

COMPUTER MATH SNAPSHOTS

THE SHAPE OF THINGS TO COME: THE COMPUTATIONAL  
PICTOGRAPH AS A BRIDGE FROM COMBINATORIAL  
SPACE TO OUTCOME DISTRIBUTION

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This column will publish short (from just a few paragraphs to ten or so pages), lively and intriguing computer-related mathematics vignettes. These vignettes or snapshots should illustrate ways in which computer environments have transformed the practice of mathematics or mathematics pedagogy. They could also include puzzles or brain-teasers involving the use of computers or computational theory. Snapshots are subject to peer review. This snapshot introduces a computer-based representation and activity that enables students to simultaneously “see” the combinatorial space of a stochastic device (e.g., dice, spinner, coins) and its outcome distribution. The author argues that the “ambiguous” representation fosters student insight into probability.

Computer Math Snapshots

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1. INTRODUCTION

This snapshot introduces the *computational pictograph*, an interactive computer-based artifact designed to help students learn mathematics. Focusing on the domain of probability, we will examine a

computational pictograph that supports students' exploration of the Law of Large Numbers. This law is explored as a coordination between the *combinatorial space* of a stochastic device – what it is possible to “get” with this device – and the *outcome distribution* of the device – what you actually “get” when you operate the device and how often you get it. To help students in this exploration, the design of the computational pictograph backgrounds the similarities between the constructs and, thus, foregrounds the critical differences between them. We will look at two implementations of the computational pictograph – the *Dice Stalagmite* and *9-Block Stalagmite* models.<sup>1</sup>

## 2. THE COMPUTATIONAL PICTOGRAPH FOR LEARNING PROBABILITY AND STATISTICS: TWO EXAMPLES

Dice Stalagmite and 9-Block Stalagmite are built in the *NetLogo* (Wilensky, 1999) modeling-and-simulation environment and are part of *ProbLab* (Abrahamson and Wilensky, 2002), an experimental unit on probability and statistics under the umbrella of the *Connected Probability* project (Wilensky, 1997). These models are named “stalagmites” to capture their behavior that is reminiscent of the growth of stalagmites, as I now explain.

### 2.1. *Dice Stalagmite*

In *Dice Stalagmite* (see Figure 1, next page) the stimulated stochastic device is a pair of dice. To introduce the model and a suggested interaction with it, I first explain the “stalagmite” nature of the simulation and then focus on the juxtaposition of the combinatorial space and the outcome distribution.

Two independent dice are “rolled” on the “ceiling” (the top of the simulation view). Next, each sample of two dice slides across to its respective pictograph column and descends to the top of the stack of samples already there (see in Figure 1, next page, on the left, that the pair “5 & 3,” highlighted by an arrow, has slid to Column “8” and is going down the chute). When the simulation is run, all the bar-chart columns grow incrementally by accumulating their respective samples, yet the central columns eventually become taller than the other columns (see Figure 1, next page, from left to right). Why are they taller? That is the “math-thematic” question. Following, we will see how the computational pictograph helps address this question. Later, we

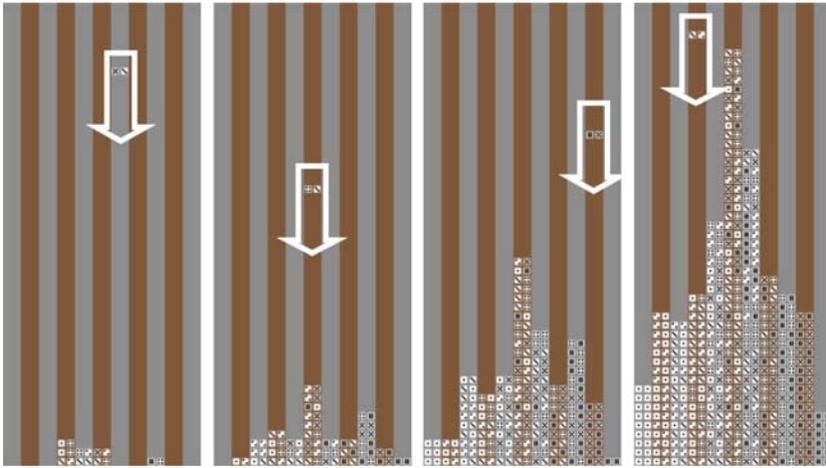


Figure 1. From left: four screen captures from Dice Stalagmite. Randomly generated pair-of-dice samples descend into their respective columns, according to the sum of their die faces (ranging from 2 to 12), giving rise to the shape that increasingly approximates a “pyramid.”

discuss combinatorial-analysis activities, outside of the computational medium, that precede these simulations and anticipate their outcomes.

The model can be set either to avoid keeping repeated samples or to keep them. When repeated samples are *not* kept, only the combinatorial space is produced (see Figure 2a, next page), and when the repeated samples are kept, an outcome distribution emerges (see Figure 2b, next page).<sup>2</sup> This juxtaposition is designed to foster the following insight: We “get” more random samples in the inner columns as compared to the outer columns, because the inner columns have relatively more unique items to choose from in the combinatorial space.

Having introduced the computational component of the design, we now situate this component within classroom mixed-media activities. In particular, we will look at a different stochastic device, the *9-block*, to clarify how students’ paper-and-crayon combinatorial-space constructions anticipate the shape of things to come in the computer simulations of probability experiments.

## 2.2. The 9-Block Stalagmite

In the *9-Block Stalagmite* model, the stochastic device is the 9-block. A *9-block* (see Figure 3, two pages down, on the left) is a 3-by-3 square grid, each cell of which can either be green or blue.

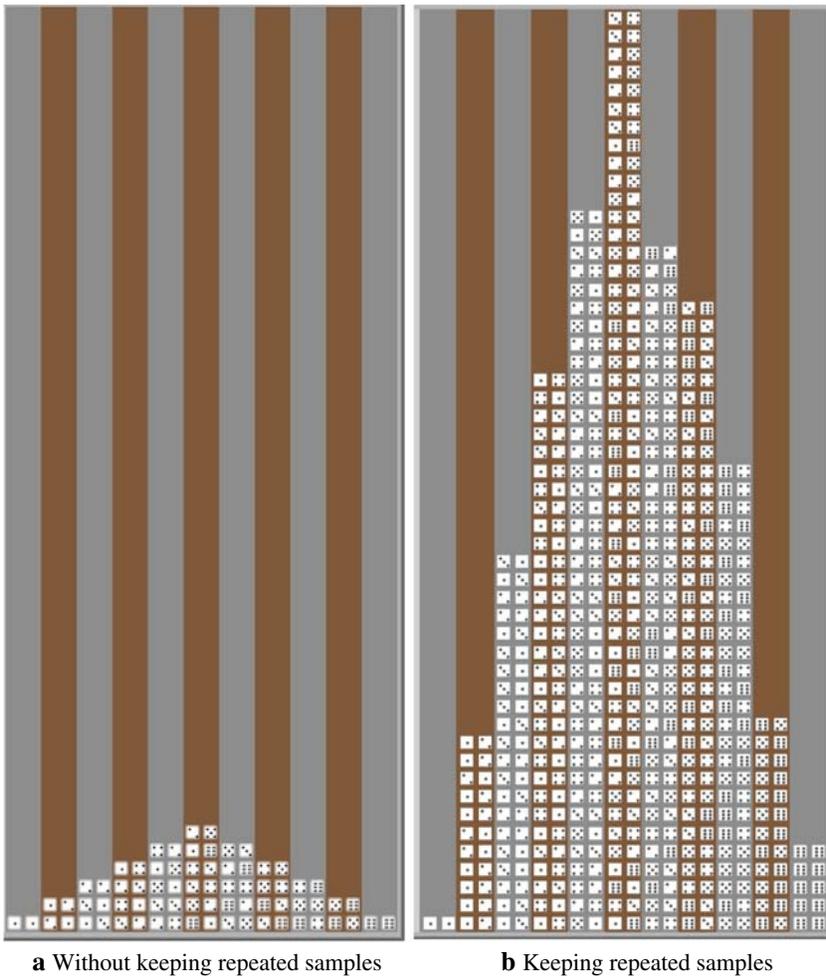


Figure 2. The combinatorial space as ‘a shape of things to come.’ Classifying the (a) combinatorial space of a pair of dice according to the sum of the dice produces a pictograph with structural properties similar to those of the (b) outcome distribution in a simulated probability experiment, in which random outcomes are arranged by the same classification.

In one classroom activity that uses the 9-block, students color in all the 512 possible configurations of green and blue (see Figure 3, next page, on the right). Students then assemble this combinatorial space of the 9-block into a *combinations tower* (see Figure 4, next page).

In order to explain adequately the 9-Block Stalagmite model and specifically in order to emphasize the unique attribute of the stalagmite

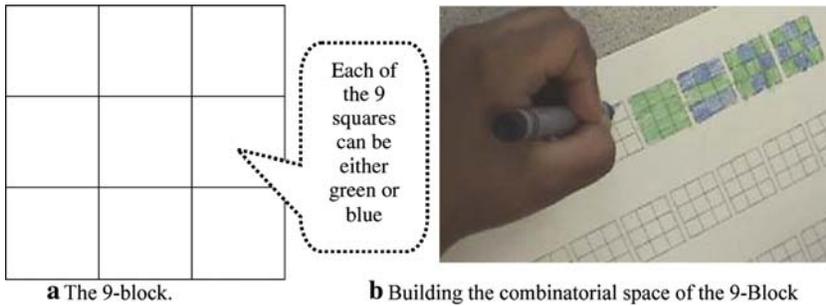


Figure 3. The 9-block: the generic form, and a classroom activity.

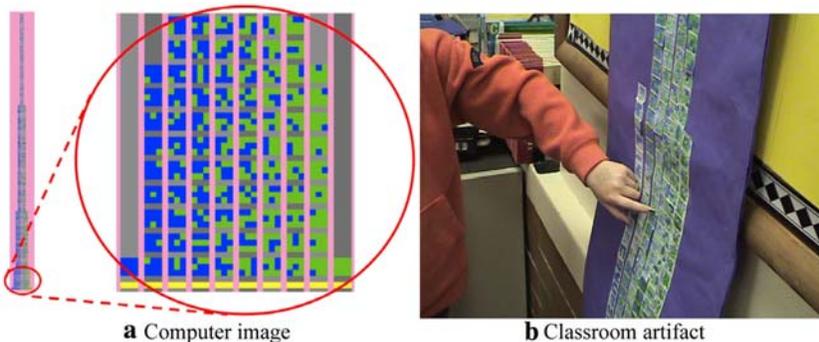


Figure 4. The *combinations tower* (on the left, the entire tower and a detail) is the combinatorial space of the 9-block. The 9-blocks are arranged in ten columns according to the number of green squares in each block. In these 10 subclasses of the combinatorial space (0 green, 1 green, ..., 9 green), there are, respectively, 1, 9, 36, 84, 126, 126, 84, 36, 9, and 1 block(s). Students create and assemble all the 9-blocks into a classroom display (see on the right).

models, I will first briefly explain a simpler model, the 9-Blocks model (not a stalagmite), that uses the same stochastic device (a 9-block).

In the *9-Blocks* model (see Figure 5a, next page), 9-blocks are generated randomly (see one sample on the top left). At each single run through the procedure, the nine squares each “choose” randomly between green and blue. In this figure, we see a sample in which 4 of the 9 squares have chosen green. The histogram records the distribution of samples according to the number of green squares in each (here, over 5000 samples have been recorded). For instance, with the selection of the current 4-green sample, Column 4 – the fifth histogram column from the left – increases by one vertical unit. An image of the combinations tower (see Figure 5b, next page, on the right)

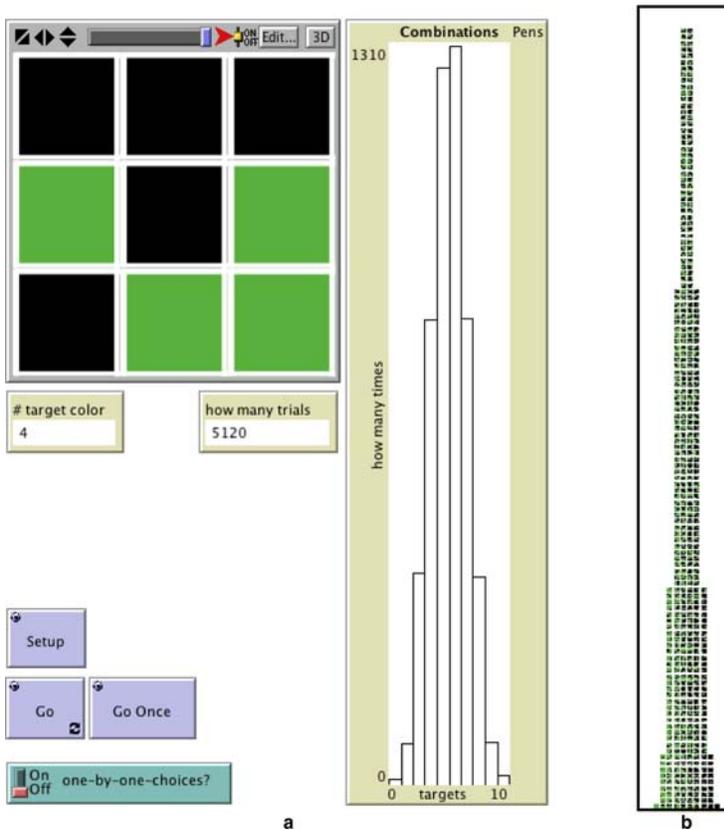


Figure 5. Learning probability through comparing structurally similar products of empirical and theoretical probability: (a) The ProbLab model 9-Blocks that simulates probability experiments; and (b) The combinations tower of all 512 different possible 9-blocks.

serves as a perceptual reference for examining the similarity between the combinatorial space (Figure 5b) and the outcome distribution (Figure 5a). As in the Dice Stalagmite experiment, this juxtaposition of combinatorial space and outcome distribution is designed to engender students' *seeing* probability by linking these constructs. Specifically, the combinations tower (Figure 5b) potentially helps students *see* why the histogram (Figure 5a) recurrently grows into its typical shape. But 'seeing chance' may be further enhanced by the stalagmite format, as we now discuss.

In the *9-Block Stalagmite* model (see Figure 6, next page), as in the 9-Blocks (non stalagmite) model, 9-blocks are created randomly. However, in this model, unlike in the 9-Blocks model yet just like in

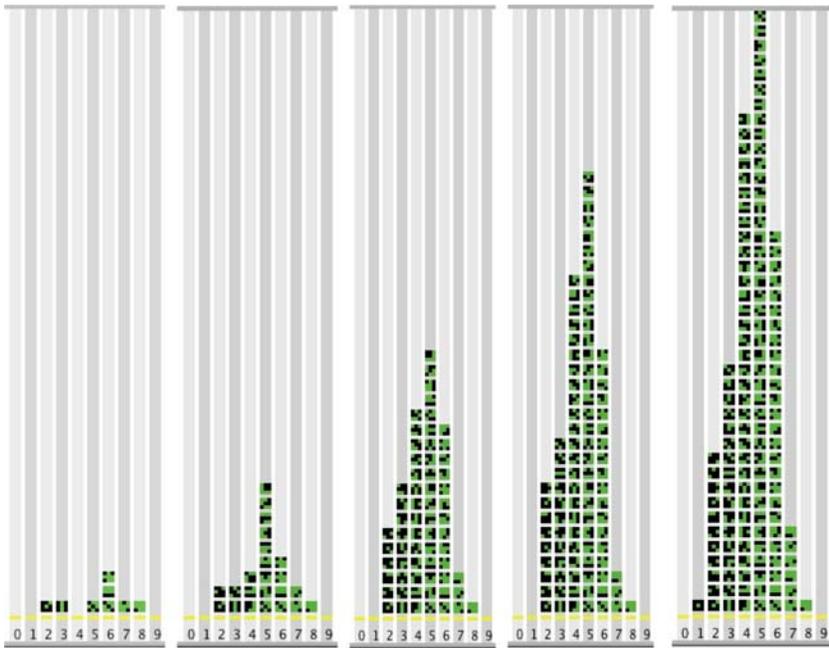


Figure 6. A detail of the ‘View’ display that is in the interface of the ProbLab model 9-Blocks Stalagmite. From left: a sequence of five screen captures during a “run” of the simulation.

the Dice Stalagmite model, the samples – the “raw data” – remain in the field of vision after they are created. That is, the samples themselves – not “stark” histogram bars – constitute the bar-chart columns that index the samples’ relative frequency. Thus, one can scrutinize the specific outcomes after the fact of generating the random samples. As I now demonstrate, such transparent outcome distributions can be readily compared to their combinatorial space in a manner that opaque histograms cannot. This comparison may engender deep conceptual understanding (Abrahamson & Cendak, 2006).<sup>3</sup>

Students may initially hypothesize that we get the typical combinations-tower shape, because the simulation is cycling deterministically through all the possible combinations for each column – “It’s simply filling them up, so obviously we should be getting the same shape.” However, scrutinizing the outcome distribution (see Figure 7, next page), students see that duplicate samples are generated “long before” the cycle has been completed (see, for example the triplet in Column 6, a column that has 84 samples “to choose from”). The question then becomes, “If *each* of the samples is generated randomly

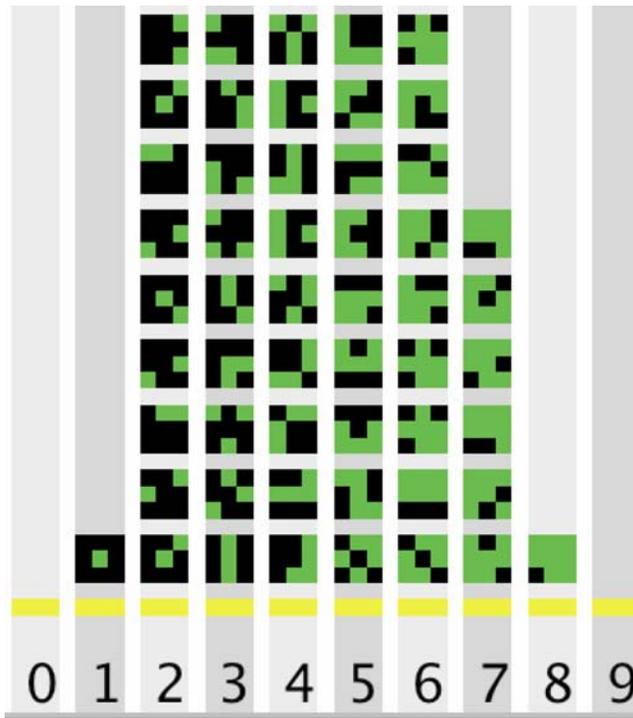


Figure 7. Detail from the ProbLab model 9-Block Stalagmite: enlargement of the base of the outcome stalagmite (compare to Figure 6, on the right). The numerals represent the number of green squares in each of the 9-blocks in the column directly above the numeral.

and independently from previous and subsequent samples, how is it that the *collection* of these random samples comes to mirror the pre-determined shape of the combinatorial space?" A version of the above math-thematic question that we have been posing to middle-school students is, "Why are Columns 4 and 5 taller than the other columns?" Students replied that there are more samples in the central columns, because "in the world" there are more samples to choose from for those columns. This 'world' is the concretized collection of elements in the combinatorial space, and this worldly ontology of 'combinatorial space' – and, moreover, the insight into the nature of probability that it enables – is articulated and further fostered through students' interaction with the computational pictograph (see Abrahamson, Janusz, & Wilensky, 2006, for a report on this study; see Abrahamson & Cendak 2006, for an explanatory model of the outcome distribution as a multiplicative-cum-stochastic transformation on the combinatorial

space – the explanatory model emerged through interviews with Grade 4–6 students who worked with the stalagmite simulations).

### 3. CONCLUDING REMARK

I have introduced the computational pictograph, a computer-based design component for reasoning through mathematical concepts. I have presented and explained the ProbLab “stalagmite” models, built in NetLogo, to demonstrate an application of the computational pictograph for the concept of probability. Finally, I suggested how the juxtaposition of a stochastic device’s combinatorial space and outcome distribution may support students’ development of deep understanding of probability.

The stalagmite computational pictograph accommodates within a single interface structure both the combinatorial space of a stochastic device and its outcome distribution. This “ambiguity,” which is specifically designed to foster insight into probability, is enabled by representing the constructs of combinatorial space and outcome distribution with shared material, semantic, and structural properties: (a) *materially*, the stalagmite assembles both constructs from the same perceptual building blocks, the raw samples themselves; (b) *semantically*, these building blocks are in both cases possible combinations produced by a stochastic device (without repeats, for the combinatorial space; with repeats, for the outcome distribution); and (c) *structurally*, the stalagmite displays the constructs as bearing the same shape. Interacting with this design, users experience the ontology of probability as a theoretical–empirical dialectic in which ‘chance’ hovers as a hybrid deterministic–random epistemological construct. Ultimately, the stalagmite simulations are designed to enable a mental handle on randomness: We can never predict with certainty what the next specific sample will be, but, overall, we can anticipate the shape of things to come.

### NOTES

<sup>1</sup> Whereas the objective of this snapshot is to introduce the principle of the computational pictograph and demonstrate an application for probability, the reader is invited to interact with existing simulations that employ this idea. See <http://www.ccl.northwestern.edu/curriculum/ProbLab/> for software, models, applets, and related publications.

<sup>2</sup> As long as the same subclasses, e.g., “sum of dice,” are chosen for both a combinatorial space and its corresponding outcome distribution, taking enough samples will result in a similarity between the profiles of the two resultant shapes. For instance, we could have arranged the compound events according to the *product* of the dice values rather than by their sum. Even a flat space, e.g., the space of a *single* die (1, 2, 3, 4, 5, and 6), will by-and-large result in a “flat” distribution of outcomes. However, classifying the compound event by the dice’s sum results in a distribution that is visually distinctive even amidst the random “bumpiness” and is thus more likely to engage students as compared to the “less interesting” case of the flat distribution. In a sense, it could be that the flat case is, in fact, more difficult to appreciate.

<sup>3</sup> This snapshot focuses on the core design of computational pictographs and does not elaborate on numerous other features of the interactive models that are designed to suggest further lines of mathematical inquiry. For instance, because the sample has 9 elements – not just 2 as in Dice Stalagmite – the distribution in 9-Block Stalagmite is closer in shape to a ‘normal’ curve. Also, in the 9-Block Stalagmite model, users can modify the chances of the squares in the sample to be green (the default chance is set at 0.5, and other chances create interesting tensions between the combinations tower and the outcome distribution). Users can also work with smaller blocks (“4-blocks”) or larger blocks (“16-blocks” and “25-blocks”). Another feature is a plot of the rate at which the model “finds” all the different blocks in the combinatorial space.

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