Running Head: MATHEMATICAL ARTIFACTS

DRAFT: March 26, 2008


Toward a Phenomenology of Mathematical Artifacts:
A Circumspective Deconstruction of a Design for the Binomial

Dor Abrahamson, Michael J. Bryant, Mark L. Howison, & Josephine J. Relaford-Doyle

Embodied Design Research Laboratory,
Graduate School of Education, University of California, Berkeley.

Graduate School of Education
4649 Tolman Hall
University of California, Berkeley
Berkeley, CA 94720-1670
USA
TEL: +1 510 525 3761
FAX: +1 510 642 3769
e-mail: dor@berkeley.edu

keywords:
mathematical cognition, design-based research, social context, theory of learning, probability, binomial, computer simulations, epistemic forms, conceptual blend, instrumental genesis, visual attention, ambiguous figures, heuristics, professional vision, disciplined perception
Abstract

We demonstrate the potential of an innovative design-based research methodology—
manipulation of normative features of didactic routines—to illuminate and support the
integration of constructivist and socio-cultural theoretical models pertaining to the roles of
epistemic and material resources at play in students’ mediated development of mathematical
concepts. We present three cases of individual middle-school students who participated in a
study of probabilistic cognition, where the design was for students to coordinate their intuition
for the outcome of an experiment with a stochastic device with its sample space that they build
through combinatorial analysis. The activity sequencing was non-normative in that only having
completed the combinatorial analysis did students recognize its pertinence to their grounding
intuition, and that recognition is the focus of our analysis. By uncovering students’ heuristics as
well as the pragmatics of the student–interviewer dyad as they communicate over properties of
the learning materials and negotiate activities embedded in the design, we implicate a major
challenge of struggling students to be their underdeveloped *epistemic disposition*—they do not
expect to be able to “solve the world” and do not skillfully use mathematical representations let
alone trust them to bear on their inquiry. To grow into STEM content, these students require
early nurturing into problem-solving socio–cognitive norms.
Toward a Phenomenology of Mathematical Artifacts:

A Circumspective Deconstruction of a Design for the Binomial

“[If someone says to you] ‘I struggled but still did not discover,’ do not believe him” [Talmud, Megila 6b], because the struggle in and of itself is a great discovery, a great find indeed.”

*Rabbi Menachem Mendel of Kotzk* (1787 – 1859)

“[T]he ultimate truth of the puzzle: in spite of appearances, it is not a solitary play: each move made by the puzzle solver, the puzzle maker made before him; each piece that he takes and takes again, that he examines, that he caresses, each combination that he tries and tries again, each blind groping, each intuition, each hope, each discouragement, were decided, calculated, studied by the other.”


Design studies constitute microworlds on learning as much as for it (A. Brown, 1992; Collins, 1992). Indeed, design studies have contributed to the growth of education research (e.g., Cobb & Bauersfeld, 1995). This empirical paper proposes a new type of contribution of the design-based research paradigm to education theory. Specifically, we examine the potential of design studies to illuminate and integrate constructivist and socio–cultural theoretical models pertaining to the roles of epistemic and material resources in the mediated development of mathematical cognition. Our ambitious objective is to contribute to the emergence of a new trans-disciplinary field of research and practice, *design theory*, that is based on a phenomenological analysis of mathematical artifacts.
Artifacts constitute the key construct of an integrated learning-and-design theoretical model, because they are both the focus of much socio–cultural theorizing and the product of design. Ideally, an integrated design theory would at once articulate how individual students develop conceptual understanding through guided interaction with learning materials and constitute a template for creating, and administering, and researching student learning with such materials.

We have selected constructivist design frameworks as the focus of our research and theorizing. We chose these frameworks, because they plausibly resonate with cognitive and socio–cultural processes inherent to naturalistic learning, for example through their focus on student generation and interaction with artifacts, and are therefore potentially conducive to illuminating these theories. Therefore, the focal activities we will be discussing were designed in accord with tenets of constructivist philosophy.

Below, we clarify the rationale of this study, discuss the pedagogical philosophy underlying the focal design, introduce this design, and then use the design to explain the theoretical resources we will be using in the data analysis. The introduction ends with a set of research questions. Following a methodology section, we present three case studies of students participating in a sequence of activities designed to ground conceptual understanding of the binomial in intuitions of chance. The paper concludes with implications of these analyses for learning theory.

*Study Rationale: Design-Based Research in Service of Cognitive and Socio–Cultural Theory Building*

This paper is on relations between methodology and theory in the social sciences, specifically design-based research methodology and cognitive and socio–cultural theory. In
particular, the paper is on relations between, on the one hand, methodology of design-based research on mathematical cognition and, on the other hand, cognitive and socio–cultural theory focusing on artifact-mediated teaching, reasoning, and learning. These relations between methodology and theory will be discussed in the context of empirical data coming from a design-based research study on middle-school students’ intuitions pertaining to the mathematical domain of probability.

Design-based research studies typically have two “deliverables,” theory and design. Namely, these studies are conducted both to advance theory of cognitive and social processes inherent to the phenomenon of learning and to create and improve learning environments, including objects, activities, and facilitation practice, that could potentially advance the learning of students at large. Usually, design-based research experiments proceed as simulated teaching-and-learning interactions, e.g., a classroom study or an interview. Because the design is being tested, it is facilitated as naturalistically as possible. By way of analogy, to develop an innovative aircraft one builds and test-flies a prototype, not an anomaly. Granted, an anomaly could be potentially revealing of aerodynamics theory, but it would never actually be used in practice. That is, just as cognitive psychologists study human perception using ambiguous figures that juxtapose distal stimuli and their perceptual construction, so designers could study mathematical learning by contriving pedagogical situations structurally rare in naturalistic development. Yet design-based researchers’ dual commitment to theory and practice compels the creation of situations that are foremost pedagogically effective as well as intellectually revealing. That said, it could be that an aircraft is designed according to a principled rationale yet, in retrospect, the structure of the emergent aircraft appears to violate normative guidelines, e.g., its wings are oriented forward rather than backward (e.g., the Russian Su-47 Bekrut). In such a case, if the
aircraft actually flies, such flight may call in question implicit aspects of the accepted theoretical models, i.e., that wings should be oriented backwards. At the very least, those empirical findings may help hone and develop existing theoretical constructs.

Such is the case of the current study, in which a violation of normative design was initially created to enhance learning, not theory, yet in retrospect it engendered situations suited for the study of theory of learning. That is, the instructional design was tailored to the facilitation of content (probability) as framed by a set of design principles (Abrahamson & Wilensky, 2007), and the insight that this design is non-normative and the realization that it may therefore advance theory building came post-hoc, serendipitously. We argue in this paper that design-based research studies are well suited for advancing theory of learning, when aspects of the design happen to differ from naturalistic patterns that are the province of these theories.

In a sense, this is a paper within a paper—a focal design-based research study on students’ probabilistic cognition is nested within meta-methodological discussion of design-based research and its contribution to building integrated models of individual cognition in social context. Yet the structural nesting of the paper is not perfect, because the meta-methodological questions emerged through the methodological work. In order to contextualize our research questions pertaining to the meta-methodological thesis, we will begin, below, by explaining the design philosophy, the design itself, and the cognitive and socio-cultural theoretical resources we integrated in our analysis of data from the implementation of the design. Once the paper dives into results from the focal study, wherein we demonstrate the application of the integrated model to three case studies, we will emerge only at the Conclusions section to explain how the focal study demonstrated the meta-methodological argument, and the Implications section elaborates thereof on prospects of its further development.
Design Philosophy

The interpretation of constructivist pedagogical philosophy underlying the design discussed below is that students learn best given opportunities to reinvent mathematical concepts as problem-solving, construction, and communication tools that draw on their naturalistic experiences (e.g., Freudenthal, 1986). Counter to some prevalent misunderstanding of these frameworks, the didactic objective of discovery-based learning, if to paraphrase von Glasersfeld (1992), is not that students reinvent the wheel from scratch but reassemble it from a kit, and only if a wheel constitutes the right tool for the job at hand. Thus, the designer circumspectly selects and creates mathematical objects, activities, and facilitation to engineer students’ content-targeted personal discovery processes. For example, participant students engage in a designed activity sequence that initially elicits their intuitive judgments and then encourages them to appropriate available mathematical artifacts—objects, procedures, and argumentation forms—to warrant or contest their claims (e.g., Abrahamson & Wilensky, 2007). It is thus intended that students learn curricular content by synthesizing intuitive knowledge and professional praxis (Case & Okamoto, 1996; Heidegger, 1962; Papert, 2000; Piaget & Inhelder, 1952; Schön, 1987; Vygotsky, 1978/1930).

Whereas discovery-based design is pedagogically appealing, we argue that some historical discovery processes are difficult to reenact pedagogically as classroom-based incremental trajectories, because—not unlike the evolution of the eye—they transpired historically as desultory coordinations among fortuitously available resources. Designing for historical reenactment is further exacerbated by specific disciplinary domains of content that are inherently empirical—inductive rather than following a sequence of axiomatic determinisms. In particular, the epistemological grounds of probability theory are tenuous relative to, say,
arithmetic, because probability theory was created to provide mathematical handles on the uncertainty of natural phenomena. This unique dichotomous composition of probability content as theoretical tools for predicting empirical phenomena creates fertile grounds for studying issues of mathematical reasoning. In particular, the distinct mathematical activities pertaining to theoretical and empirical probability—conducting combinatorial analysis vs. operating stochastic devices—may, under certain design structures, enable stark methodological decoupling and phenomenal isolation of formal and intuitive resources at play in mathematics learning as well as analysis of how these resources may be related through learning.

Following, we explain the design for probability used in this study. Then, we cite our theoretical resources through the lens of the design.

Design

*A Design for the Binomial: Wax On, Wax Off*

This paper discusses empirical data from studies in which we implemented a contradistinction of theoretical and empirical activities in a design for the binomial (see Jones, Langrall, & Mooney, 2007, for the rationale of teaching probability as a complementarity of "theoretical" and "empirical" activities). The design was for students first to intuit results of operating a physically available stochastic device, then conduct combinatorial analysis of this device, and only finally consider the relations between these two activities. That is, while students engage in combinatorial analysis, they do not necessarily appreciate that the activity is instrumental to determining expected outcome distribution in actual experimentation with the device. We thus say that students experience *suspension of pertinence*. In a sense, students were to learn to use an instrument before discovering its province of application. By way of a possibly illuminating analogy, we have nicknamed this experience the “wax on, wax off” phenomenon in
homage of the fabled cinematographic Karate Kid who did not realize he was embodying
martial-arts maneuvers while endlessly varnishing his Sensei’s car, yet had sudden insight into
the nature of the exercise when he surprised himself by dexterously blocking a punch.

In the grand scheme of the succession of design-based research studies leading up to the
focal study, the suspension of pertinence was an inadvertent design feature. This feature resulted
as a design response to students’ initial obliviousness to the relevance of permutations to
computation of chance. That is, even as they had robust intuitions for the long-term properties of
operating the stochastic device, students did not see how the activity of determining all the
possible permutations could possibly be informing of this issue. In Abrahamson (2007a), this
behavior is called “counter-intuitive reasoning,” because students focus only on the count of
favorable outcomes in a compound event (e.g., 3 Heads in a 4 coin toss), disregarding
implications of ordering (e.g., which explain why there are more ways of “getting” 2 Heads than
3). The constructivist design paradox was, thus, that students’ engagement in combinatorial
analysis could become intellectually honest only upon its completion. Indeed, all through the
activity of combinatorial analysis, students negotiated with the facilitator whether or not they
should create the permutations. This negotiation was obviated the moment students experienced
the critical insight that the sample space constitutes cogent material for developing a logical
warrant for their initial intuition.

How are we to evaluate a design wherein students engage in an activity that does not
initially appear to pertain to any problem they are attempting to solve? Granted, much of
traditional instruction positions students as passive learners of routines, e.g., solving algebraic
equations as an activity onto itself. However, we are evaluating the focal activity sequence in
light of constructivist design, wherein students “discover” targeted mathematical procedures as
problem-solving tools. It is beyond the scope of this paper to examine how design principles interact with particular content domains or to ask whether \textit{a posteriori} discovery of the pertinence of a procedure is “as constructivist” as \textit{a priori} discovery of the procedure itself. Thus, the current paper will not discuss further the design rationale or alternative designs but only its serendipitous implications for the study of learning theory.

\[\text{[insert Figures 1 and 2 about here]}\]

In the first activity of the focal design, middle-school students interacted with a box full of green and blue marbles of equal numbers from which they scooped samples of four (see Figure 1). Asked to estimate frequency distributions in projected experiments with this device, students typically indicated the ‘half–half’ green-to-blue marbles ratio in the box to implicate a ‘2 green, 2 blue’ (hence, ‘2g2b’) scoop as the most common sample and the 4g or 4b scoop as the rarest (see Figure 2). (Figure 2 is meant as a schematic representation only—neither the within-circle arrangements of the sample space nor the arrows between the circles are intended to capture the actual physical layout of the cards or students’ gestures or apparent spatial ordering of the outcomes.) During this first activity, no student suggested any form of combinatorial analysis as a means of warranting this claim.

\[\text{[insert Figure 3 about here]}\]

In the second activity, students were guided to construct the sample space of the experiment. While engaged in combinatorial analysis, students did not appear aware of its relevance to anticipating empirical distribution. Yet, once the sample space was completed and assembled in a \textit{combinations tower} (see Figure 3), i.e., by number of favorable outcomes in the compound event, students typically appropriated the sample space as a means of warranting their earlier intuitive judgment. Namely, once students had completed creating the sample space,
assigning the 16 outcomes to five categories (0 – 4) by number of green squares, and stacking these groups equi-spaced along vertical columns of common base, the students noted that some categories consisted of more outcomes than other categories and so, given the half–half color distribution of marbles in the box ($p = .5$), categories with more items would be more likely to occur randomly.

In a third activity, which we will not discuss further in this paper, students operate a set of computer-based simulations of the 4-block marbles experiment. Analyzing the dynamically emergent outcome distributions in these experiments creates opportunities for students to validate their intuitive-cum-analytic expectation of the binomial distribution.

Having introduced the design, we will now use it as a focus for presenting the theoretical resources we apply in our analysis of the implementation of this design.

**Theoretical Resources Through the Lens of the Pedagogical Design**

Striving to study students’ phenomenology of mathematical artifacts, we bring to bear and coordinate two sets of theoretical models roughly aligned with research on individual and social cognition, respectively: (a) a cognitive-science focus on sense making, problem solving, and cognitive artifacts (Arnheim, 1969; Collins & Ferguson, 1993; Hutchins, 2005; Stavy & Tirosh, 1996; Tsal & Kolbet, 1985); and (b) a socio-cultural and linguistics focus illuminating mechanisms of facilitation and interaction pragmatics (Goodwin, 1994; Grice, 1989; Schegloff, 1996; Stevens & Hall, 1998; Vérillon & Rabardel, 1995). Following, we exemplify these perspectives through the lens of the design, and then we will offer an integration of the resources which we will subsequently utilize in analyzing three cases studies.

The design analysis will focus on the combinations-tower artifact—the mathematical content it embodies, its constituency as a diagrammatic expression, and its learnability, including
prospective challenges elucidated by the theoretical constructs. We selected a single artifact out of the array of design material so as to focus the discussion, and we selected the combinations tower as this pivotal artifact, because it is an innovative learning tool, because it is a complicated representation that students construct laboriously, and because it ultimately affords the critical insight into the mathematical content embedded into the activity sequence. As we shall demonstrate, underlying the physical construction of the artifact is a variability in students’ mental constructions of this artifact, and unpacking this variability is a core objective of our analyses. Understanding students’ mental constructions will enable us to elucidate the interviewer’s actions as guiding each student to attend to particular properties of the combinations tower and coordinating these properties with aspects of the marble-scooping experiment.

Below, a brief discussion of the Heideggerian construct of ‘breakdown’ will launch a succession of theoretical perspectives that we apply in interpreting data of students’ guided interactions with the designed material. We will use the individual-cognition theoretical resources to speculate on students’ perceptual–cognitive activity in instrumentalizing the combinations tower, in particular in light of students’ initial grounding intuition that 2-green is the mode outcome. We will use the socio–cultural resources to speculate on the real-time support required for students to successfully instrumentalize the combinations tower, i.e., the interviewer’s role within the learning dyad. That is, the individual-cognition resources will address how the combinations tower can be seen and which ways of seeing it would be conducive to learning the targeted concept in alignment with the designer’s intention, and the socio–cultural resources will address how the interviewer steers the students toward these views of the combinations tower that are critical for the desired learning goals. In considering the
cognitive and socio-cultural processes engendering students’ insight, the combinations tower is an arena for examining the complementarity of these theoretical perspectives.

*Dasein-Based Research Methodology: Reversing the ‘Ready-to-Hand to Present-at-Hand’*

*Breakdown-Based Natural Learning Process*

**Introduction.** The seminal work of Martin Heidegger, *Being and Time* (1962) undergirds our phenomenological deconstruction of the design for the binomial (see Koschmann, Kuuti, & Hickman, 1998, for a comparative survey of the construct of 'breakdown' and its implications for education). Heidegger takes signs, such as an arrow, to be part of the world's equipmentality. Once routinized, a sign becomes invisible to absorbed circumspection and announces itself afresh only through disruption to these routines (i.e., through breakdown). Famously, Heidegger enumerates three mechanisms of breakdown: when equipment is either faulty, missing, or obtrusive. Below, we suggest a fourth breakdown.

**Application.** The sample space of an experiment with a stochastic generator may signify the experiment’s expected outcome distribution. Yet, we submit, this indicative nature of the sample space may be revealed to students only after they have constructed the space, when its significance leaps out as a referring-to lens toward the marble-scooping experiment. This non-normative sequencing, grounded in a cognitive–pedagogical design rationale, creates a study condition that breaks down ‘breakdown,’ which thus announces itself afresh as an object of scrutiny for education theoreticians developing design theory.

Note that reversing the Heideggerian learning process—a serendipitous aspect of the current design—goes counter to design frameworks that set students up to *expectation failure* (e.g., Schank, 1996), through which students are to become conscious of their implicit reasoning.

*Epistemic Forms and Epistemic Games*
**Introduction.** Collins and Ferguson (1993) introduce the construct *epistemic form* to capture and characterize the roles that various cognitive–inscriptional structures play in guiding scientific inquiry, e.g., a list, a table, or a system-dynamics model. These forms, once selected as advancing the modeling of the phenomenon under inquiry, organize the inquiry process by setting structural constraints; they thus guide *epistemic games*. This complementary construct, epistemic game, describes the activities of “filling out” the epistemic forms, i.e., following rules and using strategies to complete a form. For example, once the periodic table was created, the empty slots set constraints on the search for unknown elements.

[Insert Figure 4 about here]

**Application.** In our study, the target epistemic form is a distribution, specifically the expected outcome distribution of the marbles-box randomness generator. The epistemic game of filling in the distribution is the mathematical procedure of combinatorial analysis. Uniquely, in this particular design, students initially have a sense of distribution but not the form, so they cannot formulate their sense and do not know what epistemic game to play. Thus, students perform the activity of the epistemic game without initially recognizing the nature of the target form. Yet once completed, the product of the epistemic game—the distributed sample space—is spontaneously appropriated as the target form because it is recognized as encoding the initial sense (see Figure 4 for two epistemic games at play in the design).

**Professional Vision**

**Introduction.** Experienced practitioners, e.g., radiologists, chess players, or orchestral conductors, perceive their domains of scrutiny differently from novices, a capacity Goodwin (1994) calls *professional vision*. Within Goodwin’s framework, the act of seeing is not a transparent, private process but rather a socially situated activity that involves three discursive
practices: coding, highlighting, and producing and articulating material representations. Coding is the practice through which the complexity of the perceptual field is transformed and categorized into documented knowledge. Yet, in order to code, certain aspects of the perceptual field must be made salient; this is the role of highlighting, the practice of focusing on the information within the perceptual field that is determined to be most relevant, creating a figure and ground. Lastly, material representations are often constructed by professionals to complement spoken language.

Application. The marble-scooping experiment, the sample space, and the combinations tower each become a domain of scrutiny onto its own. However, the design is for students to construct the mathematical content by token of grasping how these artifacts are related. Whereas the marbles and scooper are material givens, the sample space requires construction, and then the combinations tower requires assembly. The activity of creating the sample space requires attention to the orientation of the cards and the pattern of colors. These properties are negotiated as relevant to the activity, whereas other attributes are rejected as superfluous. For example, it is not important whether the squares are completely colored in. The activity of formulating the sample space into the combinations tower requires highlighting of the 16 outcomes’ specific dimension “number of green squares.” The participant then codes the outcomes by counting the number of green squares on each card and records this code by grouping the cards in columns arranged along an ordinal sequence—an emergent, reflexive number line, in which the columns constitute both the “tickmarks” and the content. The columns, in turn, highlight the categories’ uniqueness, and their respective heights index the likelihoods of each category (for \( p = .5 \)).

We will inspect the data to establish whether and how students code the distributed sample space of the marbles-scooping experiment. That is, we will examine whether students
perceive the combinations tower as illuminating aspects of the experiment beyond “what we can get when we scoop from the box,” to mean also “how often we will get each combination.”

**Disciplined Perception**

*Introduction.* Stevens and Hall’s (1998) *disciplined perception* framework is concerned with agents’ negotiation of *orientations* toward *views*, i.e., ways of attending to material artifacts, including forms of inscription used by a particular field of technoscience, e.g., a histogram. Orientations to two or more views are coordinated in building fluency with the artifacts of a professional practice. Such orienting and coordinating is greatly facilitated by a professional who disciplines the novice’s perception through asymmetrical negotiation, where the professional’s orientations are tacitly taken to be the normative, target ways of perceiving. As in Goodwin’s framework, perception of a domain of scrutiny, including situations and professional instruments, is contextually sensitive as well as socially constructed through multimodal communication, primarily utterance and gesture.

*Application.* The guided construction and interpretation of the combinations tower requires disciplining students’ perceptions. Viewed through appropriate orientation—i.e., viewing the columns as indexing probabilities of marble-scoop outcomes—the combinations tower provides a “well-ordered medium” (Stevens & Hall, 1998). Through verbal and gestural cues, the interviewer guides the participant toward this visual orientation. In some cases, the interviewer’s interventions cause the participant “major epistemic disruption” (p. 122); during such breakdown, the participant may reorient her perception of the combinations tower, aligning it with the interviewer’s. Thus, the interviewer disciplines the participant’s perception.

*Instrumental Genesis*
Introduction. Verillon and Rabardel (1995) propose the Instrumented Activity Situations (IAS) model of instrumental genesis that elaborates on Vygotsky’s theory of conceptual learning as the appropriation of artifact-mediated routines. The authors unweave nuances of appropriation into: (a) an ‘instrumentalization’ of artifacts in the service of a problem-solving task performed by the Subject upon the Object; and, reciprocally, (b) an epistemological ‘instrumenting’ of the Subject’s utilization schemes, as though the artifact becomes the learner’s interiorized and transparent vehicle of domain-specific reasoning directed toward a class of Objects (situations).

[Insert Figure 5 about here]

Application. We have adopted the IAS model as a central epistemic form for studying our data, and have filled its slots thus: Subject—the student; Instrument—combinatorial analysis, i.e., determining a sample space and “reading” it as informing an expectation of outcome distribution; and Object—the marble box stochastic process (see Figure 5). The student begins the interview activity with only intuitive links to the scooping experiment, because the student lacks the epistemic form ‘distribution.’ The instrumentalization of combinatorial analysis is complete only if the student recognizes the pertinence of this routine to the object. Yet, an activity sequence may be set up such that the student is not initially informed of the toward-ness of the instrument (what it is ready-to-hand for, to use Heideggerian terminology). In such designs, a critical moment in students’ learning process is the ‘aha!’ insight into this pertinence.

Students who have instrumentalized the combinations tower are likely to use it as a means of expressing expected outcome distributions for p values other than .5 (Abrahamson, 2007b). However, students who remain in suspended-pertinence limbo cannot complete the process (see the Results sections).

Critical Features
Introduction. Tsal & Kolbert (1985), cognitive psychologists studying visual perception, examined what causes viewers to switch between two alternative interpretations of (= orientations toward) ambiguous figures, e.g., Jastrow’s classical ‘duck-rabbit’ drawing (see Figure 6a). They demonstrated that particular features in the figure (= in the view) trigger competing percepts. For example, as we scan the duck–rabbit picture, attending to the right side is more conducive to seeing rabbit, because the dent is more mouth-like than head-like (the dent is a critical feature of ‘rabbit’), whereas the left side is more conducive to seeing duck, because the protrusions are more beak-like than ear-like (the protrusions are critical features of ‘duck’).

[Insert Figure 6 about here]

Application. The combinations tower, too, is an ambiguous figure (see Figure 6b for one possible ambiguity). Attention to the vertical axis foregrounds the common property of within-column permutations—number of favorable outcomes—enabling consideration of their equiprobability for all $p$ values (the group probability is equally distributed among the permutations). Attention to the horizontal axis—especially along the bottom row—underscores between-group variability in number of favorable outcomes, enabling recognition of the completeness of the ordinal sequence and differentiation in expected frequency of individual outcomes (for all $p$ values other than 0 and 1). Attending to both the vertical and horizontal axes permits consideration of expected frequency distribution (1:4:6:4:1, for the $p$ value .5). Thus, the alternative disambiguations of the combinations tower are complementary for understanding it as a probability distribution: the horizontal axis spells out the ‘what’—the five possible event categories in question—and the vertical axis qualifies the ‘how many’ (when the combinations tower is interpreted as a sample space) or, eventually, for the $p$ value of .5, ‘how often’ (when we orient to the columns as indexing the expected outcome distribution).
The combinations-tower analog of the duck–rabbit critical features (locations on the stimulus) is three-fold, including location (bottom row vs. the “crown”), orientation (vertical vs. horizontal), and directionality (left-to-right vs. right-to-left, upwards vs. downwards). These orientations of view toward the combinations tower can be highlighted by a facilitator who is disciplining a student’s perception as the student instrumentalizes combinatorial analysis as the emergent epistemic game of the ‘distribution’ form.

*The ‘More–More’ Heuristic*

*Introduction.* Stavy and Tirosh (1996) describe a general problem-solving intuitive rule for inference making regarding comparative properties of two systems: ‘The more of A, the more of B,’ e.g., “Mommy is taller than me, so she runs faster than me.” Generally,

[when] two objects or systems differ in one particular quantity…. the intuitive rule is directly activated by immediate perceptual differences (e.g., length).…. [that] are often visual, …. [so the problem solver] comes to regard this quantity as the significant one in this problem. Consequently, s/he assumes that the two objects or systems differ in the same direction with regard to some other quantities. (p. 664)

Stavy and Tirosh demonstrate cogently that “[problem] solvers often view this rule as self-evident, and its use is often accompanied by a sense of confidence” (p. 663). This heuristic, which, unfortunately, may lead to erroneous inferences, “gives the solver a grip on the situation, as it creates a causal/logical relationship between the components of the system” (p. 663).

*Application.* The ‘more–more’ heuristic maps well onto our participants’ insightful inference from the sample space to the anticipated frequency of events. Namely, the 2g2b column is taller than the other columns, and so when a problem arises as specimens of which category will occur most frequently—a different phenomenal attribute of the same categories—
students may be utilizing the more–more heuristic in claiming that the 2g2b category would occur most frequently. In this case, the inference is correct (for further cautionary remarks on the pitfalls of synoptic visual inferences, see Abrahamson, in press; Arnheim, 1969; J. R. Brown, 1997; Davis, 1993).

Material Anchors for Conceptual Blending

Introduction. Operating within the cognitive-science framework of conceptual blending (Fauconnier & Turner, 2002) and building on distributed-cognition theory (Hutchins, 1995), Hutchins (2005) explores artifacts as blends of material and conceptual inputs. In such blends, the association of material structure with conceptual structure stabilizes the resulting conceptual blend.

[Insert Figure 7 about here]

Application. Once constructed by the student, the combinations tower can potentially act as a material anchor for the emergent concept of expected frequency distribution. Yet the combinations tower per se, a sample space, takes on this meaning only once the notion of frequency is blended into it (see Figure 7 for this and an earlier blend). To compose the critical blend—the combinations tower as frequency distribution—students must recognize that the combinations tower embeds information pertaining to their earlier estimate regarding the frequency of outcome categories in the marbles-box experiment. Specifically, students need to recognize that the combinations tower embeds a warrant for their earlier claim that a 2g2b scoop is most likely.

Once the combinations tower is assembled, its verticality can be coded as indexing the respective likelihoods of the five column categories. Coding the sample space in terms of the projected empiricism of the stochastic experiment enables students to formulate their earlier
intuitions for the experimental distribution. This blend is a ‘composition’ (one of three types of blends that Hutchins enumerates), because the articulated idea of frequency distribution is not present in either of the input spaces per se.

There are several phenomenological triggers to the blending of the combinations tower material anchor with input from the marbles experiment, and each of these is related to phenomenal aspects of both contexts that the student detects as similar through a process of intercontextualizing mutual constraints:

1. Same class of objects (category). Recognizing the pertinence of the combinations tower to the marble-scooping experiment could be achieved by noting the common mathematical object—event categories—treated in both contexts. Namely, the particular material organization (spatial configuration) of the combinations tower makes salient within the sample space the same phenomenal categories initially evoked in the empirical experiment, i.e., the five possible embodiments of the 4-block, where order is ignored.

2. Same specific object (2g2b category). The perceptual prominence of the 2g2b category is evoked both in the marbles box, oriented at through the 4-block, and in the sample space, oriented at through the structure of the combinations tower.

- In the case of the marbles, this prominence is achieved by formulating with the scooper the perceived symmetry/balance of green and blue in the box, (see Tversky & Kahneman, 1974, on the 'representativeness heuristic'). Such formulation of the sense of symmetry as a 2g2b marble-scooper combination is reflexive, because the scooping context itself had foregrounded the color distribution as a relevant orientation to the marbles box.
• In the case of the combinations tower, the central category column looms patently taller than the others.

3. Same process. The materiality of the sample space—16 discrete manipulatable cards, as opposed to 16 inscriptions on a sheet of paper—enables students to perceive the sample space reflexively as a sampling device, i.e., as a collection of objects affording sampling (one can shuffle the cards and then draw out a card randomly). Such a secondary experiment on the sample space is commensurate with the primary experiment with the marbles, due to the equiprobability of all outcomes in the sample space (inferred from equal numbers of green and blue marbles in the box). However, the material structure of the combinations-tower may encumber interpretation of the sample space as an “urn” with 16 equiprobable, discrete cards. Namely, the combinations-tower categories create five salient gestalts that are further juxtaposed by hetero-probability, and these perceptual–pragmatical orientations toward the sample space inhibit its alternative construction as a collection of discrete objects.

Finally, the combinations-tower blend can be elaborated when students ‘scale it up’ to predict results of many scoops, or when students consider alternative $p$ values.

Semiotic Means of Objectification

Introduction. Radford (Radford, 2003, 2008) takes a semiotic–cultural approach to cognitive and sociocultural theory in investigating the emergence of individuals’ mathematical understanding in goal-oriented social contexts. He examines mathematical reasoning as mediated, distributed, and reflexive and describes the process of objectification, through which presymbolic notions, such as unarticulated mathematical constructs, become established as referents. In his analyses of collaborative activities, Radford scrutinizes individuals’ orchestration of semiotic means of objectification—multimodal expressive tools, such as
inscriptions, linguistic devices, gesture, and tangible artifacts—in building intersubjective coherence. Radford has coined the term ‘rupture’ to describe a challenge students often experience as they shift from employing factual and contextual semiotic means of objectification toward employing symbolical notation, e.g., algebraic forms, that do not map easily onto an embodied construction of phenomena under inquiry.

Application. The student enters the card-coloring task with a presymbolic sense of the expected outcome distribution, a sense consisting of yet unarticulated “object–property compounds” (the five 4-marble events differentiated along their felt relative frequencies). However, the empty 4-block cards—the media made available to the student—do not provide the student with appropriate semiotic means of objectifying these compounds, since no single card can capture the student’s sense of frequency. For example, a 4-block card with a single green square in the bottom-left corner placed adjacent to the single card with no green squares cannot communicate a subjective sense that the 1g event is four times as likely as the 0g event. To complete the entire sample space, the student would need to objectify each event as a class of outcomes (its permutations) that collectively denote the felt frequency attribute. Yet in this passage—in coming to employ mathematically normative semiotic means of objectifying distribution—the student might experience a rupture from the factual–contextual to the indexical.

Ontological Imperialism

Introduction. In their study of music as embodied mathematics, Bamberger and diSessa (2003) demonstrate how a given musical beat—e.g., “ta, ta-ta-ta, ta”—can be represented in multiple ways that each captures some features of the presymbolic auditory experience while ignoring others (e.g., a certain representation highlights the onset of beats, while another emphasizes duration). Such selective modeling of sensuous experience is a sine qua non attribute
of any semiotic system. However, in educational contexts, such selectivity could be a double-edged sword, because symbol-based representations can become the objects that individuals use to think and construct knowledge within a certain domain. Such ontological imperialism of symbolic systems can be problematic in constructivist designs aimed to foster students’ reflective abstraction from their idiosyncratic sensory experiences, because “units of description may come perilously close to (pretending to be) units of perception” (Bamberger & diSessa, 2003, p. 132, italics in the original text).

Application. As described above, the blank 4-block cards do not provide students with adequate semiotic means of objectifying their presymbolic ‘event–frequency’ compound notions. Furthermore, the cards do capture the specific order of the 4-marble outcomes, a property that students generally ignore as irrelevant to the problem at hand. In coloring the cards, therefore, students are simultaneously unable to capture their sense of frequency and forced to represent order. This ontological imperialism of the 4-block cards is thus liable to lead to communication breakdowns between the interviewer and student or even full-blown cognitive calamities when the student is unable to negotiate the treachery of the media. On the other hand, if the student were given five receptacles (e.g., egg cups) and used actual marbles to represent the five orderless events, the media would not push the dyad to negotiate the meaning and prospective semiotic utility of attending to permutations. We will return to this pedagogical dilemma which we see as central to theory of learning and design.

Summary and an Integrated Model

Figure 8 shows a proposed integration of the key resources cited above. The IAS model constitutes the backbone for examining students’ instrumentalization of combinatorial analysis
toward the marbles-scooping experiment. The instrumentalization process is broken down into a set of three key constructions or coordinations:

1. A sample space is an epistemic form filled through the combinatorial-analysis epistemic game—creating the exhaustive “list” of all possible outcomes, possibly by first constructing an exhaustive combinations meta-list and then, within each meta-list item, playing a topical game to determine the exhaustive list of local permutations;
2. The combinations tower can be viewed either as an instantiation of an epistemic form or as a conceptual blend.

   a. *Epistemic form.* The combinations tower is a sample-space “list” meeting further constraints of a more demanding epistemic form, a non-arbitrarily organized list that formulates a set along a further dimension of interest to the inquiry process; the combinations tower is a bar chart that facilitates the tallying of the subsets and thus, their comparison. Once the form has been established—a left-to-right sequencing of a finite number of categories that ascend in order by number of green squares—the game is to “mold” the sample space into these groups, observing the construction regulations of perpendicular rows and columns as well as equal spaces between columns and concatenated outcomes within columns.

   b. *Conceptual blend.* The combinations tower is a conceptual blend anchored by a number line. This number line is first instantiated by the interviewer’s gesture—a saccadic left-to-right sweep above the desk—and then concretized/substantiated through sorting the outcomes. This blend is doubly unique:

   i. The number line is a special material anchor in that its materiality is initially only conceptual, tenuous, and projected and emerges as a
functional–organizational property of the sample space only through the act of construction;

ii. The set of cards, an apparent candidate for constituting a material anchor, cannot function as an anchor, because it is structurally unstable—the set is shuffled into a new form evoked by the number-line input.

3. The frequency distribution is a blend of the combinations tower (the assembled sample space) and the intuitive notion of distributed outcomes in the marble-scooping experiment. This construction is critical for “making ends meet” between the initial intuition of chance distribution, coming from the marbles-box context, and the structured apperception of the sample space, which comes to signify aspects of this distribution. It is through this blend that the pertinence of the combinatorial-analysis activity to the randomness experiment first becomes apparent. That is, the alignment of the combinations tower with the intuitive distribution of marble-scooping outcomes—along a shared mathematical category, object, and process—confers the combinations tower as a vehicle for seeing chance.

Each of these constructions requires aligning orientations of view toward two artifacts, and these orientations, in turn, require attention to critical features of these artifacts. This attention is facilitated through the interviewer’s highlighting the critical features of these ambiguous artifacts, and the coordination is achieved through negotiated coding of the features. Also at play are students’ heuristics that trigger causal mapping between artifacts. Pragmatical aspects of the student–interviewer—the unsaid subtext of the interviewer’s multimodal facilitation practice—will be discussed in a summary of the case studies.
Finally, note that constructing the sample space was presented to the students as an activity onto itself—they did not initially interpret the activity as conducive to warranting their intuitions of outcome distribution in the marbles experiment. Furthermore, the combinations tower was presented as a means of organizing the sample space, again with no explicit reference being made to the marbles experiment. Both these constructions—the epistemic form and the conceptual blend—were thus facilitated as intact procedures that are not instrumentalized toward the experiment. Wax on, wax off.

Research Questions

Rationale

Whereas design-based research studies typically advance both theory and design, the current post hoc empirical study treats only theory building. To shape this study, we will:

1. formulate two types of research questions: (a) from theory to data: and (b) from data to theory. Namely, we will ask research questions that are:

(a) oriented through the lens of a theoretical model toward the empirical data—the theoretical model will elucidate cognitive and socio-cultural aspects of interactions between a student and an interviewer who are manipulating learning materials; and

(b) oriented through the lens of the empirical data toward the theoretical model—we will attempt to advance the theory by proposing an assembly and integration of several theoretical models (see previous section for emerging model) and build on our data to investigate for apparent challenges of this integration;

2. address the research questions through microgenetic analysis of selected data excerpts from our design-based research study, in which a non-normative activity was implemented (the “study condition” in this pseudo-experiment)
3. articulate any insight these analyses enable vis-à-vis comparable studies implementing constructivist pedagogy, wherein activity sequencing was normative (the “control condition”)

4. articulate implications for socio–cultural theory, design-based research methodology, and mathematics education.

**Questions**

The suspension of pertinence students experience during the second activity of the design creates for mathematics-education researchers a rare opportunity for studying cognitive and socio–cultural issues through the lens of artifact-based mathematical learning. Namely, students operate a procedure without initially knowing its objective, and so procedural and conceptual aspects of the content are initially isolated. Yet the procedural and conceptual later merge in a semi-controlled moment, so that researchers can examine nuances of the design and interaction that impacted students’ insight. Thus, analysis of data from implementing this design may illuminate and hone theoretical models of socio–cultural mediation theory, because the suspension of pertinence problematizes and foregrounds these models’ assumptions. We ask:

- What criteria distinguish among the enactment of an activity and a rote procedure?
- Is there pedagogical value to practicing an activity that initially appears as self-contained?
- How do students react to an activity that appears to be irrelevant to a posed problem?
- Can we say of a student who is unaware of the pertinence of a procedure she is practicing that she is instrumentalizing the procedure?
- What material, social, and other factors contribute to the ‘aha!’ moment, when the pertinence is grasped?
• What of students who do not experience insight into the pertinence of the second activity to the first? Is this an inherent shortcoming of the design, or does it perhaps illuminate tacit aspects of mathematical learning and practice? If so, what are these aspects and how are they developed?

Methods

The nature of this study is reflexive with regards to its content. Just as students first intuited and only then warranted their mathematical assertions, so the techno-pedagogical craft of design may comprise of intuitive decisions that are only later, sometimes years later, warranted. Members of the Embodied Design Research Laboratory engaged in the design-based research project Seeing Chance (Abrahamson, PI), including several participants UC Berkeley’s Undergraduate Research Apprenticeship Program, all shared both an interest in the accumulated body of data from implementations of the focal design and a hunch that students had availed of these activities. Yet much about the design was initially difficult to articulate, in terms of explicating the ontological nature of the artifacts, agreeing on a phenomenologically viable description of the mathematical content of the design, revealing and defining behavioral patterns within and across students, and, eventually, understanding the apparently pivotal roles of the interviewer. This obscure nature of the study drove two years of intense scrutiny of the data as well as a search for intellectual resources that could inform this scrutiny. The intellectual resources progressively drawn into the study as well as the budding research questions—both about the data and about the theoretical models themselves—co-emerged through close analysis of the data, in a ‘bottom-up, top-down’ fashion.

“Bottom up,” we had just over 26 hours of video footage. These data come from a design-based research study of students’ probabilistic cognition (Abrahamson, 2007a;
Abrahamson & Cendak, 2006), in which twenty-eight Grade 4 – 6 students from a private school each participated individually in a ~1-hour-long semi-structured clinical interview (Ginsburg, 1997) administered by the researchers. Crucially for the current study, the design positioned “theoretical” and “empirical” activities as contradistinctive (Jones et al., 2007), and students were to grasp the content through appreciating the pertinence of theory to empiricism, i.e., the relevance of combinatorial analysis to actual experiments with the randomness generator.

“Top down,” we developed an interest in literature pertaining to artifact-based mediated learning. This interest grew through dissatisfaction with the analytic yield of narrow interpretations of the implementation. Namely, the radius of our attention context grew from “in the student’s head” to encompass “in the body” and further to include the materiality of the design and then the content and intention of the interviewer; then “up” to question mute yet omnipresent pragmatically aspects of the dyad and the embeddedness of this dyad within larger socio–political webs of practice. The research questions emerged as we became cognizant of the nature of our investigation and its potential implications for educational research, design, and practice.

In the discursive space between the study vectors—the bottom-up data and top-down literature—we conducted collaborative microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1991) using principles of Glaser and Strauss’s (1967) grounded theory, through which we built acumen of the entire data corpus. Emergent constructs introduced in this paper as well as the proposed integration of theoretical resources reflect iterative attempts to construe the data excerpts so as to coherently reflect the complex, dynamical relations of design, cognition, and content observed in the data, which we grew to know intimately, not unlike fans of a popular televised sitcom who can lip-sync the discourse and mime the gestures.
Three case studies were selected from the data to demonstrate a range of student understandings and difficulty with the concepts embedded in the design as well as nuances of mediation. In particular, we scrutinized excerpts that each of the four authors selected and championed as relevant to understanding the students’ reasoning. These excerpts were debated intensely over a period of a year, both within our group and with colleagues, in meetings and conferences, both verbally and through anonymous review processes. Into the debate of each excerpt, we drew the context of the entire interview as well as all other student cases with relevant attributes. Thus, we repeatedly “raked” our corpus of data, interweaving the new theoretical models.

Finally, the nature of this study is reflexive also with regards to the application of its theoretical resources. Namely, our integrated theoretical model highlights students’ difficulties in instrumentalizing combinatorial analysis toward the marbles experiment, so that we can code these behaviors toward informing further design (cf. Goodwin, 1994).

Results: Analysis of Case Studies

Below, we present and analyze three data excerpts in which 6th-grade students manifested a range of behaviors that either led or did not lead to the conceptually pivotal coordination of the design’s ‘theoretical’ and ‘empirical’ activities. In each analysis, we focus on properties of the innovative learning materials and the interviewer’s roles in guiding students to see the pertinence of the combinatorial-analysis activity to determining the expected outcome distribution in empirical experiments with the randomness generator. Thus, we are studying students’ instrumentalization of combinatorial analysis and the interviewer’s function in this learning process. Following a comparative summary of the case studies, we conclude with reflection on new synergy between design-based and classical socio–cultural research on mediated learning of
mathematics. We then offer implications of the analyses for the nature of mathematical learning and the socio–cultural mediation of effective epistemic dispositions.

Case Study 1: Difficulty in Temporary De-Instrumentalization

SH is a 6th grade student characterized by her teachers as high achieving. Early in the interview, SH identifies 2g2b as the most likely outcome, citing the equivalent numbers of green and blue marbles in the box as the origin and support of this assertion. Next, performing the combinatorial analysis, SH initially colors in only 5 cards, with 0-through-4 green squares, respectively, explaining that any other card with “different places” is “equivalent” to one of these cards. SH is unaware that the each card necessarily captures an ordered outcome, and that she has therefore created an incomplete sample space consisting of only five of the 16 distinct outcomes. She completes the sample space only after the interviewer prompts her to “think about places as well.” For the remainder of the interview, SH uses the terms “place-wise” to refer to an orientation to the particular spatial configuration of the green (and blue) squares within the 4-block (so place-wise there are 16 outcomes that fall into 5 event classes) and “color-wise” to refer to an orientation only to the number of green (and blue) squares in the 4-block (so color-wise there are only 5 possible outcomes, with another 11 that constitute variations on these 5).

As soon as the combinations tower is constructed, the interviewer asks SH, “What can you say about this?” This open-ended question makes no reference to the marbles-box context. Nevertheless, SH immediately reads off the combinations tower the probability of each color-wise outcome. That is, SH recognizes that the combinations tower encodes relevant information about group probabilities and successfully uses it to justify her earlier claim that 2g2b is the most likely outcome. SH thus instrumentalizes combinatorial analysis as a tool for determining stochastic properties of a randomness device.
SH’s understanding is robust. She initiates the following elaboration on the relation of the combinations tower to the experiment: Noting the 1-4-6-4-1 count in the combinations-tower columns, SH describes a hypothetical experiment with 160 draws that would result roughly in a ten-fold scaling up of combinations-tower groups: 10 outcomes of type 4g, 40 outcome of category 3g1b, etc. (10-40-60-40-10). Thus SH sees the combinations tower not only as indexing actual outcome distributions that are of ontologically distinct nature but as a proxy for the outcomes themselves: she sees the total set of actual experimental outcomes as a multiplicative transformation on the sample space.

[Insert Figure 9 about here]

Yet SH, who has been so intently attending to color-wise groups within the combinations tower, now faces considerable difficulty in re-attending to the 16 outcomes as unique and equiprobable, as follows. Anticipating the computer-based experiment simulations (not discussed in this paper), the interviewer selects and lifts up two cards, the top card in the 3-green column and the sole 4-green card, and holds them over the combinations tower, asking if one is more likely to come up than the other (see Figure 9). SH initially responds that the 3-green card is more likely, but after reflection arrives at the correct conclusion that they are equiprobable.

[Insert Figure 10 about here]

The interviewer wishes to establish how flexibly SH can switch between “place-wise” and “color-wise” interpretations. Thus, he again picks up two cards, one from the 2g2b column and one from the 1b3g column. Lifting them and reiterating the question, he enunciates “this specific card and this specific card.” He holds the cards in front of SH and then intentionally draws both cards across his torso even further away from the combinations tower (see Figure 10). The interviewer implicitly believes that the combinations tower makes groups so salient that
SH experienced difficulty attending to the uniqueness of the permutations as long as she was looking at the tower. By pulling these cards away he isolates them physically within SH’s perceptual field and thus, he assumes, he is helping SH isolate them conceptually. However, SH answers that the 2g2b card is more likely, pointing to it and saying “this one.” Why?

In both of these episodes SH was directed to attend to two cards that had been removed from the combinations tower and judge their relative likelihoods. Why in the second episode would she answer that a single card is more likely than another, given that in the first episode she had correctly stated that any two single cards in this sample space are equiprobable? To answer this we must examine the manner in which the two questions, virtually identical in wording, were presented. We find that they differ along one critical dimension: in the first episode the interviewer held the two cards in fixed positions directly over their respective columns, whereas in the second episode he purposefully removes the cards from the environment of the tower.

To determine how these subtle differences could produce such different responses from SH, we must analyze the combinations tower itself. The combinations tower is a polysemous figure; a between-column orientation makes group properties salient, yet a within-column orientation renders each card unique through juxtaposition with its permutations. However, if a card is removed from the column, it is unclear as to whether it is to be understood as a unique outcome or as a representation of the group to which it belongs. In the first episode, SH’s correct interpretation is promoted by the cards’ slight physical separation from their respective groups that helps maintain a sense of their respective group memberships; positioning the cards above their respective columns is a semiotic means of objectifying both their uniqueness and their group membership. SH’s gestures during her response indicate that this positioning helps her see the card as separate from the combinations tower while still remembering its group affiliation.
(see Figure SH-1). In the second episode, the cards’ complete removal from their respective columns de-emphasizes their uniqueness within their event class and prompts SH to interpret them as group *representatives*, rather than individual group members. Within this understanding, she correctly responds that the 2g2b card is more likely.

SH’s response in the second episode manifests *inadvertent metonymy*, by which a student interprets a permutation as representing an entire event class. Namely, SH’s robust orientation-of-view to the verticality of the combinations tower as indexing relative likelihoods interferes with the alternative construction of 16 distinct outcomes. SH’s interpretation results in a *false contract*, or communication breakdown, between the student and the interviewer. The interviewer intends for the cards to serve as semiotic means of objectifying ordered outcomes, and so assumes that asking about “specific cards” necessarily implies unique permutations. For SH, at this specific moment in the interview, order is not a significant attribute of the card—it was pivotal during the combinatorial analysis, because she then saw order as an attribute that advanced the goal of creating all the possible outcomes, and it was instrumental in reading the *entire* combinations tower as indexing frequency. Yet her attention to order appears still contextually bound—she cannot alternate fluently between the varying orientations toward the card, and her default orientation is to construe the single outcome as metonymic of its entire containing class. The interviewer and student thus agree on the referent of speech but differ as to its sense. Both senses are viable, yet the interlocutors cannot negotiate an agreed sense before they acknowledge the breakdown and articulate their idiosyncratic senses for this shared referent.

In order to elucidate the apparent tension SH is experiencing between place-wise and color-wise interpretations of the combinations tower, the interviewer takes a new approach. He asks,
“What if I flip them all over [flips over two of the cards in the bottom row, so they show only blank sides] and there’s cards ‘A,’ ‘B,’ ‘C,’ ‘D’ [scribbles with his hand in the air above the cards, mimicking writing on them] and we mix them up on the table [gestures mixing above the tower], and… just pulling out, like out of a hat [gestures pulling out of the shuffled set]—Would any of the cards be more likely to come out than the others?”

SH seems confused, and then asks “color-wise or place-wise?” to which the interviewer emphatically responds, “No, just cards” (perhaps a clearer and more appropriate response would have been “place-wise”). Despite these attempts to get SH to disengage from a group orientation to the combinations tower, still she replies, “One of these rows,” gesturing upwards along the 2g2b column. The interviewer asks again, “how about specific cards, is there any specific card more likely than the others?” Perhaps sensing that she is not satisfying the interviewer (Grice, 1989; Schegloff, 1996), SH makes another guess: “Ones with two of one color next to each other on a side.” SH is attempting to re-instrumentalize the combinations tower in accord with her understanding of the interviewer’s request; however, because she is still oriented towards the probabilities of groups, she regroups the cards based on a different property.

At this point the interviewer physically flips over all the cards, actually labels them with the alphabetic characters A – P, and shuffles the cards into a stack. From this stack he draws out two cards randomly and asks if one has a better chance of being drawn than the other. SH confidently responds, “No, they’re all just one out of 16.” The interviewer then flips the cards over to the colored sides and asks, “Does it matter that on this side they have different colors?” SH pauses, then answers,
“I guess not, because we’re not talking about that side, we’re talking about the letters. When you talk about those sides [colored sides], that’s two-and-two [points to 2g2b card, color-side up] and that’s three-green-one-blue [points to 3g1b card].”

This response supports our claim that SH is manifesting inadvertent metonymy: she construes each of the two specific cards as its respective, activity-specific, socially constructed group, rather than one of 16 ordered outcomes. The breakdown then intensifies when SH claims that the probabilities of drawing each lettered card would change based on whether or not she knows its flip-side colors. This tension is resolved only in the following intervention.

The interviewer describes and mimes a machine that randomly selects a card from the stack. When the cards are letter-side up, SH easily responds that “they’re equivalent” in likelihood of being drawn by the machine. The interviewer then flips the entire stack over to the color-sides and reiterates, “I’m still a machine, picking one randomly. Does [any card] have a better chance?” SH answers hesitantly, “I guess not, when you’re talking about independent little cards.” Perhaps the machine’s patent “group blindness” enabled SH—adopting the machine’s point of view—to disengage herself from the group-based interpretation of chance.

The interviewer concludes, “We’re working back and forth between these things. As individual cards, you say they have the same chance, but as members of families, some families have a better chance. Can you live with both things?” SH quickly answers, “Yeah, because... yeah, they’re different.” During the interview wrap-up conversation, SH is asked whether she had been confused. She explains, “We were talking about many different things, so it got confusing at one point... When I found out that we were talking about the independent, not the families, then it was fine.”
In sum, SH is a case of a high-achieving student who experienced great difficulty in temporary de-instrumentalization of combinatorial analysis or un-blending of the inputs.

Case Study 2: Non-Normative Material-Anchor Input as Hindrance to Expressing Correct Intuition

LF is a 6th grade student whose teacher characterized as middle achieving. At the beginning of the interview, when asked to predict the outcome of scooping from the marbles box, LF lists the possible outcomes by number of green squares: “Four of one, none of the other; three of one, one of the other; two and two; and one and three.” In listing these categories sequentially—four, three, two, one—LF spontaneously demonstrates systematic combinatorial analysis at the combinations level (albeit he omits the “zero” item). When probed to elaborate on his analysis, LF says, “Knowing that there’s an equal amount of marbles, the chance is probably that we will get two and two.” LF thus infers a most probable event based on an evaluation of all possible events. However, prompted to express more precisely the chance of getting 2b2g, LF proceeds to re-enumerate all possible events, determines that there are seven events [sic] and answers, “One out of seven.” LF thus ignores the permutations and miscounts the combinations. However, he thus demonstrates intuitive familiarity with the Bernoulli/Laplace principle of expressing chance as a probability by determining the fraction of favorable events out of all possible events. Impressively, LF also appreciates the effect of replacement on outcome distribution: “If you scoop it this time, the chance of getting one blue and three green…. Then dumping it back in—next time the chance of getting one blue and three green is the same amount.” In sum, though we witness tension between LF’s initial intuitive prediction (2g2b is most common) and subsequent analytic inference (2g2b has a 1/7 probability), LF is apparently unperturbed by this intuitive–analytic conflict. As we shall see later, though, LF’s obliviousness
to the permutations will have repercussions for his construction and interpretation of the sample space. Specifically, LF’s focus on combinations in the context of the marbles box foreshadows his subsequent difficulty in ascribing permutations any relevance toward achieving the goal of determining the distribution of expected outcomes.

[Insert Figure 11 about here]

LF departs the marble-box context with a set of object–property compounds, each scoop coupled with his sense of its frequency relative to the other scoops. During the card-coloring activity (see Figure 11), LF encounters the learning issue of whether or not permutations are relevant to predicting the results from the marble scooper. He tentatively concludes that permutations should not be included in his analysis, declaring his work complete when he has created only five cards, one from each number-of-green category. Studying these five cards for over ten seconds, LF utters, “I think that’s it…” and then spends almost another ten seconds staring at the cards before following up with a slight “yeah.”

Like SH, LF is oblivious to the fact that each of the 4-block cards he has created can be taken to represent an ordered outcome. Yet, this parallel obliviousness to permutations has different repercussions for SH and LF. With SH the ontological imperialism lead to a false contract between her and the interviewer but did not cause conceptual breakdown—that is, when we factor into our analysis of SH’s utterances her presumed understanding of the conversation, SH never makes mathematically incorrect statements. LF, on the other hand, experiences deeper conceptual difficulty; the imperialism of the media goes beyond a negotiable communication breakdown. Namely, when presented with the empty cards as the available means of objectification, it appears to LF as if he will need but a single card for each of these five events; as if the ‘order’ dimension that emerged from his interaction with the available media does not
carry any import for the analytic or explanatory process.

Presumably, having thus created a five-card sample space, LF might still “read” the cards as nevertheless *connoting* the initial notion, by which the frequency of events varies and $2g2b$ constitutes the most likely event. That is, LF might complement the physically constructed cards with the yet-uninscribed mentally experienced sense of their relative frequencies. Yet, LF amends his earlier claim: he states that the events are equally likely at $1/5$ probability each. Thus, the presymbolic frequency property of the five events is *lost in translation* (Brar, Galpern, & Abrahamson, 2006): LF is unaware of the transcriptional implications of the shift in semiotic systems—from the factual to the contextual—for the necessary production of new signs; he thus omits to inscribe with the cards critical aspects of meaning implicit to the earlier context of the marbles box yet absent in the context of the cards. In sum, LF re-blended frequency into the media prematurely—before the complete sample space is constructed—resulting in his new claim of a flat distribution, which mis-represents his initial claim of a varied distribution.

[Insert Figure12 about here]

At the behest of the interviewer, LF creates the remaining 11 permutations. In so doing, LF has difficulty in building the entire $2g2b$ category, believing he is done with only four cards (he omits the two permutations with horizontal rows of like color). He says, “Once I got to four I knew there couldn’t be any more…. because there’s only four spaces [in the scooper].” With the combinations tower eventually completed, LF is asked to restate his expectation of the experimental outcomes. He maintains his recent conclusion that the five events are equally likely, saying, “These [other 11 cards] don’t really matter…. If the placement mattered, these [other 11 cards] *would* matter, but these [other 11 cards] are all the same thing…. It’s these same original five or any one of these” (see Figure 12). At this point, the interviewer asks again,
“Looking at [the combinations tower], you say the chance of getting 2g2b is still one-out-of-five?” LF appears to consider changing his answer, possibly to please the interviewer’s subtext of doubt. Ponderously, LF replies, “Well actually…,” but eventually answers affirmatively (1/5), consistent with his prior statement.

[Insert Figure 13 about here]

Figure 13 serves to summarize LF’s seeing of the sample space. Initially, he sees only the bottom-row five cards as pertinent to the mathematical activity, as though they constitute the complete sample space. The 11 upper-potion cards are apparently mere embellishment with no bearing whatsoever on establishing the expected outcome distribution of the marbles-scooping experiment. Though LF had earlier attended to the columns’ heights—while building the combinations tower, he observed that the columns “go up in steps”—he completely dismisses their relative heights as pertinent to issues of probability. LF’s treatment of the permutations as irrelevant to determining the outcome distribution marks that he has not mastered the combinatorial-analysis procedure. In terms of Verillon and Rabardel’s IAS framework, LF has not fully instrumentalized the combinatorial analysis, so that the Subject–Instrument relation has not been completely forged. However, LF does appear to understand that the procedure is an instrument for determining outcome distributions, so that the Instrument–Object relation is by and large in place.

In terms of conceptual-blending theory, LF has input intuitive frequency into the blend yet applied it to a non-normative material anchor (orienting his view only toward the bottom row of five outcome cards in the combinations tower rather than to the entire five columns). To amend the blend, LF has to re-input his sense of frequency into the entire sample space. Thus, to help LF complete the instrumentalization of combinatorial analysis, the interviewer has to enable
LF to re-experience and sustain his initial sense of frequency, from the marbles box context, yet
gently re-orient this sense so as to encompass the vertical extensions of the columns.

[Insert Figure 14 about here]

The interviewer responds to LF’s ‘one-in-five’ belief by commenting that, under LF’s
rule, a 100-draw sample would yield about 20 outcomes per group. As he says this, the
interviewer gestures at each column as hypothetically encoding the false assumption (see Figure
14). Specifically, using a pen as a deictic device, the interviewer gesturally highlights the
verticality of the columns while verbally coding its hypothetical implication to the expected
distribution: “So you would say that if we drew 100 times, we would get, say, 20 of these
[vertical pen at 4g column], 20 of these [vertical pen at 3g1b column], 20 of these [at 2g2b
column], 20 of these [1g3b], and 20 of these [4b].” By suggesting the columns as categorical
gestalts, the interviewer endorses LF’s focus on the five categories yet implicitly qualifies it,
prompting LF to stretch his visual-attention spotlight (Posner, 1980) upward from the bottom
row to encompass the entire columns, the critical features of the artifact that encode the variable
distribution that LF had lost in translation. As we now explain, highlighting the variable
verticality of the combinations tower and coding it as indexing likelihood by category appears to
tap LF’s initial sense, coming from the context of the marbles box, concerning variable
likelihood by category, and this evoked sense prompts LF to reevaluate his own proposed flat
distribution (20, 20, 20, 20, 20) vis-à-vis the patent variation in column heights (1, 4, 6, 4, 1). LF
will thus unshackle the ontological imperialism…

Immediately, LF clues in to the rhetorical subtext—he experiences logical dissonance and
switches to the normative inference. Manifestly, LF experiences an ‘aha!’ moment, saying,

“Out of all the possibilities you can get, six-out-of-sixteen are two-and-two, and these
[each of 0g and 4g cards] are only one-out-of-sixteen…What I was saying, one-out-of-five chance [exasperated]…. You’ll get these [2g2b] more than these [0g or 4g] because there’s six of these and there’s only one of these.”

LF’s ‘aha!’ moment includes not only a realization that 2g2b is the most likely event because it has the most permutations—a additive ‘more–more’ deduction—but that the relative heights of the columns code the groups’ respective frequencies proportionally. Thus, the inadvertent metonymy of each of the five 4-blocks along the combinations tower’s bottom row is instantiated as a group of discrete objects; each of these five 4-blocks is stretched into an articulated category of 4-block combinations, event-classes with 1, 4, 6, 4, and 1 permutations, respectively, that are implicated as indicative of likelihood, thus resonating with the initial proportional judgment pertaining to the same phenomenal attribute in the context of the marbles box. Prompted to re-attend to the combinations tower, LF unblends the frequency coding from the bottom row of cards and reblends it onto entire set of cards. By connecting the combinations tower to his initial intuitive judgment, LF completes the Subject–Instrument relation—the combinatorial-analysis procedure becomes fully instrumentalized as a tool for scrutinizing the class of mathematical objects ‘randomness generators.’ Namely, LF now sees the combinations tower as not only what-we-can-get but also how-often-we-get-what.

Finally, we offer an observation on the relations between speech and gesture in remedial instructional rhetoric, with implications for gesture studies. We have explained how LF aligned a flat-distribution outcome expectation (1/5 probability for each event category) with a non-normative orientation toward the sample space (5 outcomes only). In so doing, LF swayed from trusting his own senses—that 2g2b would be the most common event—to trusting an inference from conducting a mathematical procedure. Granted, mathematical instruments
often check intuitive inference, so LF would not necessarily err by curbing his intuitions, and yet LF does not consider that he may have mis-instrumentalized combinatorial analysis so as to produce an incomplete sample space.

The interviewer’s role then becomes twofold: to rekindle LF’s initial intuition and help him instrumentalize combinatorial analysis correctly. Both interventions are necessary: rekindling LF’s intuition per se would create for LF an unresolved sense–sensibility contradiction between his initial seeing of the marbles box and his current interpretation of the mathematical construction; at the same time, LF’s perception of the combinations tower needs disciplining. Moreover, for pedagogical considerations the interviewer wishes to present the evidence for LF’s own arbitration, so that LF experience appropriation of the mathematical instrument rather than just passively receive the teacher’s instruction. The interviewer responds to this dilemma by honing the emergent sense–sensibility contradiction so as to scaffold LF’s negotiation of these resources toward resolution, as follows.

The interviewer communicates the sense–sensibility contradiction multi-modally, evoking the initial intuition in gesture while revoicing the erroneous inference in verbal utterance: the interviewer points to each of the five variably tall columns while enunciating, “20 of these, 20 of these, etc.” This is a case of a teacher initiating a *speech–gesture mismatch*, which Church and Goldin–Meadow (1986) have discovered in student behavior as marking preparedness for insight and Singer and Goldin–Meadow (2005) have documented in teacher behavior as supporting student insight. Albeit, the particular species of mismatch, in our case, is a *contradiction mismatch* rather than the *complement mismatch* that Goldin–Meadow et al. have revealed. The interviewer performs the contradiction mismatch as a pedagogical–rhetorical ploy—he scaffolds the student’s insight by modeling a generative contradiction between two
inferences. The interviewer’s selection of the gesture modality to represent the normative, rather than the non-normative, orientation may be a tacit pedagogical practice rooted in a teacher’s intuition that the embodied information will dominate in the student’s reasoning; as though the student’s naturalistic reasoning process is to evaluate whether a candidate verbal formulation resonates with embodied apprehension; as though learning is intuition groping for epistemic form (Abrahamson, 2007b).

Case Study 3: Limited Epistemic Gaming Capacity

RG is a 6th grade student characterized by his teacher as low achieving. He is the single participant who does not interpret the combinations tower as warranting the intuition that “half-half” would be the most likely outcome. That is, although RG recognizes the combinations tower as encoding “these are what we can get” from scooping (the sample space), he does not see it as encoding “how often we get each event” (frequency distribution). In fact, RG does not appear to instrumentalize the added values of the combinations tower at all. SH and LF, in comparison, offered analyses of the combinations tower that involved stochastic properties of the marbles-box experiment, such as probability and outcomes from repeated sampling. Granted, some of their analyses led to non-normative mathematical inferences, yet RG’s analyses do not include any stochastic properties or related mathematical constructs. RG’s difficulty in instrumentalizing the combinations tower as a frequency distribution can be traced back to the beginning of the interview, as follows.

Initially examining the marbles box, SH and LF each observed that 2g2b should occur most frequently. Thus, they constructed outcome categories as mathematical objects, and in particular the 2g2b event had been marked as unique. Plausibly, having schematized the outcomes by categories assisted SH and LF in managing the combinatorial analysis. Moreover,
perhaps they were primed to compare columns in the combinations tower and attend in particular to the 2g2b column. SH and LF’s particular category-based orientation to the combinations tower, schematized in a recent experience (marbles box) and sustained through engagement in a related task (building and sorting the sample space), primed their category-based attention and orientation to the new context (combinations tower), thus facilitating the association of the combinations tower with the marble box (same object). RG’s experience is different:

[Insert Figure 15 about here]

When RG initially examines the marbles box, he does not name any outcome as most likely, arguing for the impossibility of prediction in situations involving chance. Following, RG experiences great difficulty in creating the sample space and assembling the cards into the combinations tower. In particular, RG does not appreciate that the columns should be organized sequentially by number of green (0 – 4). Next, he assembles the 2g2b cards into two columns instead of just one (see Figure 15a). Guided to consolidate the two 2g2b columns into one column, he places it incorrectly between the 0g and 1g3b columns. Also, he aligns the 3g1b column with the top of the 2g2b column, instead of its base (see Figure 15b). Thus, RG is challenged by the activities involving sorting, order, and form, as though he is unversed in the tacit grammar of epistemic games played in building and filling mathematical representations. Consequently, RG is unaware of the emergent number-line input (0 – 4)—the “x-axis” of the combinations tower. Pivotal, the verticality of the combinations tower is not prominent in RG’s visual orientation toward the combinations tower.

Only later, after having constructed the combinations tower, RG reexamines the marbles box and infers a relation between green-to-blue color distribution (‘half–half’) and greatest outcome likelihood by number of green (2g2b). Yet then, still, RG does not see how the
combinations tower encodes pertinent information that might be used as an analytical warrant for
this intuitive assertion. In fact, RG appears complacent leaving an intuitive judgment
unwarranted—that is, he does not construe his proportional judgment as demanding further
support, such as in the form of analytic proof. It is as though RG does clue in to the subtext of
the interviewer’s probes that were designed to encourage RG to utilize the available media so as
to engage in analysis toward building a proof (on the social construction of proof, see Barnes,

In an effort to discipline RG’s perception of the combinations tower, the interviewer cites
the normative orientation of view. Namely, the interviewer explains how ‘some people’ notice
that 2b2g is the tallest column; he counts up the six 2g2g cards, enunciating “One, two, three…”
while pointing to each card in turn (see Figure 15c). RG responds that this is “just a prediction”
that cannot justify whether “half–half” is most likely.

The RG case study is an example of cascading disadvantages. Failing to initially see the
randomness generator the way his fellow classmates saw it, RG is subsequently behind because
he does not self-orient to the combinations tower as his classmates do spontaneously. But more
deeply, RG falls behind because he is not equipped with a host of tacit expectations, heuristics,
norms of discourse, and skills that constitute the fabric of mathematical reasoning. RG is not
equipped to assume mathematical agency (Veeragoudar Harrell, 2007). RG is unacculturated and
underpracticed at playing epistemic games—he does not expect to be able to determine universal
regularities through empirically noting connections between designed activities or conducting
explorative experiments, does not apply problem-solving practices, and under-performs in
building and interpreting mathematical representations.

Finally, it is worthy to note that throughout his participation in the activities described
above, RG’s demeanor is little more than utter disengagement. He does not resist the
interviewer’s instructions, yet carries them out with what appears to be very low interest,
possibly the lowest of all our 28 participants. And yet, once the interview enters the latter part of
the protocol—computer-based simulations of the marble-scooping experiment, RG’s demeanor
changes drastically to become the most enthused of all our participants. Beaming, he closely
monitors the stochastic accumulation of outcomes by categories, vociferously rooting for the
middle column, and is gleeful and astonished when, in one of the runs, the 1-green column
accumulated the plurality of outcomes: “One won!” (see Gee, 2003, for a discussion of gaming,
engagement, and learning, which is beyond the scope of this paper)

Summary of Findings

By and large, all students correctly intuited relative likelihoods of event classes in the
marbles experiment, i.e., that 2g2b is the most likely event, 4g and 4b are equally rare, and 1g3b
and 3g1b are somewhere in between in terms of their expected frequency. Yet, the students did
not initially have an available epistemic form—distribution—to articulate and inscribe their
probabilistic intuition. What would it take to articulate intuitive distribution using a normative
mathematical representation, such as a histogram?

Synthesizing an intuitive sense of distribution with its normative representation requires
formulating this intuition, such as articulating it in spatial–numerical epistemic form (e.g., a
diagram). This form must necessarily organize the outcome categories according to some
commonly used rationale (e.g., spatially ordered sequence) as well as encode the relative
expected frequency of outcomes by category (the property in question). In one type of standard
form—a histogram displaying frequency distribution—organization is achieved through sorting
the outcome categories spatially according to the quantitative dimension that distinguishes them,
e.g., number of green squares in each 4-marble draw, along an emergent number line. Respective likelihoods are indexed by vertical extension—lines or columns—positioned perpendicularly to the linear order of categories, so that perceptual comparison is enabled. And yet, study participants are not versed in the epistemic games of sorting and graphing quantitative information. How, then, is it possible that students identify the combinations tower as a warrant for their intuitive sense of distribution?

When students first encounter the combinations tower, they assimilate/ground it into their intuitive scheme of distribution. Namely, students recognize the combinations tower as a means of articulating their qualitative sense of the event categories’ expected relative frequencies, a qualitative sense that would be verbalized as the ‘more’ taller, the ‘more’ frequent. This orientation of view toward the combinations tower implies attending to the relative heights of the column’s vertical sweep, necessarily backgrounding the particulars of each column, i.e., the specific permutations in each group. And yet the relative tallness (the continuous upward sweep) is constituted by a greater number of permutations (discrete objects). So the combinations tower readily affords a very familiar activity, counting, as a means of consolidating the resonance of intuition and form in terms of an empirical rule: the more elements, the more frequent. Thus, the activity of counting permutations perfunctorily becomes the standard procedure for articulating intuitive distribution, even for students who initially repudiated the pertinence of combinatorial analysis to determining the expected outcome distribution. The instrumentalization of combinatorial analysis is completed when students rephrase the induced empirical rule in terms of the combinatorial activity they have been immersed in: the more permutations, the more frequent.

In sum, students who fully instrumentalized the combinatorial-analysis process are those
who connected between the sample space and the randomness experiment, giving rise to the new
contuct of distribution. Thus, the construct of distribution emerges as a reconciliation of two
activities that are mathematically commensurate yet phenomenologically disparate: (a)
perceptual judgment of a randomness generator’s anticipated outcomes; and (b) combinatorial
analysis of this same device. From this perspective, the critical role of a facilitator, once each of
these activities has been enacted, is to indicate to the learner the pivotal orientations of view
toward each artifact (marbles box, combinations tower) and how they are to be related. Thus
does the student: fully instrumentalize an analytic procedure (Vérillon & Rabardel, 1995); draw
the appropriate inputs into the conceptual blend (Hutchins, 2005); learn to play the domain-
specific epistemic game (Collins & Ferguson, 1993); and coordinate tacit and explicit knowledge
(psychological and epistemological, Papert, 2000; perceptions and concepts, Piaget & Inhelder,

What does such coordination of percepts and concepts critically hinge on? The above
scrutiny of our case studies suggests the following intriguing conclusion. Experiencing the
pertinence of combinatorial analysis to determining the expected outcome distribution of a
randomness generator can be attributed to issues of mediation, pragmatics, socio–mathematical
norms, and heuristics rather than logical deduction or necessity (Barnes et al., 1996;
Wittgenstein, 1956). Thus, numerous para-mathematical factors are at play in the
teaching/learning of mathematics, including discursive cues, intuitive perceptual judgments,
general problem-solving strategies, and fluency in the tacit grammar of school math (cf. Lave,
1992), and the teacher’s role is to assist students in developing each of these tokens of successful
performance and in applying them confluently within particular mathematical content domains.
Denuded of these multiple layers of implicit scaffolding and hidden rules of socio-mathematical
conduct, learning the content of probability as constructed in this design is no more (and no less) than operating a set of routines grounded in coordinated perceptual judgments—of the stochastic device, of the sample space.

Discussion: On Instrumentalizing Order, or

In Praise of a Socially Constructed Modicum of Neo-Platonism

We have scrutinized three case studies of students who each engaged in a sequence of activities involving learning tools designed to foster the development of probabilistic cognition. In our analyses, we interpreted the rationale of the instructional activity sequence as designed to foster students’ instrumentalization of combinatorial-analysis routines as tools for determining and rigorously articulating expected outcome distributions of a randomness generator under scrutiny. We have noted that not a single student had initiated combinatorial analysis or, while enacting it, realized its pertinence as an activity performed in order to warrant their intuitive perceptual convictions regarding the expected outcome distribution.

The case-study students, selected for analysis as representatives of the range of mathematical aptitudes exemplified by the 28 Grade 4 – 6 study participants, differed in their performance on the key learning issue of coordinating the two key activities. Our analysis of the microgenetics of students’ insight—or lack thereof—have suggested that much para-mathematical factors impact the learning of content, including discursive cues, general problem-solving strategies, and general school-wise proficiency. In fact, when one strips away these numerous facets of implicit scaffolding, not much seems left that can be implicated as “content per se.” Thus, our line of research shifted, through engaging in this extended collaborative process of inquiry, from a “cognitivist” search for the logico–mathematical deductive/inductive construction of a particular domain of content to a broader examination of mathematical content,
reasoning, and inquiry as a situated and mediated practice. This examination was conducted through a consideration of the potentially mundane aspects of mathematical learning and juxtaposition of case studies, whose emergent uniqueness enabled detection of para-mathematical aspects of the pedagogical dyad. Following, we propose that a deep-rooted psychological trait determined these students’ aptitude to experience the pivotal insight.

In implicitly expecting two focal artifacts in the learning environment, the combinations tower and the marbles box, to be meaningfully related, SH manifested a schooled epistemic disposition, i.e., the tacit, general problem-solving heuristic or belief that “the world makes sense,” so that ritualized social activity is trusted as meaningful, as ultimately nurturing, as challenging and satisfying. To participate successfully at school in organized practice, one must therefore engage para-mathematical routines, such as what we have called ‘suspension of pertinence.’ Reflexively, participating in the designed activity and discovering the meaningful relation between elements of the design reaffirmed and bolstered SH’s favorable epistemic disposition, such that she is likely to manifest similar behavior in future encounters with designed activities. Thus, the makes-sense epistemic disposition is—at least partially—mediated through participation in social activities. An epistemic disposition is different from epistemic knowledge (Perkins, Crismond, Simmons, & Unger, 1995), a problem-general mental tool kit. Epistemic disposition is a prerequisite to actuate epistemic agency in problem solving; to leverage any available epistemic knowledge and play the epistemic game (see also Inagaki & Hatano, 1973, on 'intellectual curiosity,' cited in Miyake, Miyake, & Shirouzu, 2006). Thus, epistemic disposition is a requisite component engendering intellective competence or intellective character (J. Greeno, cited in Gordon, 2001; Gordon & Bridglall, 2006).
Yet, whereas such epistemic disposition is certainly conducive to favorable assessment within the school system, is it ultimately fostering minds that can engage successfully in scientific inquiry, or do designs “that make sense” in fact plant false expectations that would be disappointed when the student is faced with “real world” ill-structured scientific problems or even placed in looser pedagogical frameworks? Literally, who would be the better future STEM practitioner, SH, the “Platonist” who trusts in the world’s underlying design, LF, the “agnostic” who harbors a salubrious skepticism that is not easily swayed by the para-mathematical cues of mediated practice, or perhaps RG, the “iconoclast” who repudiates social engineering completely?

[Insert Figure 16 about here]

Perhaps partaking in activities within designed learning environments instills scientific habits of mind by virtue of the accessible examples of worldly order—an order that, one comes to believe, underlies the most complex natural phenomena, whether or not the watchmaker is assumed to be blind. Thus, by instrumentalizing STEM artifacts, humans covertly instrument their minds to tacitly expect natural phenomena to fall into epistemic forms. Such is the epistemic model that authors’ of cryptic word puzzles entrain in the avid puzzler—it all makes sense, and it is up to you to determine this sense (Perec, 1978). As an earlier author remarked, setting in stone a fiat of inquiry, “Ask and it will be given to you; seek and you will find” (Matthews, 7:7-8). Such, too, is Paul Erdös’s not-too-whimsical neo-Platonist conviction that mathematical proofs yet to be discovered are already inscribed in ‘The Book.’ It is such a disposition that lends comic effect to Jean Arp’s “Les Trois Cannes” (1927; see Figure 16). This work of art tantalizes because our projected instrumentalizations of these would-be tools are
rebuffed, so that we are faced with the absurdity of futile artifacts. Such is our social experience: we expect forms to function.

Is the capacity to suspend a sense of pertinence a virtue (of inquiry) or a vice (of schooling)? That is, what are possible tradeoffs of having students engage in activities that initially appear to serve only the contained goal of completion? Humans appear to be attracted to gaming activities wherein the challenge, intrinsic motivation, guiding scheme, and inherent satisfaction is self-contained completion, such as assembling a jigsaw puzzle. Perhaps the inquisitive mind is instrumented through engaging in solving closed problems, through which general problem-solving heuristics are socialized, appropriated, and honed (Arieti, 1976; Csikszentmihalyi, Rathunde, Whalen, & Wong, 1993; Erikson, 1950/1993; Gopnik, Meltzoff, & Kuhl, 1999; Root-Bernstein & Root-Bernstein, 1999). Salomon and Perkins (1998) call our attention to the roles of social mediation in fostering students’ ‘high-road learning,’ a disposition to problem solving that is auto-regulated, intentional, and conceptually oriented. Perhaps, then, suspension of pertinence is a valuable aspect of inquiry-oriented epistemic disposition that we should foster in our young. Such fostering could be enacted through modeling the pleasure of engaging in “futile” problem solving.

Conclusions: On the Unbearable Social Construction of Mathematical Learning

This paper set out to demonstrate a specialized potential of design-based research methodology to illuminate cognitive and socio-cultural theory of learning. We have demonstrated that designers, as marshals of learning environments, are in a unique position to investigate situated mathematical reasoning. Namely, designers can modulate students’ learning experience by manipulating normative features of didactic routines. Specifically, the study discussed in this paper featured an implemented design that violated two norms of schooled
mathematics: The first activity broke off without providing students any feedback as to the veracity of their intuitive judgment nor a solution procedure to pursue its validation. The second activity was initiated as ostensibly unrelated to the first. Yet, the pertinence of the second activity to the first became apparent through insight, so that these two phenomenologically disparate activities emerged as mathematically commensurate, in line with the design rationale. Whereas the first activity did not require any construction, the second activity was labor intensive. And yet, both activities ultimately elicited an intuitive perceptual judgment. The mathematical commensurability of the two activities—that the second activity pertinently provided an apparent logical warrant for the first—emerged essentially through para-mathematical processes. That is, students came to recognize what the tool they had been instrumentalizing was in fact being instrumentalized toward. Wax on, wax off.

We conclude that the para-mathematical aspects of students’ learning processes may well have remained obscured from our professional vision were it not for the non-normative activity design. Namely, the requirement that students self-connect procedure to problem elicited a range of responses that, juxtaposed, revealed to us critical aspects of learning that had hitherto passed as transparent, as too mundane to see. Armed with these new lenses, dyad interactions announced themselves afresh, revealing for our scrutiny tacit negotiation mechanisms vital to sanctioned participation in mathematical practice.

The research problem addressed in this study was serendipitous—it emerged through the process of analyzing data from a design-based research study of students’ mathematical cognition. The “social” properties of the experimental procedure—that a researcher was actively involved in facilitating the procedure protocol—was regarded more as a methodological necessity and perhaps as foreshadowing classroom interactions than as an opportunity to attend
to social aspects of mathematical teaching and learning. That is, the design and sequencing of activities had been created in light of a pedagogical rationale—it had not been pre-mediated to enable the particular analyses reported herein. Namely, the design was guided by the premise that students connect strongly to mathematical content when they experience opportunities to appropriate it as a solution procedure. The particular design problem that suggested a sequencing rift between problem and procedure had been that whereas middle-school students have qualitatively sound probabilistic intuitions for the behavior of randomness generators that produce compound events (e.g., students know that ‘6–6’ is a rare two-dice roll), these students do not perceive the pertinence of combinatorial analysis to the rigorous investigation of these events’ expected frequency distribution. Currently, members of the research team recognize that socio-construction of mathematical knowledge is inextricably intertwined into individual reasoning (Greeno, 1998), and that any attempt to methodologically cleanse an experimental intervention from ostensibly non-core mathematical content is by default impossible. Rather, in the tradition of in-situ educational research, we hope to have demonstrated that it is possible to unthread the texture of in-vivo social interactions so as to pull out the many personal and interpersonal strings. In particular, we have attempted to present a circumspective phenomenology of mathematical artifacts.

Implications for Education Research: Design-Based Research as a Powerful Trans-Disciplinary Paradigm for Theorizing on Individual Learning in Social Contexts—Manifesto of an Emergent Phenomenology-of-Mathematical-Artifacts Design Theory

The study reported in this paper was a case of utilizing design-based research methodology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to generate theoretical-cum-pragmatic models of mediated learning processes. A successful design model is plausibly a
powerful learning model. That is, to the extent that design models are effective, it is because they capture, synthesize, and harness psychological prerequisites of cognitive development. Design-based research studies are uniquely equipped to pursue such circumspect modeling due to their intrinsic intersecting of content, pedagogy, and technology and due to the nature of their data-to-theory-to-design feedback loops, which are characteristically both rapid and iterative.

Furthermore, results of the study reported in this paper question the tacit distinction between the modeling of theory and design by demonstrating homomorphism between learning models and design models in terms of premises, agents, structure, and a focus on artifacts. A design is an artifice—an artificial administration of artifacts—and when a design ‘works,’ when it has sounded the depths of human learning and resonated with this process, this artifice reflexively becomes a phenomenon onto itself, a probe for the study of human learning. In such a study, the researcher is privileged as both participant–observer and introspective analyst of empirical data—from the drawing board to the coding table—with which the researcher is intimately and richly familiar. Specifically, the designer is poised to conduct phenomenological deconstruction of mathematical artifacts.

A circumspective Phenomenology of Mathematical Artifacts design-theory framework harbors the promise of implicating nuances of students’ learning difficulties. In particular, our epistemological shift away from considering content per se to embracing the variegated complexity of para-mathematics re-positions the strife of struggling students as requiring broad nurturing into the epistemic disposition treasured by society at large and thus critical for these students’ prospects—the socially mediated epistemic disposition that ‘the world makes sense.’

To the extent that our research process proved useful, it may be worthwhile to reflect on the cognitive trajectory exemplified by this collaborative process. We submit that our insight into
the pertinence of socio–cultural theory to design-based research was itself a conceptual blend between two pre-existing theoretical structures which, through engaging in phenomenological deconstruction of mathematical artifacts, emerged as mutually informing, co-constraining, and co-constructing. That is, our intuitive notions of ‘what artifacts should work,’ articulated in the form of a guiding design framework, learning axes and bridging tools (Abrahamson & Wilensky, 2007), were re-positioned vis-à-vis theoretical models of ‘how people learn with artifacts’ (e.g., Vérillon & Rabardel, 1995), so as to produce an integrated model that is potentially more effective. As we discuss below, this research trajectory demonstrates a gravitation of design-based research methodology toward assuming a more central role in the heart of learning-sciences frameworks.

Learning Sciences researchers with a focus on design have been operating for several decades under the tacit impression that two distinct types of theoretical models are at play: learning models and design models. The learning models have been viewed as theoretical and broad sweeping, because they attempt to articulate intellectual structures that lend coherence to an ever growing cadre of empirical studies, whereas the design models have been viewed as pragmatic and local, because they purport to function as principled frameworks for planning and actuating effective content-oriented instruction, guided by a pedagogical philosophy. We wish to question this distinction between learning models and design models and offer a reconciliation of these models in the form of an integrated model that is broad enough to capture essential agents and mechanisms at play in teaching-and-learning situations yet narrow enough such that it constitutes a template for guiding concept-targeted design.

Learning models have historically differed in their disciplinary foci on individual cognition or social–cultural construction, yet the post-antinomy models (Cole & Wertsch, 1996),
e.g., *situativity* (Greeno, 1998), position the student as an individual cognizer tasked with developing the conceptual understanding that underlies the praxis and semiosis that is the heritage of its society. In these broad-scope learning models, artifacts play a special acculturating role by tapping students’ skills and instrumenting a collective of individuals toward sufficiently sharable perspectives on worldly phenomena (Hutchins, 1995; Rogoff, 1990; Vygotsky, 1978/1930; Wittgenstein, 1953). Design models, on the other hand, position mathematical artifacts as topical catalysts of insight, embodied instantiation of pedagogical philosophy engineered to scaffold domain-specific conceptual understanding and nurture procedural fluency (Abrahamson & Wilensky, 2007; Clement, 1993; Fuson, 1998; Gravemeijer, 1994; Papert, 1980; von Glasersfeld, 1987). Thus, design models have been regarded as the ‘how to’ manuals for implementing within complex learning environments implications of the erudite theories of learning (with some scholars authoring models both of learning and of design). Yet, we argue, theories of learning and design are too intertwined and too mutually informing so as to warrant separate models. Thus, it is not just a question of modeling parsimony but also of practicality that these models blend.

Authors’ Note

The Embodied Design Research Laboratory operates as a team. The first author is the director of the lab, the second and fourth authors participated through UC Berkeley’s Undergraduate Research Apprenticeship Program, to which we are grateful, and the third author is a graduate student in the SESAME program. The data addressed in this paper were collected as part of Abrahamson’s tenure as a National Academy of Education/Spencer Postdoctoral Fellow; the data were analyzed with the support of a UC Berkeley Committee on Research (COR) Junior Faculty Research Grant (JFRG) awarded to Abrahamson. We thank other
members of EDRL, and especially Colleen Lewis and Sneha Veeragoudar Harrell, for their feedback on the manuscript. Thank you also to Tobin White for his comments.

References


Phenomenology of Mathematical Artifacts


Figure 1. Scooping four marbles from a box with equal numbers of green and blue marbles.
Figure 2. Schematic representation of students’ intuitive sense of distribution.
Figure 3. The combinations tower is the distributed sample space of the 16 unique scoops:

(a.) a student’s construction; and (b) a schematic image for expository convenience.
Figure 4. Epistemic forms and games in the first two activities of the design sequence.
Figure 5. Application of the Verillon and Rabardel (1995) instrumental-genesis model to the case of the design for the binomial.
Figure 6. The combinations tower as an ambiguous figure: (a) duck or rabbit?; (b) rows or columns?
Figure 7. Conceptual blending: Assembly of the combinations tower (above) and coming to see frequency distribution in the combinations tower (below).
Figure 8. Integrated models: Cognitive assembly stages sequence in the design for the binomial.
(pointing to 4 green card) “That itself is a group, and itself is 10…”

“…But that itself is also 10 (pointing to 3 green 1 blue card)…”

“…because 40 divided by 4, and there are 4 (gestures downwards from card along entire group), is 10. So I guess they’re equivalent.”

Figure 9. SH reflects on prior claim that cards have different probabilities
Int: “This specific card and this specific card…”

(moves cards across his body, away from the tower)

“…just…”

“…Is there more chance of getting one of these two?”

Figure 10. Interviewer pulls cards away from tower
"Figure 11. LF completes his combinatorial analysis of the 4-Block: He has created five cards."
Figure 12. LF performs a sweeping gesture encompassing the top 11 cards (the “crown”) of the combinations tower. He says that these cards are irrelevant to the combinatorial analysis.
Figure 13. The combinations tower (left) as an ambiguous figure: attending to all 16 cards (above) or only to the bottom row of 5 cards (below).
Int.: 20 of these, 20 of these, 20 of these, 20 of these, and 20 of these

**Figure 14.** The interviewer disciplines LF’s perception of the combinations tower.
Figure 15. (a, b) RG is initially unaware of the emergent number line embedded in the combinations tower; (c) The interviewer attempts to discipline RG’s perception by highlighting the plurality of outcomes in the 2b2g column.
Figure 16. Instruments devoid of instrumentation—a Dadaist’s comment: Jean Arp’s *Les Trois Cannes*. 