (1977) *ecological psychology.* Applying dynamical systems to ecological psychology enables sports scientists to explain the learning of physical skills as the complex self-organizing of subject–environment dynamical systems. Consider the case of an infant learning to walk. From a dynamical systems perspective, it is neither the case that she gets input from her body, figures it all out in her brain, then carries this out with the body nor that the infant must achieve biological maturation of the brain before she can plan and execute the complicated motor-action coordination of motility. Rather, walking happens. Thelen and Smith (1994) reported on elegant studies demonstrating that walking is an emergent form of iterated motor actions initially absent of central cerebral control. When infants much younger than normative walking age are dressed in a harness and stood over a moving walkway, walking-like actions manifest spontaneously, sequentially, each leg at its turn jerking forward to prevent an anticipated stumble. Then at some point one might discern anticipatory agency in raising the knees so as to maintain greater stability. But the body is at the vanguard of motor-action learning. The same principle has been demonstrated in robotics—machines that were never programmed to walk nevertheless develop walking-like behaviors when placed in appropriate environments. In the absence of any executive commanding unit, new and robust motor-action routines self-organize via situated interaction in constrained settings. As Clark (1999) wrote, "The active body of the robot is here providing the functional equivalent of the missing second layer of neural processing" (p. 52).

Thus, in a dynamical systems approach, decision-making and learning processes are modeled not as generating a sequence of disembodied symbolical propositions, such as abstracted inferences and decisions, but as emerging from the agent's goal-oriented, situated, adaptive interactions in the environment (Araújo, Davids, Chow, Passos, & Raab, 2009). Moreover, the emergent quality of self-organizing complex adaptive systems implies that learning processes are highly dependent on organismic qualities; for example, a shorter basketball player might develop a different style of throwing the ball to the basket compared to her taller teammate. Finally, the same bottom-up emergent quality of naturalistic learning processes implies that they are not linear but stochastic. Thus, the order of events along natural learning processes varies both within and across individuals. Yet whereas each inter- and intra-personal trial is sensitive to initial conditions and susceptible to random encounters, individuals tend to gravitate toward similar systemic solutions that satisfy task objectives amid multiple constraints. For example, athletes reared by the same coach, in the same space, under equivalent regimes and diets, and with the same event goals will develop a comparable style that nevertheless bears an idiosyncratic signature.

We are thus characterizing human learning as systemic, emergent, nonlinear, distributed, and self-adaptive. In so doing, we are building on approaches to human development inspired by nonlinear dynamics. Nonlinear dynamics is a branch of physics that provides a formal representation of any system evolving over time. The behavior of any living system can be plotted as a trajectory into a state space (all possible states of the system and the paths to them). We can identify different stably dynamical states of a system, known as *attractors,* equivalent to functional states of coordination. For example, the human movement system self-organizes via constant subject–environment interactions in recurrent perception–action loops. One might consider the motor actions of a professional laborer wielding a sledgehammer. No two strikes are ever identical, and greater variation is manifest as goals and constraints vary, and yet all strikes are contextually adapted instantiations of one and the same systemic attractor, an agent's dynamical solution to the situated problem of satisfying task objectives perceived as similar (Bernstein, 1996).

Systemic conceptualizations of human–environment relations have yielded powerful theoretical constructs, and some of these have found their way into seminal discourses of the learning sciences. Perhaps the most familiar of these constructs is that of an affordance imported from ecological psychology into the learning sciences by way of the human–computer interaction design literature (Norman, 1998). In a subject–environment system, affordances, understood as opportunities for action (Gibson, 1977), emerge within goal-directed activity. The concept of affordances is pivotal to understanding decision making from the perspective of ecological dynamics: Opportunities for goal-oriented action emerge within the dynamics of the system that may produce shifts between functional states of coordination. For example, when an athlete standing in the soccer field apprehends an approaching ball, he may perceive affordances for goal-oriented actions of kicking and thus shift into an appropriate functional state of coordination.

Yet if the human movement system adaptively self-organizes via subject–environment sensorimotor interaction, how should we conceptualize the role of social intervention in the development of functional coordination? That is, how does teaching work?

Coaching Ecological Dynamics: Constraints-Led Nonlinear Pedagogy

When a dynamical system consists of human agents engaged in goal-oriented activity, its self-organizing behavior can be affected or "channeled" (Araújo & Davids, 2004, p. 50) by different types of constraints. Imagine a hiker walking toward a distant destination along a path that varies in terrain from paved road to sand, mud, slush, and snow. These changes in the path's substance can be regarded as constituting different *environmental* constraints on the execution of walking (compare the knee work for snow and asphalt). Our hiker may well become fatigued by this trudge, thus introducing *organismic* constraints on the journey. At this lower energy level, she slouches ahead. Now, in turn, the many miles still ahead might demand of the hiker even faster walking so as to arrive still before dusk in time to pitch a tent. Thus, the *task* constraint changes.

Indeed, Newell (1986) identified three sources of constraints affecting the behavior of the system on either a short time scale (decision making while performing a skill) or a longer time scale (the process of learning a skill): organism, environment, and task. *Organismic constraints* are present at the biochemical, biomechanical, neurological, and morphological levels; *environmental constraints* can be ambient/global to action (gravity, temperature) or local/focal to action (availability of tools); and *task constraints* include the goals of the action as well as any socially agreed rules (Newell, 1996, p. 404). Thus, contrary to classic cognitivist models, the behavior of the system is not premeditated or controlled by internal directives or explicit rules. Within the ecological dynamics framework, the human agent's intentions concerning goals to be achieved are just one among other constraints affecting systemic behavior.

From a pedagogical point of view, the introduction of appropriate constraints into the system is an issue of paramount importance. Introducing constraints can help learners become perceptually attuned to relevant affordances for performing a specific skill. This is a common phenomenon in sport activities. According to converging studies, "athletes' perceptual sensitivity may indicate perceptual learning by attunement to the appropriate informational variables available as a result of perceptual–motor experience in their sport" (Weast, Shockley, & Riley, 2011, p. 704). For example, a high jumper becomes through practice increasingly sensitive to select informational variables relevant to optimizing his running approach so that he clear the bar. Still, though it is ultimately the athlete who performs the jump, his technique was honed over the years of practice by a coach. In particular, the athlete's perceptual–motor experience and attunements can be channeled by pedagogical intervention. It is important to note that coaches do not teach directly (linear pedagogy) but create conditions appropriate for the emergence of the athlete's learning (nonlinear pedagogy; see below).

Davids, Button, and Bennet (2008) and Renshaw, Chow, Davids, and Hammond (2010) have proposed a nonlinear pedagogical approach to motor-action learning based on introducing and modifying constraints in the learning environment. The objective of *nonlinear pedagogy* is to optimize systemic opportunities for athletes to develop robust skills that are both task appropriate and tailored to their organismic constraints. Key to the success of nonlinear pedagogy is creating physical and cultural conditions that enable and encourage athletes to engage in subjective exploration and self-discovery, wherein variability and flexibility are regarded as positive outcomes (Vereijken & Whiting, 1990). This pedagogy of diverse personalized solutions is in stark contrast with traditional didactics, wherein athletes are directed to perform repeatedly predetermined ideal technical solutions and wherein exploration and self-discovery are not fostered.[3]

Consider a piano teacher who would like his student to strike the keys with fingers extending flat rather than curved at the knuckles. He could explain this technique and demonstrate it to her; then, once the student attempts to emulate his action, he could manually correct her hand shape. Alternatively, the teacher

[3]Note that nonlinear pedagogy resonates strongly with the work of Soviet neurophysiologist and founder of biomechanics Nikolai Bernstein (1896–1966): Whereas linear pedagogies emphasize repetition of idealized motor actions and therefore view variability as noise or error distributed around the ideal, Bernstein (1996) viewed motor learning as a form of problem solving and insisted that variability in conditions fosters the development of *dexterity,* that is, flexibility to adapt motor performance to diverse ad hoc contexts.

could place a book on the student's hands as she is playing and ask that the book never fall. The book constitutes a constraint on the student's actions. It perturbs the dynamic stability of the existing student–keyboard music-oriented dynamical system. In response, the student must adapt her situated motor-action schemes in tune with the modified task/environment. She accommodates her sensorimotor relation with the keyboard by developing a new motor-action coordination of playing the piano with straight hands. Thus, via the mediation of the book constraint, the piano comes to afford a new way of acting on it. At this point, the book may no longer be necessary and could thus be removed. The book played the role of an enabling task constraint: Initially it was restrictive, and yet it shaped a new way of relating to the piano. It is important to note that it is the student who solved the problem, and presumably her solution was best adapted to her organismic idiosyncrasies, such as the length and suppleness of her fingers. Moreover, the teacher never told or showed the student *how* to solve the problem.

Readers familiar with constructivist pedagogical philosophy might note a similarity between nonlinear, constraints-led pedagogy and the principle of fostering opportunities for individuals to reinvent cultural–historical knowledge (Kamii & DeClark, 1985). It is important that from an objective materialist perspective neither the piano itself nor the book itself changed in this process—only the student changed, and this change could perhaps be documented through neuroimaging techniques. However, from the ecological dynamics perspective what changed was the student–piano *relation* (Roth, 2015)—a new affordance was created. From the ecological dynamics systemic perspective it does not make sense to say that the student per se has changed, because what the student learned is intrinsically situated and mutually adaptive (Malafouris, 2013; Schwartz & Martin, 2006). As far as the student is concerned, the *world* has changed—it now bears new opportunities for action (new meaning, Cisek, 1999; new horizons, Dreyfus & Dreyfus, 1999; new affordances, Gibson, 1950; a new enactive landscape, Kirsh, 2013). Moreover, the new technique is necessarily situated whether the student is actually playing, miming playing, marking playing (Kirsh, 2010b), or just imagining playing.

We value these phenomenological conceptualizations of learning because they enable us both to practice and research what we view as student-centered pedagogy. Moreover, these pedagogical views from the sports sciences enable us to better position our work within the broader literature on cultural strategies for fostering valued motor-action skill. As we now explain, the technologically rich learning environments at the center of our design-based research efforts can be viewed as spaces for enacting a form of ancient cultural practice—nonlinear, constraints-led pedagogy for fostering individuals' development of motor-action routines. These spaces, as we will see, have been called *fields of promoted action.*

Designing for Discovery: Fields of Promoted Action

An enduring concern for educators operating in nonlinear pedagogy is to design and implement ecological conditions, tasks, and resources that facilitate students' self-exploratory activity, resulting in targeted skill outcomes. Cultural anthropologists of motor-skill development call these ecological conditions *fields of promoted actions* (Reed & Bril, 1996, p. 438). A *field of promoted action* is a socially constructed and monitored microecology in which a novice, such as an infant, is presented with a specific motor problem as well as constraints—organismic, environmental, and task based—that encourage self-exploratory behavior leading to the discovery and practice of culturally valued action solutions. Fields of promoted action thus foster students' agency and customization in their own learning process: students are guided to reinvent viable ways of being in the world, where these ways were never dictated, demonstrated, explained, or cued.

By way of example, consider the sport of boxing. Hristovski, Davids, Passos, and Araújo (2012) engaged boxing novices in a basic training session in which boxing experts taught the novices a front-punch. As the novices practiced front-punching a punching bag, the researchers discreetly calibrated the task constraints: They moved the punching bag laterally within the boxing novices' field of action. Once the punching bag passed a critical point, where the front-punch apparently became too awkward to execute, the novices responded by performing a back-fist punch. They thus reinvented a cultural–historical maneuver that they had not used prior to that session.

Figure 3 depicts and summarizes the notion of a constraint-based field of promoted action. By way of illustration, let us return to the piano lesson example,

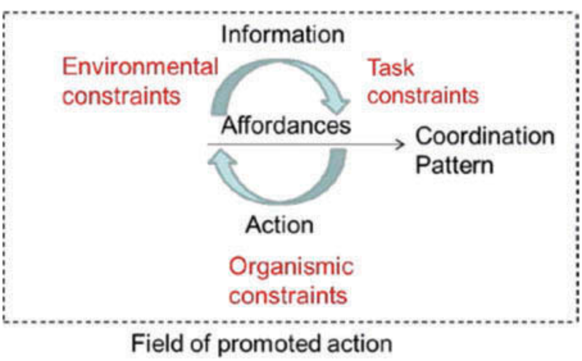


FIGURE 3 Nonlinear pedagogy: introducing constraints into a field of promoted action. The inclusion of different constraints affects the recurrent perception–action loops of the organism–environment dynamical system, bringing forth new affordances for performing a specific task-oriented coordination pattern. Adapted from Newell (1996).

in which the instructor interpolated a book as an environmental constraint on the student–piano relation. Therein, "Action" would refer to the goal-oriented physical motions (hand–arm–torso movements directed at striking the piano keys), and "Information" would include multisensory perceptions (auditory input from the key strikes; visual, haptic, and tactile sensations from the keys and book; somatic, kinesthetic, and proprioceptive perceptions in the torso; and so on). Disequilibrated by the new constraint, the system begins adapting so as to continue generating task-oriented behavior: A modified Action–Information loop begins to evolve as the student iteratively tunes her actions vis-à-vis the information they elicit and keeping within all constraints. Thus, the environment, in particular the piano, comes to afford a new coordination pattern, so that the system achieves new dynamical stability.

We have now discussed fields of promoted action, and we have considered the idea of a coach introducing productive constraints into a student's task-oriented perception–action loops. As educational researchers, we are aware of the manifold forms that instruction can take. We now turn to look at some of these forms of intervention from a systemic perspective. We will be talking specifically about various forms of feedback that a coach/teacher might offer an athlete/student.

Implementing Nonlinear Pedagogy: Real-Time Augmented Information as Productive Constraints Promoting Discovery

A model of learning as tuning action to accommodate constraints need not imply that all the teacher does is set the proper field and then leave the student to explore (cf. Kirschner, Sweller, & Clark, 2006). The didactic setting is far more interactive. A paramount concept developed by Newell (1996) is that of *augmented information,*[4] a subcategory of environmental constraints. This subcategory includes various forms of supplementary information, such as didactical input coming from a coach or, via technological extension, from some feedback mechanism. It is critical to note that learners engaged in self-exploratory activities do not always have access to this information. Newell (1996) stated that "augmented information acts as an environmental constraint to action. The different categories of information provide varying boundary conditions to the search

[4]Augmented information itself breaks down into three different types that are thus sub-subcategories of environmental constraints: (a) *Prescriptive information* refers to the multimodal specification of the target action; (b) *information feedback* is what the performer receives from his or her own movement, whether during the action (concurrent) or as its result (terminal); and (c) *transition information* is any kind of external input that reacts to the agent's performance of an action by way of offering qualifying instructions toward the prospective reenactment of that action (e.g., "Turn more energetically this time!").

through the perceptual-motor work space in the realization of new task goals" (p. 424).

The notion that information coming from a coach constitutes a *constraint* may present itself as somewhat odd. Yet recall that the word *constraint* in this context does not bear its colloquial meaning as a negative factor. Per systemic views of learning, the novice is searching for effective modes of behavior. Any relevant information, whether from sensory perception or from a guiding cultural agent, facilitates the search by negating the vast combinatorial branches of possible action.

Augmented information comes in a variety of modalities, including speech and/or gesture but also direct physical contact. Becvar Weddle and Hollan (2010) described the variety of didactical strategies used by dental hygiene experts in guiding novices into the practice. In so doing, they offered the phrases "molding" and "directing" (p. 128). *Molding* is the didactical practice of literally manipulating the learner's body so as both to guide and constrain him or her through a dynamical envelope of situated physical performance. Molding may appear to an onlooker to be the most direct way possible of getting a learner to do the thing itself correctly. However, from an ecological dynamics perspective molding is not about prescribing rote physical actions. Rather, molding should optimize learners' opportunities to experience and solve a motor coordination problem. As they are taken through the motions, learners seek agency by engaging the environment, attending to relevant information, grasping at features, sensing relevant informational invariants co-occurring with effective actions, adjusting their motor-action coordinations to emulate the molded action patterns, and so building effective perception–action loops. Thus, learners develop perceived affordances for accomplishing new tasks (see also Abrahamson et al., 2012; Churchill, 2016; Ginsburg, 2010; Ingold, 2000). *Directing* captures expert guidance communicated primarily in the speech/gesture modalities. It is the most salient form of augmented information in mainstream mathematics classroom instruction (Alibali & Nathan, 2012).

From the ecological dynamics perspective, all forms of intervention pose constraints (augmented information) on students' solution of motor-action problems. These forms include, but are not limited to, physical guidance, joint enactment, and metaphoric framing (Abrahamson, Sánchez-García, & Smyth, in press). Yet whereas it is the instructor who introduces augmented information, students themselves are able to develop new structures that productively constrain their own interactions. We now discuss these structures, attentional anchors.

Attentional Anchors

Systemic approaches to the analysis of mathematics teaching and learning foreground the inherently *relational* nature of students' situated cognitive activity.

It is these agent–environment relations, developed through goal-oriented activity in dedicated task spaces, that give rise to students' targeted conceptual knowledge. These desirable relations emerge from having students discover and practice specific perception–action routines. From a system perspective, then, the pedagogical enterprise is to educate students' perception, so that "cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (Varela et al., 1991, p. 173). Now, if this systemic framework is to have traction on empirical data coming from educational research, what might we look for as evidence that these cognitive structures exist and are indeed emerging? In particular, what form of hypothetical construct might we implicate as evidence of this learning process? This construct would need to be suitable for a discussion of guided discovery-based learning within dedicated environments. We propose the attentional anchor as this construct. The current section introduces the construct and explains its role in our study.

Consider a cellist working on the quality of his sound production. He is playing long bow strokes, listening carefully to nuances of sonority and attending meditatively to the body's intimate sensations. At one point he realizes that he can feel in his bowing hand (the right hand) whether the left hand is holding down the string loosely or firmly on the fingerboard, and he notices that, reciprocally, he can feel in his left hand whether the right hand is holding the bow loosely or firmly. He becomes attuned to the hands feeling each other as mediated through the string and bow. The hand-to-hand instrumented bond becomes a collective unit that vibrates more or less according to the firmness of grips. The cellist monitors for the effect of this vibrating bond on the quality of sound, and he experiments with the acoustic results of adjusting the bond via micro-operations of each hand. The bond, which had been latent to the environment and outside the scope of tacit consciousness, is now the new thing that the cellist manipulates as his means of controlling the quality of sound. The bond is not the hands themselves—the hands subtend the bond that lives and quivers between them; the bond is external to the body, a current or buzz that the hands at once operate and feel. Of course the cellist's attention may still wander back and forth between his individual hands and the bimanual bond, but all sensorimotor activity is now subservient to and conditional on controlling the bond. The bond is now a bona fide phenomenological entity, an interactive ontological unit, a new gestalt tool for controlling the quality of sound. The bimanual instrumented bond has become an attentional anchor that the cellist appropriates into his musical toolbox. This attentional anchor acts as a constraint on the infinite search for optimal sound production. Under this constraint, the cellist develops a new motor-action coordination. The cello thus comes to afford a new way of playing, and following much practice the attentional anchor might become second nature and fall below consciousness.

An *attentional anchor* is a hypothetical construct first articulated by Hutto and Sánchez-García (2015). Their paper applied philosophical tenets of radical-enactivist cognition to the explication of empirical phenomena researched by sports scientists. In turn, the construct of attentional anchor is central to our thesis on the sociocultural ontogenesis of mathematical concepts. What is more, if our claims bear merit, then attentional anchors could be considered in designing and researching educational activity quite broadly for any disciplinary domain whose instruction is launched in fields of promoted action.

Attentional anchors lie at the intermediate level of interaction between subject and environment. Similar to affordances, their nature is relational, hybrid. Numerous studies in skill acquisition of sport techniques suggest the comparative advantage of directing learners' focus away from internal kinesiological components toward external environmental structures (Wulf & Su, 2007; Zarghami, Saemi, & Fathi, 2012). For example, when swimmers practice the arm stroke in crawl style, they perform better by focusing on pushing the water back than on pulling their hands back (Stoate & Wulf, 2011).

Attentional anchors may be a specific object (real or imagined), area, or other aspect, behavior, or characterization of the perceptual manifold that an agent detects, invokes, selects, and uses to monitor perception–action couplings for the activity at hand. Attentional anchors include the location, sensation, and effect of the subject's orientation during motor-action performance. The attentional anchor emerges and interpolates itself into the agent–environment relation to serve as an enabling environmental constraint—it becomes a new systemic element that hones and channels attention for action. The attentional anchor reduces operational complexity, rendering ergonomic and feasible an otherwise overwhelming task (Kelso & Engstrøm, 2006; Newell & Ranganathan, 2010). The agent acts on the attentional anchor as its control panel and dashboard, which the agent experiences as overlaid on the perceptual field—the attentional anchor becomes the mediating proxy for both operating on the environment and interpreting feedback from the environment. Specifically, the attentional anchor enables learners to operate on the environment via managing complex information invariants.

Attentional anchors can be immaterial. In general, constraints can be embodied in concrete objects, such as a line of plastic cones across a soccer field that articulate a slalom dribbling path. But constraints can be immaterial, invisible, and even imaginary, so that students themselves must install the constraints into their field of promoted action. For example, a juggling coach may instruct the novice to imagine a tall rectangle rooted in her hands and soaring above her head. This invisible geometric shape becomes an attentional anchor—a phenomenologically real percept that mediates the juggler's control of the balls (see Liao & Masters, 2001, on biomechanical metaphors, which are imagined geometric shapes that athletes are instructed to project into the perceptuomotor space).

Attentional anchors and motor-action skill emerge dialectically, iteratively: Even as attentional anchors materialize between agent and environment as mediating control structures, they in turn afford the development of those very motor-action coordinations that would be necessary for operating on the environment via these control structures. In terms of the theory of instrumental genesis (Verillon & Rabardel, 1995), attentional anchors are Artifacts that the Subject projects between herself and her Objective.

Attentional anchors might be discovered by individuals, as in the anecdotal case of the cellist, above, or suggested by instructors in the form of augmented information, such as if another string player reading this text now attempts to implement this attentional anchor. It is important to note that although individuals may cultivate their own attentional anchors discreetly, under the radar of consciousness, we submit that attentional anchors can be reified via reflection and discourse. Our empirical examples argue this point.

Summary

We have now outlined the theory of ecological dynamics as well as its concomitant nonlinear pedagogy and constraints-led model. The theory of ecological dynamics draws from a tradition of examining natural and social phenomena from a complexity perspective (Edelman, 1987) and specifically implicating motor-action roots of cognitive development (Savelsbergh, Vereijken, & Zaal, 2005). The application of system dynamics theory to the modeling of educational settings and activities has precedents in the literature of the learning sciences (Barab et al., 1999; Clancey, 2008; Davis & Sumara, 2008; Greeno, 1998). Furthermore, the relevance of sports sciences to the theory of embodied cognition has previously been proposed (Beilock, 2008). Yet we view ecological dynamics as serving particularly well our attempts to implicate motor action as bringing forth mathematical ideas and to witness this fragile, spontaneous process in the microevents of students' guided, goal-oriented interactions with manipulable features of learning environments.

More specifically, ecological dynamics, along with the hypothetical construct of attentional anchors, may enable us to explain three types of intriguing phenomena that we have observed repeatedly in our empirical data yet appear to defy existing theoretical frameworks: (a) the spontaneous emergence of new ontological entities into the dynamics of student–environment relations, along with these new entities' affordances for action; (b) students' adoption of environmental features, such as mathematical frames of reference, as tools for enhancing physical interaction, discourse, and reflection; and (c) the interaction mechanics of teachers' intervention tactics for steering students toward effective engagement of burgeoning mathematical structures. These three phenomena are treated in the next section by further considering empirical data collected during experimental implementation of the MIT-P.

THE ECOLOGICAL DYNAMICS OF THE MIT: PARADIGMATIC EXAMPLES FROM IMPLEMENTING AN ACTION-BASED, CONCEPT-ORIENTED LEARNING ACTIVITY

New educational technology may give rise to new forms of teaching and learning not readily explicable by existing theory. Looking at embodied-interaction technology, here we present a set of vignettes paradigmatic of our three research problems: (a) the spontaneous emergence of higher order structures facilitating sensorimotor activity, (b) sensorimotor assimilation of symbolic artifacts en route to mathematical discourse, and (c) forms of tutorial intervention unique to mathematical fields of promoted action. Our empirical data are drawn from a study that implemented the MIT-P with 22 Grade 4–6 students who participated either individually or in pairs in task-based semistructured clinical interviews (Howison et al., 2011). The vignettes were selected both to feature a variety of students differing in age and mathematical achievement and to provide snapshots of three different stages along the activity design.

In all cases, the researcher and student worked in a quiet room on the school campus. They sat side by side so that the researcher could see at once both the student's manual actions and the effects of these actions on the computer monitor. For continuity across the vignettes, we focus on student behavior around the numerical item of a 1:2 ratio, that is, where the technological interaction system is set up so that the screen will turn green only if the ratio between the respective heights of the left and right hands is at 1:2 above the bottom edge of the monitor.

The Emergence of an Attentional Anchor Mediating Agent–Environment Dynamics

The MIT-P system potentiates multiple embodied entries into the mathematical field of proportionality, such as difference, speed, rate, and multiplicative relations (for an exhaustive list of students' green-making strategies, see Abrahamson et al., 2014; Reinholz et al., 2010). By *embodied entries* we are referring to the emergence of motor-action control structures leading to mathematical signification, where this emergence is heavily designed for yet locally spontaneous—it is unmodeled, undirected, and uncued by the tutor. As a case in point, we now treat dynamical control strategies oriented on the spatial interval between the two hands.

Irit is a Grade 5 female student assessed by her teacher as a "high achiever." During the first 4 min of the MIT-P activity, Dor (the researcher–tutor) explains to Irit the task objective and resources. Initially the screen is red, and the two remote-control devices lie on the desk. Irit lifts the controls. In an attempt to make the screen green, she waves them up and down in several different patterns. Fourteen seconds later, she strikes green and freezes her hands. Irit has placed the cursors

FIGURE 4 Irit (child on right) is working with the Mathematical Imagery Trainer for Proportion, with Dor (researcher–tutor) looking on. The ratio is set at 1:2. The cursors are at about a quarter (left) and half (right) way up the screen, and so the screen is green.

at about a quarter way up the screen (left hand) and halfway up (right hand), that is, at a 1:2 ratio, and so the screen has turned green. The following conversation ensues (see Figure 4). (Verbal utterances appear in bold for legibility.)

Irit: **They have to be a certain distance.**
Dor: **They have to be a certain distance.**
Irit: (She lowers her right hand to about quarter height, which is the same height as the left hand, and so the screen becomes red. She then lifts her right hand back to the previous location, and the screen becomes green again.) **They can't be together.**
Dor: **They can't be together, you're saying.**
Irit: (30 s of silent exploration) **But this one** (right) **has to be higher.**
Dor: **So they have to be a certain distance, and right has to be higher.**

As she engages in solving the embodied-interaction problem, Irit so far has made two observations about what she believes are effective strategies for achieving the task objective of making the screen green. The first observation highlights the distance between the two cursors as a property of the visual display correlated with the desired effect of green. Specifically, the observation "They have to be a certain distance" suggests some absolute magnitude for this interval. At the same time, the observation suggests that Irit believes that this certain distance is not tied to a particular screen location but could be implemented in other locations. After some experimentation, Irit further determines a second observation: "This one [right] has to be higher."

Dor then augments on the direct feedback that Irit received from the interaction: He suggests that she explore higher and lower regions of the screen. Raising both hands higher, Irit keeps her certain distance constant (a fixed distance between her hands), and so the screen turns red. Eventually the following exchange ensues:

> Dor: **So what happens to the distance?**
> Irit: **Oh! No, it gets shorter if you go down more, and then it gets tall** . . . **longer if you go up.** . . . **It has to be** . . . [the] **higher this one** (right hand) **goes, the lower this one** (left hand) **has to go. The distance gets bigger up higher, and smaller down here.**

The interval between Irit's hands has thus become entified out of negative space. It articulated into being, foregrounded as an auxiliary stimulus wedged between agent and task. It emerged through goal-oriented explorative interaction as a thing—a handle, a lever, a utensil crafted ad hoc out of thin air; an entity that was objectified, described, and referenced; a thing whose own features were modified along select dimensions, and these modifications in turn were coordinated and correlated. The interval thus evolved into a ready-to-hand tool mediating situated implementation of motor intentionality.

The interval emerged for Irit because doing so favorably collapsed two motor-action schemes into a single scheme—from moving two hands to manipulating one thing. This simpler scheme is thus oriented on a physically extraneous focusing medium, the interval. Indeed, as they manipulated the interval, our study participants never spoke about what each individual hand should do but rather about the behavior, handling, and impact of the interval. The interval, an external object, served the students as an attentional anchor that promoted their performance.

In this section we treated the first research problem: *spontaneous microgenesis of imaginary objects mediating sensorimotor control of interactive systems.* As we see in the next section, once symbolic artifacts are interpolated into the working space as potential frames of reference, the interval—and in particular its contoured grasp—creates new opportunities for mathematical signification.

Sensorimotor Assimilation of Symbolic Artifacts en Route to Mathematical Discourse

When the interviewer was ready to light up the grid on the computer monitor, he would prepare the child for the introduction of this new element in the visual display by stating, "Now, I'm about to add something here—let's see what this does for us." In Reinholz et al. (2010), we described the new behaviors we witnessed once the grid was introduced onto the display, such as the *a*-per-*b* strategy, in which the students might state, "For every 1 unit that I go up on the left, I go up 2 on the right." We noted the pedagogical value of these changes in strategy—the activity was apparently steering the students from situated qualitative description to general quantitative redescription (Abrahamson et al., 2011). Here we examine students' microbehaviors as the attentional anchor frames initial perceptions and uptake of the grid.

Eden and Uri are sixth-grade male students identified by their mathematics teacher as "high achieving." They worked as a pair on the MIT-P problem, so that the control of the two cursors was distributed between them: Eden sat on the left and controlled the left-hand cursor, and Uri sat on the right and controlled the right-hand cursor. Sitting side by side, both students faced the screen. We join the dyad about 20 min into the interview. They had been working on the continuous screen (no grid), and now the interviewer was about to switch on the grid.

Below we use the following abbreviations: RT = right-hand tracker (the remote-control device), LT = left-hand tracker, Rc = right-hand cursor (that is operated by RT), and Lc = left-hand cursor. Also, particular gridlines are indexed as in graphs by their order relative to the base line (e.g., "3-line" is the third gridline up from the datum line). The symbol // indicates simultaneous talk.

Eden: (The grid is switched on) **Grid.**

Uri: **Yeah.** (Grabs RT, lifts Rc up to 1-line. Simultaneously, Eden, too, lifts Lc up to 1-line. On the way up, between 0-line and 1-line, the screen flashes green for a moment but then turns red. Eden lowers Lc back down, holds it at .5 units. The screen turns green.) **Oh so you can like show where** . . . **Let's see, so** (Rc up from 1-line to 2-line) // **if you're on here** . . .

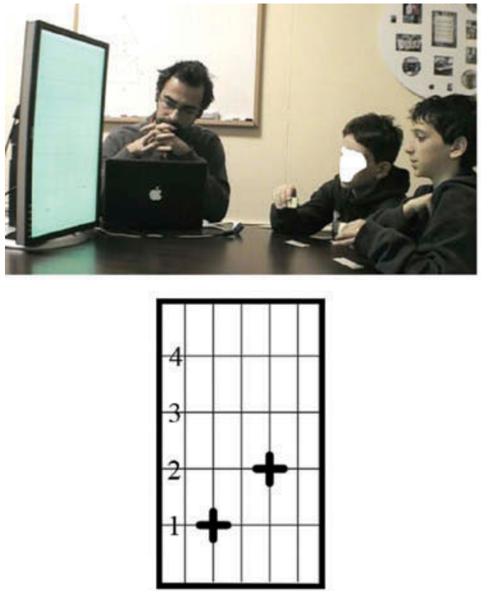
Eden: // **maybe it has to be two** . . . (Lc up to 1-line; see Figure 5a) **an entire box apart.**

Uri: (Rc up to 3-line) **If I go here** . . .

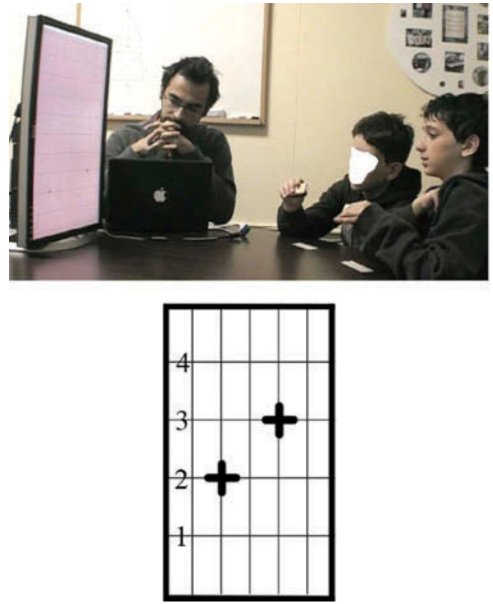
Eden: (Lc up to 2-line; screen goes red; see Figure 5b) **Then maybe you should raise it** (Uri raises Rc to just below 4-line; screen flashes green). **So maybe the higher you go, the more boxes it is apart.**

Soon after, the dyad shifted from their higher–bigger strategy, which is continuous and qualitative, to an *a*-per-*b* strategy, which is discrete and quantitative.

What is intriguing to us in this short excerpt is a nonevent—the students' casual projection of the interval onto the gridlines. The grid is a constraint introduced into the student–screen relation, but it readily affords the existing motor-action coordination. When Eden says, "Maybe it has to be two . . . an entire box apart," the pronoun *it* refers to the interval between the cursors. The gridlines offloaded and materialized the attentional anchor by offering distal proxies for its lower and upper bounds. The attentional anchor was thus reified in the public domain in the form of a perceptually stable, externally present, deictically referable, bounded entity. This shift was smooth. Yet once the interval operation had been delegated to the grid lines, an abrupt shift occurred that culminated with the new *a*-per-*b*



(a)



(b)

FIGURE 5 (a) After the introduction of the grid, Uri (middle) and Eden (far right) find green with Rc at 2-line and Lc at 1-line, respectively. Noticing the distance between Rc and Lc, Eden predicts that the fixed distance subtends "an entire box." The diagram is a partial schematic recreation of the screen (actually, the y-axis ran to 10, and there were no numerals at this point). (b) Immediately, Uri and Eden reposition Rc and Lc to 3-line and 2-line, respectively. The screen turns red. On noticing that the fixed distance theory does not obtain, Eden says, "Then maybe you should raise it. So maybe the higher you go, the more boxes it is apart." This diagram too was recreated for clarity. Rc = right-hand cursor; Lc = left-hand cursor.

scheme. This scheme achieved stability, because it enhanced the coproduction of green by regulating the dyadic turn-taking coordination of motor actions.

Phase shifts in motor-action coordination might be smooth or abrupt, depending on whether the existing routines can be maintained with the introduction of new task/environmental constraints (Kelso, 1984; Kostrubiec, Zanone, Fuchs, & Kelso, 2012). Following the introduction of the grid we witness first a smooth and then an abrupt shift.

We are excited by this new theoretical insight into how action grips symbol, because we perceive ecological dynamics as minding the epistemic gap between action and symbol. That is, the theory offers an explanation for how situated physical actions give rise to disciplinary content: Students engage symbolic artifacts as reified attentional anchors that first mediate action yet then shift activity discourse into semiotic registers. *Ecological dynamics thus views conceptual development as the spontaneous, situated adoption of symbolic artifacts as action tools.* Symbolic artifacts bear hybrid ontology, in the sense that they are both perceptual and semiotic entities (Uttal, Scudder, & DeLoache, 1997). They are "transitional objects" (Papert, 1980, p. 161)—both sensory and abstract. We might grab a symbol for its perceptuomotor affordance for action yet only subsequently leverage its semiotic potential for planning and communicating prospective actions, elaborating reasoning, and supporting argumentation. We "language" attentional anchors, articulating them via available frames of reference into the grammar of explicit reflection, and in so doing we reinvent and internalize cultural meanings. Thus, the meaning of a mathematical symbol is established as the felt sense of its afforded action (Cisek, 1999, p. 136).

We have now treated our second research problem: *adoption of symbolic artifacts as regulatory enactive, epistemic, and discursive devices.* In the final subsection we zoom out to foreground the instructor's active role in monitoring the field of promoted action.

The Instructor's Multimodal Intervention as Environmental Constraints on Action

If learning is the education of perceptuomotor attention (Gibson, 1966), then teachers can play pivotal roles in this process. Indeed, the MIT-P tutors developed a variety of instructional techniques to assist study participants in solving the motor-action coordination problem (Abrahamson et al., 2012). Here we describe and exemplify techniques consisting particularly of expert–novice physical coenactment.

One coenactment method is to distribute the operation of the control devices, one person per device. Figure 6a shows a tutor structuring a student's search in the physical problem space. The tutor fixates the location of the left-side virtual object on the computer monitor, waits for the student to determine the corresponding

a. b.

FIGURE 6 Hands-on mathematics instruction. (a) Comanipulating virtual objects via distributed coenactment. (b) Molding via joint coenactment.

right-side location that affects the target state (green screen), then iterates the procedure until the student assumes agency in leading the activity. Another method is to co-operate both control devices—that is, with both people each handling both devices. Figure 6b shows a tutor coaching a student to move in a new way. As the student raises her arms, the tutor gently constrains the motions by either speeding up or slowing down each of the hands so as to achieve the target state, all the while responsively enabling the student gradually to assume agency in enacting the coordinated actions.

Shani (see Figure 6) is a fifth-grade female student indicated by her mathematics teacher as "low performing." The vignette begins at a moment when Shani is holding the cursors up at about 1.5 (Lc) and 3 (Rc), and so the screen is green. After some lull, in which Shani appears unsure how to proceed, Dor positions himself to the left of Shani, and the following interaction ensues:

> Dor: **I have an idea. I'll take charge of this one, alright?** (takes LT, with Shani still handling RT) **And I'll put it . . . I dunno . . . here.** (places Lc at 3-line; screen goes red; Shani is still holding Rc at 3-line) **Can you make the screen green when he's here?** (Shani lifts Rc up to near 6-line, adjusting for darker green at 6-line, and holds it there. The dyad holds their positions for about 16 s.) **Okay, how about when I go here** (Dor lifts Lc to 4-line, the screen goes red, and Shani promptly lifts to near 8-line, adjusting for darker green at 8-line; see Figure 6a. The dyad keeps holding the cursors at those precise locations.)
>
> Shani: **Wait a minute. A while ago you asked me how many greens there are. It . . . could really be infinite.**

Dor: **Oh.**

Shani: **Like, because, if it really is all about the distance between them** (gestures diagonal line between cursors), **which is, like . . . I think it is, because it's getting darker, depending on that** . . . **uhhm** . . . **, then it really doesn't matter where on the screen it *is*.**

Shani realizes that there is an infinite number of green positions on the screen.

Dor: **Okay. And what else do we know about the distance? Like if I came *here*** (lifts Lc to 5-line, which is halfway up the screen, and screen goes red), **what would you want to do?**

Shani: **I'd go up** (lifts Rc to 10, which is the highest gridline on the screen; the screen goes green), **and then it would become green again.**

Dor: **Aha, okay. And if I'm down, let's looks what happens say if I'm all the way down here.** (Lowers Lc to 1-line; screen goes red)

Shani: (lowers Rc to 2-line; screen goes green) **It looks like 2. Yeah.**

Dor: **1 and 2.** (Dyad continues holding the cursors at 1 and 2)

Shani: **So basically, like, uhhmm** (points to screen), **if you put either one on a point, you'd be able to find a green.**

Dor: **Ha! What if this one is at 2?** (raises Lc to 2, screen goes red) **Where do you think that one should be?**

Shani: (Raises Rc to 4-line; screen goes green) **Oh they're getting farther apart as it goes up** (gestures an interval between right-hand thumb and middle finger, which she points toward the screen)

Dor: (Dragan, a confidante, looks at Dor, who raises eyebrows discreetly.) **Oh.**

This is the first time Shani assumes agency in leading the joint activity Dor has initiated.

Shani: **Like, the last** . . . (points downward on screen to previous positions; Dor responds by lowering Lc to 1-line, Shani lowers Rc to 2-line) **Like, yeah, here it was, like, 1 and 2, but** (raises Rc to 4-line; simultaneously, in continuous green, Dor raises Lc for green at 2-line) **then it was 2 and 4.**

Dor: **Huh. What do you think it will be if I bring it up to 3?** (Lc to 3-line, red)

Shani: **Probably 6.** (Rc-to 6-line, green)

Dor: **Huh.** . . . **How** . . . **Why did, why did you get 6? Yeah, you're right!**
Shani: **Well it's going up** . . . **by a box.** (gestures box interval toward screen)

Shani is referring to the vertical interval between the cursors that increased by 1 grid unit between 1-and-2, 2-and-4, 3-and-6, and so on.

Dor: **Oh, I see.** (Raises Lc to 4-line, red)
Shani: **So now this would be 8 probably** (Rc to 8-line, green).
Dor: **Yeah** . . . **What's going up by a box? I kind of get what you mean, but not completely. What is going up by a box?**
Shani: **Well the x** . . . (points to cursor, which is shaped as a +) **Well the bottom one** (Lc) **is going up by 1 box, but the top one** (Rc) **is going up by 2.**
Dor: **Ohhhh** . . . **okay.**

In the course of speaking about the interval between the cursors, which is increasing by one box each move (the higher–bigger principle), Shani finds herself shifting to the relative elevation of each of the two cursors (the *a*-per-*b* principle). Soon after, Shani notes that the different displacements of *each* cursor (the *a* and *b*) account for the changing interval *between* the cursors.

Dor hands Lc over to Shani. She performs a perfect continuous green while raising both cursors, followed by a near-perfect green lowering of the cursors. The performance appears to enact *a*-per-*b* simultaneously rather than sequentially. Dor then asks Shani what else she has noticed about the numbers, and Shani immediately detects and validates that Rc is always double Lc, and she attributes this new relationship to the fact that Lc goes up by 1 for every 2 that Rc goes up. She says that now she can predict, given a left-hand location, where the right hand should be.

In a matter of several minutes, Shani has thus both detected and connected four insights: (a) There are infinite green locations on the screen, (b) the size and elevation of the interval between the cursors should correlate for green, (c) the cursors rise at different yet coordinated rates, and (d) the cursors' green locations are related by a consistent multiplicative factor of 2. The distributed coenactment appears to have been important for Shani to generate this rapid succession of interconnected insights.

Dragan then removes the grid and numerals from the screen, leaving only the cursors, and asks Shani to try to perform green while moving her hands up and then down again. Shani places her hands at about where the 4-line and 8-line had been. She then moves up to the invisible 5-line and 10-line. After an imperfect green run up and down the screen, Shani comments that it is more difficult to do without the grid.

Immediately then we arrive at the case of joint coenactment (Figure 6b).

Dor: (Positions himself behind Shani's seat) **Why don't you hold them for a moment** (Shani grabs the remotes, and Dor gently holds Shani's left hand in his left hand, and her right hand in his right) **I want to see if I can track your hands all the way up in green. I'm going to find a green place** . . . (places cursors at about the invisible locations of 1-line and 2-line and, once stable at dark green, begins lifting both hands slowly, in green).

Shani: **So this one should be m** . . . **so my right should be moving faster** . . .,

Dor: **Oh, I see.**

Shani: . . . **so then it can be going up 2 spaces on the grid** . . . **on the graph, while the other one's only going 1.**

Dor: ***Oh*, I see.** (They continue to raise their hands, get to the top, then come down again, all in perfect green.) **Cool.**

Shani thus discovered yet another visualization of the effective motor action: The right hand must move faster than the left hand. Moreover, she coordinated between speed and rate—or, if you will, between "smooth and chunky images of change" (Castillo-Garsow, Johnson, & Moore, 2013)—by accounting for the hands' correlative continuous speeds in terms of the hands' coiterated respective composite units (1 and 2).

We have thus presented a case of a tutor who molded a child's embodied practice. As in the previous case of distributed coenactment, the child's agency in joint coenactment trends from relative passivity toward leading the joint production: Once a general sense of pace is coestablished, the child falls in with the actions, so that eventually the tutor may remove his hands. In this particular episode, molding the child's manipulation—a direct physical constraint—enabled her for the first time to attend to the hands' relative speeds. She also linked speed and rate by revisualizing this speed-based performance as a simultaneous enactment of *a*-per-*b* strategy—for every 1 unit that the left hand rose, the right hand rose 2 units *at the same time* (Abrahamson et al., 2014).

In this section we encountered two ways in which a teacher might introduce supplemental constraints on the novice's sensorimotor activity in the field of promoted action. These ways involved coenacting a cultural practice with a novice either by distribution of the task as a coordinated coproduction or by joint operation. Both ways were oriented on systemically steering the student to detect new attentional anchors empirically. That is, the student is to discern systemic information structures bearing dynamical invariance and manipulate these structures directly as new mediating objects. The choreography of coenactment shifts from sequential turn taking to simultaneous coperforming, in which both agents are constantly and dynamically interdependent on each other's mutually responsive coordination (Masciotra, Ackermann, & Roth, 2001). As in the internal martial arts practice of push-hands, the two participants in this joint coproduction of the green screen continuously yet silently negotiate leadership. Similar to a pair of athletes in a two-person sport, such as rowing or luge, tutor and student reach an intimate level of intersubjective sensorimotor coordination by anticipating and closely tracking each other's actions (Bekkering et al., 2009; Gallese & Lakoff, 2005). The tutor might then hand over the tracker and let the student enact the new strategy solo.

We have now treated our last research problem: *educators' active role in facilitating effective sensorimotor engagement.*

CONCLUSION

We have introduced the theory of ecological dynamics, which originates in kinesiology and sports science. We argued for the purchase of this theory in mathematics-education research. In particular, we argued that ecological dynamics offers an analytic lens geared to filling the enduring theoretical gap in tracking and explaining students' ontogenesis from goal-oriented sensorimotor action to conceptual reasoning. Our argument was contextualized in artifacts and findings from a design-based research project that has been developing and evaluating a pedagogical activity involving the solution of motor-action coordination problems. A set of vignettes from an empirical study detailed and elaborated the argument.

These are still early days in the application of motor-action theories to mathematics education research, and many questions remain. Nevertheless, we cautiously submit that the theory of ecological dynamics offers a useful framework for designing, implementing, and analyzing activities in which students develop fundamental understandings of mathematical notions via solving and reflecting on motor-action inquiry problems. From this view, mathematical meanings emerge from the guided signification of situated motor-action coordinations (for parallel perspectives on action coordinations in language development, see Glenberg & Kaschak, 2002). The theory thus supports an integration of seminal constructivist and sociocultural perspectives on human learning.

Students can develop mathematical coordinations via engaging in carefully designed activities. Given appropriate cultural mediation, students can fairly instantaneously learn to move and therefore think in new ways that then become elaborated, refined, and reformulated as disciplinary discourse. This thesis suggests children's universal capacity to deeply understand mathematical concepts, regardless of prior academic accomplishment, because it shifts the site of critical mathematical learning away from the symbolic semiotic register toward situated sensorimotor engagement with manipulation problems.

Per the ecological dynamics view of mathematics education, latent stimuli in the field of promoted action—features, patterns, and even imaginary features, clusters of features, and mixed actual and imagined features—become salient and meaningful through their apparent capacity to enable the satisfaction of task objectives. These stimuli are thus temporarily foregrounded as objects (gestalts) that the agent evaluates as potentially instrumental in mediating the achievement of task objectives. These objects are psychological constructions of available features of the environment (objectifications; Radford, 2000)—the objects suggest themselves as attentional anchors enhancing motor-action performance. For example, an empty space between two hands can emerge into consciousness in the course of embodied interaction, spontaneously coalescing from negative space into situated pertinence. Thus, nothing becomes thing, an interval whose size is controlled, monitored, measured, and correlated with contextual goals. The attentional anchor is therefore a self-generated auxiliary stimulus that pops up between agent and environment. Attentional anchors are ad hoc groundings of evolving operatory schemes, contrived handles extending motor intentionality to better grip the world. We adopt and exercise attentional anchors because they prove useful in getting a job done. An attentional anchor is an invented constellation of selected features in the perceptual field. It is utilized as an instrument for controlling a situation. In turn, attentional anchors serve as presymbolic kernels of new concepts as the activity shifts into disciplinary frames of reference.

Ecological dynamics brings back motor action into learning (Savelsbergh et al., 2005). And if learning mathematics begins from learning to move in a new way, then learning mathematics could be not too unlike learning to become a dental hygienist (Becvar Weddle & Hollan, 2010), musical instrumentalist (Haviland, 2011; Simones, Rodger, & Schroeder, 2014), potter (Churchill, 2016), knitter (Lindwall & Ekström, 2012), carpenter (Ingold, 2000), dancer (Katz, 2013), or martial artist (Sánchez-García, 2013). In all of these spatial–dynamical manual practices, novices must develop habits not only of professional perception per se (Arcavi, 2003; Goodwin, 1994; Stevens & Hall, 1998) but of perceptuo*motor* orientation to their respective domains (and see Merleau-Ponty, 1945/2005, p. 59, on visual perception as grasping). In the case of complex production tasks, a perceptuomotor orientation evolves dialectically with the emergence of attentional anchors that mediate a tighter grip on the world by affording ergonomic motor-action coordination (Hutto & Sánchez-García, 2015). Discursive treatment of attentional anchors then organizes the social enactment of cultural practice. Inter alia, discourse increases the specificity of novices' evolving actions and contextualizes these actions in frames of reference from the greater activity structure.

Thus, shared meanings of disciplinary concepts are consolidated (Goodwin, 1994; Hutchins, 1995; Radford, 2010; Trninic & Abrahamson, 2013).

Educators play pivotal roles in creating and managing students' discovery, manipulation, and transition to new artifacts. Instructors scaffold student activity by implementing task or environmental constraints that bound the agent's search for effective motor-action coordinations. Even when the instructor tightly molds students' engagement in embodied interaction, it is ultimately the students who must discover for themselves both the specifying information structures and their motor-action affordances. Later in the learning activity, the instructor introduces additional constraints in the form of figural artifacts bearing semiotic potential—the agent hooks these artifacts for their pragmatic or epistemic utility, yet in so doing shifts into mathematical register.

Embodied-interaction learning environments transform the practice of mathematical teaching, rendering it similar to coaching in the overtly physical disciplines, such as music, dance, or carpentry. Yet for these new pedagogical methodologies to enter educational institutions, we would have to rethink multiple aspects of mathematics teachers' professional practice, beginning from epistemology and through to assessment. Embodied interaction might help remediate what Thompson (2013) diagnosed as "the absence of meaning" in mathematics education. To the extent that meanings emerge from presymbolic multimodal operatory schemes afforded by actual or imagined perceptual features, ecological dynamics stands to stimulate this concerted discussion (Hutto, Kirchhoff, & Abrahamson, 2015).

Embodied-interaction activities offer solutions for researchers and teachers alike who wish to both observe mathematical thinking as it is occurring and offer students opportunities to reflect on their actions. Thus, embodied-interaction activities take us one step closer in responding to a lament expressed by Von Glasersfeld (1983):

> [A] useful analogy [for the teaching of mathematics content] might be found in the teaching of athletic skills. . . . Unfortunately, we have no tachistoscope or camera that could capture the dynamics, the detailed progression of steps, of the mental operations that lead to the solution of a numerical problem. (pp. 51–52)

Von Glasersfeld (1983) concluded that the only way to become apprised of children's mathematical thinking is to conduct *teaching experiments,* Steffe's combination of pedagogical design with Piagetian task-based semistructured clinical interviews.

Von Glasersfeld might have been heartened to learn that eye-tracking technology can indeed "capture the dynamics, the detailed progression of steps, of the mental operations that lead to the solution of a numerical problem." As we track the gaze of children attempting to make the screen green, we can literally see the emergence of attentional anchors as perceptual attractors prior to their verbal or gestural articulation as mathematical objects (Abrahamson et al., in press; Shayan, Abrahamson, Bakker, Duijzer, & Van der Schaaf, 2015).

As the theory and practice of embodied interaction develop, an important avenue of research is to determine the conceptual scope of these technological designs beyond simple functions in the Cartesian plane (Nemirovsky, Kelton, & Rhodehamel, 2013). For example, what would embodied interaction look like for algebra or calculus? As we ourselves pursue these questions, we hope to have stirred some interest in the endeavor.

This line of work—developing, analyzing, and evaluating mathematics learning from an ecological dynamics perspective—seems urgent to us. Even as children are increasingly engaging both in and out of school in embodied interaction within technological environments, such as manipulating touchscreen tablets, their coordinated motor actions are scarcely considered in terms of the conceptual development they might foster. And yet embodied interaction is precisely where conceptual development begins. As we consider future directions for theories of learning as well as the opportunities for learning that will coemerge with these theories, we believe that in more than one sense learning is moving in new ways.

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REFERENCES

Abrahamson, D. (2002). When “the same” is the same as different differences: Aliya reconciles her perceptual judgment of proportional equivalence with her additive computation skills. In D. Mewborn, P. Sztajn, E. White, H. Wiegel, R. Bryant, & K. Nooney (Eds.), *Proceedings of the Twenty Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 1658–1661). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Abrahamson, D. (2009). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics, 70*(1), 27–47. [Electronic supplementary material at http://edrl.berkeley.edu/publications/journals/ESM/Abrahamson-ESM/]

Abrahamson, D. (2012a). Discovery reconceived: Product before process. *For the Learning of Mathematics*, *32*(1), 8–15.

Abrahamson, D. (2012b). You’re it! Body, action, and object in STEM learning. In J. V. Aalst, K. Thompson, M. J. Jacobson, & P. Reimann (Eds.), *Proceedings of the International Conference of the Learning Sciences: Future of learning (ICLS 2012)* (Vol. 1, pp. 283–290). Sydney, Australia: University of Sydney/International Society of the Learning Sciences.

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction*, *2*(1), 1–16. doi:10.1016/j.ijcci.2014.07.002

Abrahamson, D. (2015). The monster in the machine, or why educational technology needs embodied design. In V. R. Lee (Ed.), *Learning technologies and the body: Integration and implementation* (pp. 21–38). New York, NY: Routledge.

Abrahamson, D., Gutiérrez, J. F., Charoenying, T., Negrete, A. G., & Bumbacher, E. (2012). Fostering hooks and shifts: Tutorial tactics for guided mathematical discovery. *Technology, Knowledge, and Learning*, *17*(1–2), 61–86. doi:10.1007/s10758-012-9192-7

Abrahamson, D., & Howison, M. (2010). Embodied artifacts: Coordinated action as an object-to-think-with. In D. L. Holton, (Organizer & Chair) & J. P. Gee (Discussant), *Embodied and enactive approaches to instruction: Implications and innovations*. Paper presented at the annual meeting of the American Educational Research Association, April 30–May 4, Denver, CO.

Abrahamson, D., Lee, R. G., Negrete, A. G., & Gutiérrez, J. F. (2014). Coordinating visualizations of polysemous action: Values added for grounding proportion. *ZDM*, *46*(1), 79–93. doi:10.1007/s11858-013-0521-7

Abrahamson, D., & Sánchez-García, R. (2014, June). *Learning is moving in new ways: An ecological dynamics view on learning across the disciplines*. Paper presented at the Embodied Cognition in Education symposium, Utrecht, The Netherlands.

Abrahamson, D., & Sánchez-García, R. (2015). A call to action: Towards an ecological-dynamics theory of mathematics learning, teaching, and design. In T. G. Bartell, K. N. Bieda, R. T. Putnam, K. Bradfield, & H. Dominguez (Eds.), *Proceedings of the 37th Annual Meeting of the North-American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA)* (Vol. 12, pp. 1261–1268). East Lansing, MI: Michigan State University.

Abrahamson, D., Sánchez-García, R., & Smyth, C. (In press). Metaphors are projected constraints on action: An ecological dynamics view on learning across the disciplines. In C.-K. Looi, J. Polman, U. Cress, & P. Reimann (Eds.), *"Transforming learning, empowering learners," Proceedings of the International Conference of the Learning Sciences (ICLS 2016).* Singapore: National Institute of Education.

Abrahamson, D., Shayan, S., Bakker, A., & Van der Schaaf, M. F. (in press). Eye-tracking Piaget: Capturing the emergence of attentional anchors in the coordination of proportional motor action. *Human Development*.

Abrahamson, D., & Trninic, D. (2011). Toward an embodied-interaction design framework for mathematical concepts. In P. Blikstein & P. Marshall (Eds.), *Proceedings of the 10th Annual Interaction Design and Children Conference (IDC 2011)* (Vol. "Full papers," pp. 1–10). Ann Arbor, MI: IDC.

Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: Signifying sensorimotor enactment in fields of promoted action. *ZDM*, *47*(2), 295–306. doi:10.1007/s11858-014-0620-0

Abrahamson, D., Trninic, D., Gutiérrez, J. F., Huth, J., & Lee, R. G. (2011). Hooks and shifts: A dialectical study of mediated discovery. *Technology, Knowledge, and Learning*, *16*(1), 55–85.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*, *21*(2), 247–286. doi:10.1080/10508406.2011.611446

Araújo, D., & Davids, K. (2004). Embodied cognition and emergent decision-making in dynamical movement systems. *Junctures*, *2*, 45–57.

Araújo, D., Davids, K. W., Chow, J. Y., Passos, P., & Raab, M. (2009). The development of decision making skill in sport: An ecological dynamics perspective. In D. Araújo & H. Ripoll (Eds.), *Perspectives on cognition and action in sport* (pp. 157–169). Hauppauge, NY: Nova Science.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*(3), 215–241. doi:10.1023/A:1024312321077

Barab, S. A., Cherkes-Julkowski, M., Swenson, R., Garrett, S., Shaw, R. E., & Young, M. (1999). Principles of self-organization: Learning as participation in autocatakinetic systems. *Journal of the Learning Sciences*, *8*, 349–390. doi:10.1080/10508406.1999.9672074

Becvar Weddle, A. B., & Hollan, J. D. (2010). Professional perception and expert action: Scaffolding embodied practices in professional education. *Mind, Culture & Activity*, *17*(2), 119–148. doi:10.1080/10749030902721754

Beilock, S. L. (2008). Beyond the playing field: Sport psychology meets embodied cognition. *International Review of Sport and Exercise Psychology*, *1*(1), 19–30. doi:10.1080/17509840701836875

Bekkering, H., De Bruijn, E. R. A., Cuijpers, R. H., Newman-Norlund, R., Van Schie, H. T., & Meulenbroek, R. (2009). Joint action: Neurocognitive mechanisms supporting human interaction. *Topics in Cognitive Science*, *1*(2), 340–352. doi:10.1111/j.1756-8765.2009.01023.x

Bernstein, N. A. (1996). Change in movement and skill: Learning, retention, and transfer. In M. L. Latash & M. T. Turvey (Eds.), *Dexterity and its development* (pp. 3–235). Mahwah, NJ: Erlbaum.

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, *33*(3), 31–37.

Chandrasekharan, S., & Nersessian, N. J. (2015). Building cognition: The construction of computational representations for scientific discovery. *Cognitive Science*, *39*(8), 1727–1763. doi:10.1111/cogs.12203

Churchill, E. (2016). *Skill learning, parsing, and narrated enactments: Decomposing and blending action at the potter's wheel.* Manuscript in preparation.

Cisek, P. (1999). Beyond the computer metaphor: Behavior as interaction. *Journal of Consciousness Studies*, *6*(11–12), 125–142.

Clancey, W. J. (2008). Scientific antecedents of situated cognition. In P. Robbins & M. Aydede (Eds.), *Cambridge handbook of situated cognition* (pp. 11—34). New York, NY: Cambridge University Press.

Clark, A. (1999). *Embodiment: From fish to fantasy*. St. Louis, MO: Washington University in St. Louis.

Confrey, J., & Scarano, G. H. (1995). Splitting reexamined: Results from a three-year longitudinal study of children in grades three to five. In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA)* (Vol. 1, pp. 421–426). Columbus, OH: ERIC Clearinghouse.

Davids, K., Button, C., & Bennet, S. (2008). *Dynamics of skill acquisition: A constraints-led approach*. Champaign, IL: Human Kinetics.

Davis, B., & Sumara, D. (2008). Complexity as a theory of education. *Transnational Curriculum Inquiry, 5*(2). Retrieved from http://nitinat.library.ubc.ca/ojs/index.php/tci

de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics*, *80*(1–2), 133–152. doi:10.1007/s10649-011-9364-8

Dreyfus, H. L., & Dreyfus, S. E. (1999). The challenge of Merleau-Ponty's phenomenology of embodiment for cognitive science. In G. Weiss & H. F. Haber (Eds.), *Perspectives on embodiment: The intersections of nature and culture* (pp. 103–120). New York, NY: Routledge.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1–2), 103–131. doi:10.1007/s10649-006-0400-z

Edelman, G. M. (1987). *Neural Darwinism: The theory of neuronal group selection*. New York, NY: Basic Books.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, *22*(3–4), 455–479. doi:10.1080/02643290442000310

Gibson, J. J. (1950). *The perception of the visual world*. Boston, MA: Houghton Mifflin.
Gibson, J. J. (1966). *The senses considered as perceptual systems*. Boston, MA: Houghton Mifflin.
Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, acting and knowing: Toward an ecological psychology* (pp. 67–82). Hillsdale, NJ: Erlbaum.
Ginsburg, C. (2010). *The intelligence of moving bodies: A somatic view of life and its consequences*. Santa Fe, NM: AWAREing Press.
Glenberg, A. M., & Kaschak, M. P. (2002). Grounding language in action. *Psychonomic Bulletin & Review*, *9*(3), 558–565. doi:10.3758/BF03196313
Goodwin, C. (1994). Professional vision. *American Anthropologist*, *96*(3), 606–633. doi:10.1525/aa.1994.96.issue-3
Greeno, J. G. (1994). Gibson's affordances. *Psychological Review*, *101*(2), 336–342. doi:10.1037/0033-295X.101.2.336
Greeno, J. G. (1998). The situativity of knowing, learning, and research. *American Psychologist*, *53*(1), 5–26. doi:10.1037/0003-066X.53.1.5
Greeno, J. G. (2015). Commentary: Some prospects for connecting concepts and methods of individual cognition and of situativity. *Educational Psychologist*, *50*(3), 248–251. doi:10.1080/00461520.2015.1077708
Haviland, J. B. (2011). Musical spaces. In J. Streeck, C. Goodwin, & C. LeBaron (Eds.), *Embodied interaction: Language and body in the material world* (pp. 289–304). Cambridge, UK: Cambridge University Press.
Howison, M., Trninic, D., Reinholz, D., & Abrahamson, D. (2011). The Mathematical Imagery Trainer: From embodied interaction to conceptual learning. In G. Fitzpatrick, C. Gutwin, B. Begole, W. A. Kellogg, & D. Tan (Eds.), *Proceedings of the annual meeting of the Association for Computer Machinery Special Interest Group on Computer Human Interaction: "Human Factors in Computing Systems" (CHI 2011), Vancouver, May 7-12, 2011* (Vol. "Full papers," pp. 1989–1998). New York, NY: Association for Computing Machinery Press.
Hristovski, R., Davids, K., Passos, P., & Araújo, D. (2012). Sport performance as a domain of creative problem solving for self-organizing performer-environment systems. *The Open Sports Sciences Journal*, *5*(Suppl. 1–M4), 26–35. doi:10.2174/1875399X01205010026
Hutchins, E. (1995). *Cognition in the wild*. Cambridge, MA: MIT Press.
Hutto, D. D., Kirchhoff, M. D., & Abrahamson, D. (2015). The enactive roots of STEM: Rethinking educational design in mathematics. *Educational Psychology Review*, *27*(3), 371–389. doi:10.1007/s10648-015-9326-2
Hutto, D. D., & Sánchez-García, R. (2015). Choking RECtified: Embodied expertise beyond Dreyfus. *Phenomenology and the Cognitive Sciences*, *14*(2), 309–331. doi:10.1007/s11097-014-9380-0
Ingold, T. (2000). *The perception of the environment: Essays on livelihood, dwelling, and skill* (2nd ed.). London, UK: Routledge.
Kalchman, M., Moss, J., & Case, R. (2000). Psychological models for the development of mathematical understanding: Rational numbers and functions. In S. Carver & D. Klahr (Eds.), *Cognition and instruction: 25 years of progress* (pp. 1–38). Mahwah, NJ: Erlbaum.
Kamii, C. K., & DeClark, G. (1985). *Young children reinvent arithmetic: Implications of Piaget's theory*. New York, NY: Teachers College Press.
Katz, M.-L. (2013). Growth in motion: Supporting young women's embodied identity and cognitive development through dance after school. In M.-L. Katz (Ed.), *Moving ideas: Multimodality and embodied learning in communities and schools* (pp. 31–46). New York, NY: Peter Lang.
Kelso, J. A. S. (1984). Phase transitions and critical behavior in human bimanual coordination. *American Journal of Physiology: Regulatory, Integrative and Comparative*, *246*(6), R1000–R1004.
Kelso, J. A. S., & Engstrøm, D. A. (2006). *The complementary nature*. Cambridge, MA: MIT Press.
Kim, M., Roth, W.-M., & Thom, J. S. (2011). Children's gestures and the embodied knowledge of geometry. *International Journal of Science and Mathematics Education*, *9*(1), 207–238. doi:10.1007/s10763-010-9240-5

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist*, *41*(2), 75–86. doi:10.1207/s15326985ep4102_1

Kirsh, D. (2010a). Thinking with external representations. *AI & Society*, *25*, 441–454. doi:10.1007/s00146-010-0272-8

Kirsh, D. (2010b). Thinking with the body. In S. Ohlsson & R. Catrambone (Eds.), *Proceedings of the 32nd Annual Meeting of the Cognitive Science Society* (pp. 2864–2869). Austin, TX: Cognitive Science Society.

Kirsh, D. (2013). Embodied cognition and the magical future of interaction design. *ACM Transactions on Computer-Human Interaction*, *20*(1), 1–30. doi:10.1145/2442106.2442109

Kostrubiec, V., Zanone, P.-G., Fuchs, A., & Kelso, J. A. S. (2012). Beyond the blank slate: Routes to learning new coordination patterns depend on the intrinsic dynamics of the learner–experimental evidence and theoretical model. *Frontiers in Human Neuroscience*, *6*. doi:10.3389/fnhum.2012.00222

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 33–40). Hillsdale, NJ: Erlbaum.

Liao, C., & Masters, R. S. (2001). Analogy learning: A means to implicit motor learning. *Journal of Sports Sciences*, *19*, 307–319. doi:10.1080/02640410152006081

Lindwall, O., & Ekström, A. (2012). Instruction-in-interaction: The teaching and learning of a manual skill. *Human Studies*, *35*(1), 27–49. doi:10.1007/s10746-012-9213-5

Malafouris, L. (2013). *How things shape the mind*. Cambridge, MA: MIT Press.

Masciotra, D., Ackermann, E., & Roth, W.-M. (2001). "Maai": The art of distancing in karate-do mutual attunement in close encounters. *Journal of Adult Development*, *8*(2), 119–132. doi:10.1023/A:1026498019155

Merleau-Ponty, M. (2005). *Phenomenology of perception* (C. Smith, Trans.). New York, NY: Routledge. (Original work published 1945)

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the Twenty Seventh Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 105–109). Honolulu, HI: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education*, *44*(2), 372–415. doi:10.5951/jresematheduc.44.2.0372

Newell, K. M. (1986). Constraints on the development of coordination. In M. G. Wade & H. T. A. Whiting (Eds.), *Motor development in children: Aspects of coordination and control* (pp. 341–361). Amsterdam, The Netherlands: Martinus Nijhoff.

Newell, K. M. (1996). Change in movement and skill: Learning, retention, and transfer. In M. L. Latash & M. T. Turvey (Eds.), *Dexterity and its development* (pp. 393–429). Mahwah, NJ: Erlbaum.

Newell, K. M., & Ranganathan, R. (2010). Instructions as constraints in motor skill acquisition. In I. Renshaw, K. Davids, & G. J. P. Savelsbergh (Eds.), *Motor learning in practice: A constraints-led approach* (pp. 17–32). Florence, KY: Routledge.

Norman, D. A. (1998). *The design of everyday things*. New York, NY: Basic Books.

Nunes, T., & Bryant, P. (1996). *Children doing mathematics*. Cornwall, UK: T. J. Press.

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York, NY: Basic Books.

Papert, S. (1993). *The children's machine: Rethinking school in the age of the computer*. New York, NY: Basic Books.

Papert, S. (1996). An exploration in the space of mathematics educations. *International Journal of Computers for Mathematical Learning*, *1*(1), 95–123. doi:10.1007/BF00191473

Papert, S. (2004). [Keynote speech]. Retrieved from http://vimeo.com/9092144
Piaget, J. (1968). *Genetic epistemology* (E. Duckworth, Trans.). New York, NY: Columbia University Press.
Post, T. R., Cramer, K. A., Behr, M. J., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching and assessing of rational number concepts. In T. P. Carpenter, E. Fenema, & T. Romberd (Eds.), *Rational numbers: An interaction of research* (pp. 327–362). Hillsdale, NJ: Erlbaum.
Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics*, *42*(3), 237–268. doi:10.1023/A:1017530828058
Radford, L. (2005). The semiotics of the schema: Kant, Piaget, and the calculator. In M. H. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and sign: Grounding mathematics education* (pp. 137–152). New York, NY: Springer.
Radford, L. (2010). The eye as a theoretician: Seeing structures in generalizing activities. *For the Learning of Mathematics*, *30*(2), 2–7.
Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *REDIMAT—Journal of Research in Mathematics Education*, *2*(1), 7–44.
Reed, E. S., & Bril, B. (1996). The primacy of action in development. In M. L. Latash & M. T. Turvey (Eds.), *Dexterity and its development* (pp. 431–451). Mahwah, NJ: Erlbaum.
Reinholz, D., Trninic, D., Howison, M., & Abrahamson, D. (2010). It's not easy being green: Embodied artifacts and the guided emergence of mathematical meaning. In P. Brosnan, D. Erchick, & L. Flevares (Eds.), *Proceedings of the Thirty-Second Annual Meeting of the North-American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 32)* (Vol. 6, pp. 1488–1496). Columbus, OH: PME-NA.
Renshaw, I., Chow, J. Y., Davids, K., & Hammond, J. (2010). A constraints-led perspective to understanding skill acquisition and game play: A basis for integration of motor learning theory and physical education praxis? *Physical Education & Sport Pedagogy*, *15*(2), 117–137. doi:10.1080/17408980902791586
Roth, W.-M. (2015). Excess of graphical thinking: Movement, mathematics and flow. *For the Learning of Mathematics*, *35*(1), 2–7.
Ryle, G. (1945). I.—Knowing how and knowing that: The presidential address. *Proceedings of the Aristotelian Society*, *46*, 1–16. doi:10.1093/aristotelian/46.1.1
Sánchez-García, R. (2013). Teaching/learning interactions as social phenomena: The case of Aikido as a community of practice. In A. Rodríguez, D. Moscoso, & J. Fernández (Eds.), *"Sociology and sport in face of new challenges"—Proceedings of the 10th European Association for Sociology of Sport Conference (EASS 2013)* (Vol. "Abstracts," p. 40). Córdoba, Spain: Spanish Association for Sport Sociology.
Savelsbergh, G. J. P., Vereijken, B., & Zaal, F. T. J. M. (2005). Putting the "motor" back into development: The legacy of Esther Thelen (1941–2004). *Infant Behavior and Development*, *28*, 97–98. doi:10.1016/j.infbeh.2005.02.006
Saxe, G. B. (2002). Children's developing mathematics in collective practices: A framework for analysis. *Journal of the Learning Sciences*, *11*(2–3), 275–300. doi:10.1080/10508406.2002.9672140
Schwartz, D. L., & Martin, T. (2006). Distributed learning and mutual adaptation. *Pragmatics & Cognition*, *14*(2), 313–332. doi:10.1075/pc.14.2
Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences*, *16*(4), 565–613. doi:10.1080/10508400701525253
Shayan, S., Abrahamson, D., Bakker, A., Duijzer, C., & Van der Schaaf, M. F. (2015). The emergence of proportional reasoning from embodied interaction with a tablet application: An eye-tracking study. In L. Gómez Chova, A. López Martínez, & I. Candel Torres (Eds.), *Proceedings of the 9th International Technology, Education, and Development Conference (INTED 2015)* (pp. 5732–5741). Madrid, Spain: International Academy of Technology, Education and Development.

Simones, L. L., Rodger, M., & Schroeder, F. (2014). Communicating musical knowledge through gesture: Piano teachers' gestural behaviours across different levels of student proficiency. *Psychology of Music, 43, 723–735*. doi:10.1177/0305735614535830

Skemp, R. R. (1983). The silent music of mathematics. *Mathematics Teaching*, *102*(58), 287–288.

Sophian, C. (1995). *Children's numbers*. Dubuque, IA: Brown.

Spinillo, A. G., & Bryant, P. E. (1991). Children's proportional judgments: The importance of "half." *Child Development*, *62*, 427–440. doi:10.2307/1131121

Spinillo, A. G., & Bryant, P. E. (1999). Proportional reasoning in young children: Part-part comparisons about continuous and discontinuous quantity. *Mathematical Cognition*, *5*(2), 181–197. doi:10.1080/135467999387298

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107–149). New York, NY: Cambridge University Press.

Stoate, I., & Wulf, G. (2011). Does the attentional focus adopted by swimmers affect their performance? *International Journal of Sports Science and Coaching*, *6*(1), 99–108. doi:10.1260/1747-9541.6.1.99

Thelen, E., & Smith, L. B. (1994). *A dynamic systems approach to the development of cognition and action*. Cambridge, MA: MIT Press.

Thompson, P. W. (2013). In the absence of meaning . . . In K. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 57–94). New York, NY: Springer.

Trninic, D., & Abrahamson, D. (2013). Embodied interaction as designed mediation of conceptual performance. In D. Martinovic, V. Freiman, & Z. Karadag (Eds.), *Mathematics education in the digital era: Vol. 1. Visual mathematics and cyberlearning* (pp. 119–139). New York, NY: Springer.

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology*, *18*, 37–54. doi:10.1016/S0193-3973(97)90013-7

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Vereijken, B., & Whiting, H. T. (1990). In defence of discovery learning. *Canadian Journal of Sport Sciences*, *15*(2), 99–106.

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education*, *10*(1), 77–101. doi:10.1007/BF03172796

Vilar, L., Araújo, D., Davids, K., & Renshaw, I. (2012). The need for "representative task design" in evaluating efficacy of skills tests in sport: A comment on Russell, Benton and Kingsley (2010). *Journal of Sports Sciences*, *30*(16), 1727–1730. doi:10.1080/02640414.2012.679674

Von Glasersfeld, E. (1983). Learning as constructive activity. In J. C. Bergeron & N. Herscovics (Eds.), *Proceedings of the 5th Annual Meeting of the North American Group for the Psychology of Mathematics Education* (Vol. 1, pp. 41–69). Montreal, QC, Canada: PME-NA.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press. (Original work published 1930)

Weast, J. A., Shockley, K., & Riley, M. A. (2011). The influence of athletic experience and kinematic information on skill-relevant affordance perception. *Quarterly Journal of Experimental Psychology*, *64*(4), 689–706. doi:10.1080/17470218.2010.523474

Wulf, G., & Su, J. (2007). An external focus of attention enhances golf shot accuracy in beginners and experts. *Research Quarterly for Exercise and Sport*, *78*, 384–389. doi:10.1080/02701367.2007.10599436

Zarghami, M., Saemi, E., & Fathi, I. (2012). External focus of attention enhances discus throwing performance. *Kinesiology*, *44*(1), 47–51.

Zhang, J., & Patel, V. L. (2006). Distributed cognition, representation, and affordance. *Pragmatics & Cognition*, *14*(2), 333–341. doi:10.1075/pc.14.2