Rationale for a Ratio-Based Conceptualization of Slope: Results From a Design-Oriented Embodied-Cognition Domain Analysis

We report encouraging results from the Planning Phase of a co-participatory design-based research project that brings together researchers and teachers interested in incorporating educational technology into high school mathematics classrooms. The following two ideas, which were pivotal to aligning our perspectives, co-emerged in our discourse following curricular reviews and cognitive domain analyses that specifically investigated the potential role of proportional reasoning in learning algebra content: (a) product-based conceptualization of proportional equivalence is pedagogically advantageous over process-based conceptualization; and (b) ratio-based conceptualization of slope is pedagogically advantageous over rate-based conceptualization. We detail findings from our collaborative reflection process and outline principles for the Design Phase, during which our embodied-interaction systems will be further developed to incorporate a product-based conceptualization of slope.

1. Objectives

This theoretical paper stems from a sequence of participatory-design sessions that involved mathematics-education researchers and high-school teachers collaborating to effect change by bridging theory to practice (Engestrom, 2008). We are jointly planning a classroom study that would scale up and further develop an earlier tutorial intervention (Brown, 1992; Lamberg & Middleton, 2009). Specifically, our goal is to interleave experimental proportions activities (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011) into an existing algebra curriculum (Moses, Kamii, Swap, & Howard, 1989). Entering this researcher–teacher dialogue, the researchers hoped to gain access to authentic research sites, where they could implement their design vis-à-vis the complexity of professional practice, whereas the teachers believed in the potential of reconceptualizing the domains of proportion and slope as well as the particular embodiment proposed for these notions.

Here we lay out tentative findings from the preparatory collaboration, essentially reporting: (a) our cognitive domain analyses for the content of proportions and slope; (b) results from curriculum reviews of these concepts; and (c) implications for the design of new proportions-based activities for algebra curricula. We maintain that although we have not yet collected empirical data, this report should already stand as a contribution to rethinking algebra.

In its broadest sense, our quest for design that makes algebra accessible to all students resonates with the conference Call. Namely, by collaborating with teachers who are deeply vested and involved in the Algebra Project, we aspire to offer “theoretical analyses as well as research-based arguments...” for “educational policies and practices [that] might reduce poverty” via a commitment to “investigate why educational policies and practices often fail to address poverty” (AERA website).
2. Theoretical Perspectives

An accepted general rationale for the instruction of proportion in middle school is that students need to coordinate across two progressions of value sequences. As such, an understanding of $1:2 = 2:4 = 3:6 = \ldots$ is grounded in pairing values from the progressions 1-2-3-… and 2-4-6… For example, Piaget, Grize, Szeminska, and Bang (1968) state that, “Proportionality cannot be understood until there has been a …. coordination of two laws of progression corresponding to two functionally linked variables” (p. 137).

 Whereas we concur with Piaget et al. (1968), we wish in this paper to promote a position that qualifies their rationale as treating only one of two alternative conceptualizations of proportion. We will call their conceptualization of proportion process based, because it highlights coordinated transformation actions from one ratio to the next, that is, the co-iteration of composite units, for example, 1 and 2, respectively, by which each subsequent ratio unfolds. Espousing a phenomenological epistemology that draws on a grounded-cognition theory of learning (Hall & Nemirovsky, 2012), we wish to supplement with a product based conceptualization of ratio, in which equivalent ratios are experienced as sensuously identical, for example, the consistent hue of green created by a 1:2 mixture of blue and yellow as compared to a 2:4 mixture. Both conceptualizations are vital for learning, yet process-based proportion should be grounded in experiences of product-based proportion.

The proposed epistemological differentiation of process-versus-product conceptualizations of ratio, we believe, bears far-reaching repercussions for both the theorization and practice of mathematics education from middle school and beyond. More broadly, we submit that the grounded-cognition perspective on mathematics learning has much to offer in addressing issues of equity and social justice, because the perspective implicates critical lacunae in extant algebra curriculum, due to which designers’ best intentions never become realized. As Johnson (1999) states:

No matter how sophisticated our abstractions become, if they are to be meaningful to us, they must retain their intimate ties to our embodied modes of conceptualization and reasoning. We can only experience what our embodiment allows us to experience. We can only conceptualize using conceptual systems grounded in our bodily experience. And we can only reason by means of our embodied, imaginative rationality. (p. 81)

More poignantly, Dreyfus and Dreyfus (1999) warn that, “Until cognitive scientists recognize this essential role of the body, their work will remain a mixed bag of ad hoc successes and, to them, incomprehensible failures” (p. 118).

3. Modes of Inquiry

As design-based researchers, we are learning scientists whose design practice constitutes a context for developing theoretical models (e.g., Sandoval & Bell, 2004). As such, our technological products are intertwined with our theoretical perspectives (Collins, Joseph, & Bielaczyc, 2004). The following introduction of our earlier design (Abrahamson et al., 2011) will therefore both extend our theoretical discussion and anticipate our proposed design for slope.

The subject matter of ratio and proportion is didactically essential, because it underlies high-school STEM content and professional scientific reasoning. However, many students in elementary school and beyond experience difficulty in understanding and using the core notions
of proportionality. In particular, students incur difficulty in developing fluency with proportions that builds upon, yet is differentiated from, simpler non-multiplicative notions, notations, nomenclature, and procedures (Karplus, Pulos, & Stage, 1983; Lamon, 2007).

We approached this design problem by analyzing the target content from an embodied-cognition perspective that implicates human reasoning as grounded in traces from spatio–dynamical experiences (Barsalou, 2010). An appeal of this epistemological position for designers of reform-oriented mathematics education is in its categorical implication of mundane interaction as furnishing personal resources for learning and reasoning. Indeed, the theory resonates with tenets of genetic epistemology, and in particular the constructivist thesis that conceptual activity is embodied in perceptuomotor schemas, as captured in the statement, “Mathematics uses operations and transformations…which are still actions although they are carried out mentally” (Piaget, 1971, p. 6). The grounded-cognition hypothesis further resonates with the growing consensus that mathematics is a situated, multimodal, multi-media, and multi-semiotic praxis (Bamberger, 1999; Bautista & Roth, 2012; Lemke, 1998, 2003; Nemirovsky, 2003; Núñez, Edwards, & Matos, 1999; Radford, 2002; Rotman, 2000; Skemp, 1983).

Designers have historically sought to augment on mundane interactions so as foster new resources for learning content (Diénès, 1971; Froebel, 2005; Kamii & DeClark, 1985). From a grounded-cognition perspective, we conjectured specifically that some mathematical concepts are difficult to learn because everyday experiences do not occasion opportunities to embody and rehearse perceptuomotor schemas underlying those concepts. In particular, we conjectured that students’ canonically incorrect solutions for rational-number problems—“additive” solutions (e.g., "2:3 = 4:5" or "2/3 = 4/5" - cf. Behr, Harel, Post, & Lesh, 1993)—indicate insufficient kinesthetic–visual action images to ground proportion-related concepts (cf. Fischer, Moeller, Bientzle, Cress, & Nuerk, 2011; Goldin, 1987; Pirie & Kieren, 1994). We thus sought to ‘phenomenalize’ the concept of proportion (cf. Pratt, Jones, & Prodromou, 2006) in the form of an interactive device that would afford learners opportunities to develop and generalize its principles.

We thus engineered a computer-supported inquiry activity for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation (Abrahamson et al., 2011). The Mathematical Imagery Trainer (“MIT”) is a technological design engineered to support deep learning of mathematical content, such as proportions (see Figure 1).

![Figure 1](image-url)
In Abrahamson, Lee, Negrete, & Gutiérrez (in press), the researchers among us demonstrated the pedagogical potential of having students ground a process-based solution procedure for the MIT-P in product-based sensory experiences of proportional equivalence. Namely, students working on the MIT-P discovered the solution strategy of co-iterating composite units in their left- and right-hand gestures, respectively, as a means of maintaining a green screen. This rationale “struck home” with the teachers among us, as follows.

To begin with, they as well as their colleague teachers have long believed that the concept of slope is pivotal to the entire domain of Algebra. Yet their gut feeling has been that the standard introduction of slope does not enable students to comprehend what exactly it is about slope that remains constant throughout the extent of the linear function. Over the months of our collaboration, it gradually emerged to us that the key to the new design would be in somehow leveraging the product-vs.-process rationale for proportion so as to create responses to the absence of meaning in students’ introduction to slope.

To begin with, we wished to corroborate our hunches about extant curricular content. We thus reviewed common texts for proportion and, separately, for slope. In order better to articulate the potential utility of adapting the MIT-Proportion into an MIT-Slope, we also reviewed several technological designs for algebra. It was hoped that through discussing findings from these curricular reviews vis-à-vis our earlier design and the teachers’ insights, we would come to hone continuities from our proportion work to algebra instruction.

4. Findings

In the following, we first offer results from our curriculum analysis and then, based on inferences from this analysis, we elaborate on our proposal for an alternative approach to slope.
Models of Proportional Equivalence

<table>
<thead>
<tr>
<th>Model</th>
<th>Prerequisites for Learning the Model</th>
<th>Description</th>
<th>Example</th>
<th>Evaluation</th>
</tr>
</thead>
</table>
| Scaled Ratios | $\frac{a}{b} = \frac{na}{nb}$, where $n$ is a real number | This common textbook approach poses finding the scale factor of both the numerator and the denominator; if the ratios are equal, proportionality is assumed. | Determine whether the ratios 4/5 and 24/30 are equivalent.  
\[
\begin{align*}
4 &= 4 \\
5 &= 5 \\
24 &= 24 \\
30 &= 30
\end{align*}
\]
\[\frac{4}{5} = \frac{24}{30} = \frac{4}{5}
\]
The ratios are equivalent, therefore they form a proportion.  
\[
\frac{4}{5} = \frac{24}{30}
\]
Emphasizes the process of scaling by $n$ to obtain equivalent ratios. |
| Equation of the form | $\frac{a}{b} = \frac{c}{d}$ | The $\frac{a}{b} = \frac{c}{d}$ equation model, is a definition which allows for solving one unknown quantity through using "cross multiplication," ie. $ad = bc$. | extreme | mean  
\[
\begin{align*}
4 &= 24 \\
5 &= 30
\end{align*}
\]
\[\frac{4}{5} = \frac{24}{30}
\]
Assumes equivalence of ratios in order to solve for values using the process of cross multiplication. |
<table>
<thead>
<tr>
<th>Multiplication Table</th>
<th>Multiplication Table</th>
<th>Through inherent symmetry in the design, the multiplication table allows students to conceptualize proportion in a familiar context.</th>
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<tbody>
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<td>(Abrahamson &amp; Cigan, 2003)</td>
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Grounds proportion in familiar artifacts but does not allow for proportion to be experienced as a perceptual sensation preserving equivalence.

Students start with a Multiplication Table (pictured above) and then move to a Rate Table, then to a Ratio Table, and ultimately to a mini-multiplication-table puzzles.
### Models of Slope

<table>
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<td>Rate of Change</td>
<td>• Ordered Pairs</td>
<td>The Rate of Change model, also known as the $\frac{\text{rise}}{\text{run}}$ model, in which slope is defined as a ratio of the change ($\Delta$) of $y$ values over $x$ values.</td>
<td><img src="image" alt="Rate of Change Example" /></td>
<td>Focuses on the differences in respective consecutive entries for each available. Allows for pattern recognition through repeated addition.</td>
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<td>• Cartesian Coordinate Plane</td>
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<td>• Two-Column Table</td>
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<td>Slope Triangle m = $\frac{y_2-y_1}{x_2-x_1}$</td>
<td>• Ordered Pairs</td>
<td>The Slope Triangle model relies on the geometric representation of a linear function on the Cartesian coordinate plane. To solve for $m$, students must find two points, $(x_1, y_1)$ and $(x_2, y_2)$, to substitute into the given equation $m = \frac{y_2-y_1}{x_2-x_1}$.</td>
<td><img src="image" alt="Slope Triangle Example" /></td>
<td>This model, similar to the “cross multiply” model of proportional equivalence, is a consistent formulaic approach to solving for slope of a line.</td>
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<td>• Cartesian Coordinate Plane</td>
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<td>Embodied “Position Over Time” SMALLab (Johnson-Glenberg et al., 2009)</td>
<td>• Cartesian Coordinate Plane • Embodied Technology</td>
<td>This model of learning slope and linear functions uses multiple representations in mixed-reality technology. By utilizing 3-dimensional tracking technology, the activity records student position in respect to time plotted on a Cartesian coordinate plane as they move along a number line.</td>
<td><img src="image" alt="Graph showing position over time" /></td>
<td>This model emphasizes slope defined as rate through tracking individual positions over time. The mixed-reality setting grounds the concept of slope in embodied action through graphical representation.</td>
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</tbody>
</table>
5. Conclusions

In agreement with our assumption, the curricula review demonstrated that current curricula: (a) introduce proportion as a process-, not product-based relation; and (b) introduce slope as a rate-, not ratio-based concept. As we move to create our review-and-enrichment proportions gateway activities for algebra, we seek a design that builds on our product-based grounding of proportions to offer a ratio-based entry to slope.

Technology appears to play a promising role in implementing the above conjecture in the form of actual curricular materials, because “concrete” virtual objects afford embodied experiences that traditional text cannot (Sarama & Clements, 2009). In turn, the increasing ubiquity of mobile technology and remote sensing devices suggests that our proposed design should keep with the Algebra Project tenets of broad accessibility.

Figure 2. Prototype sketch of the Mathematical Imagery Trainer for Slope (MIT-S). The student raises and lowers the hands to adjust the x and y values. The meandering line is the student’s attempt to trace along the target straight line. The interactive solution to this $y = x$ function is to raise both hands at the same speed.

Our future design will thus build on our Mathematical Imagery Trainer work and take the form of an embodied-interaction activity, in which students discover coordinated “ratio gestures” that inscribe linear-function graph lines (see Figure 2). We eagerly anticipate the imminent implementation and research of this study in the teachers’ classrooms.

References


