WHEN "THE SAME" IS THE SAME AS DIFFERENT DIFFERENCES: ALIYA RECONCILES HER PERCEPTUAL JUDGMENT OF PROPORTIONAL EQUIVALENCE WITH HER ADDITIVE COMPUTATION SKILLS

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The paper introduces the "eye-trick", an optical illusion, and argues for its viability as a didactic means to mediate between young students' naturalistic perceptual judgments and mathematical descriptions of proportional equivalence classes (e.g., 2:3=4:6=6:9=8:12=...etc.).

Cognitive-psychology (Suzuki & Cavanagh, 1998), biological (Thinus-Blanc, 1988), and developmental (Piaget & Inhelder, 1946) studies all suggest a human capacity to perform *perceptual* judgments of proportional equivalence, e.g., between two geometrically similar rectangles. Such performance appears to rely on what Cobb and Steffe (1998, p. 55; see also Gelman, 1993) call "concepts in action, enactive concepts, rather than [on] abstract concepts embodying a structural relationship between...quantities", as evidenced in students' notoriously low achievement in *numerical* proportion problems (e.g., Kaput & West, 1994). By embracing students' domain-appropriate 'enactive' knowledge, we hope to create "instruction [that] is in harmony with [learners'] schemes" (Cobb & Steffe, 1998, p. 48), and may thus pre-empt "discontinuities between the child's procedures and the child's concepts" (p. 58; see also Vygotsky, 1978; and Freudenthal, 1981, on mathematization). Specifically in the domain of ratio and proportion, the "eye-trick", a perceptual illusion (see below), may afford students an opportunity for "logico-mathematical structuration that....*goes beyond perception*" (Piaget & Inhelder, 1969, p. 49, my italics).

The motivation of this work is our belief that ratio and proportion is an advantageous conceptual entry to rational numbers (e.g., see Confrey, 1998) because ratios do not require embedded numbers, as fractions do. Fractions are *parts-to-1-whole*, and thus present the perceptual-logical challenge of 'inclusion' (e.g., Singer & Resnick, 1992). The simpler visual physical instantiations of *whole-to-whole* ratios in geometrically similar shapes suggest a simpler approach. This work was done as part of our larger project to utilize the multiplication table as a source for teaching/learning rate, ratio, and proportion as coming from iterated addition (see also Abrahamson, 2002; Cobb & Steffe, 1998; see Abrahamson & Cigan, 2002, for an outline of our curricular unit).

Method

The eye-trick involves two proportionate pictures (e.g., of heights 2cm&3cm and 4cm&6cm, Figure 1a). Children are asked to shut one eye to eliminate their stereo-

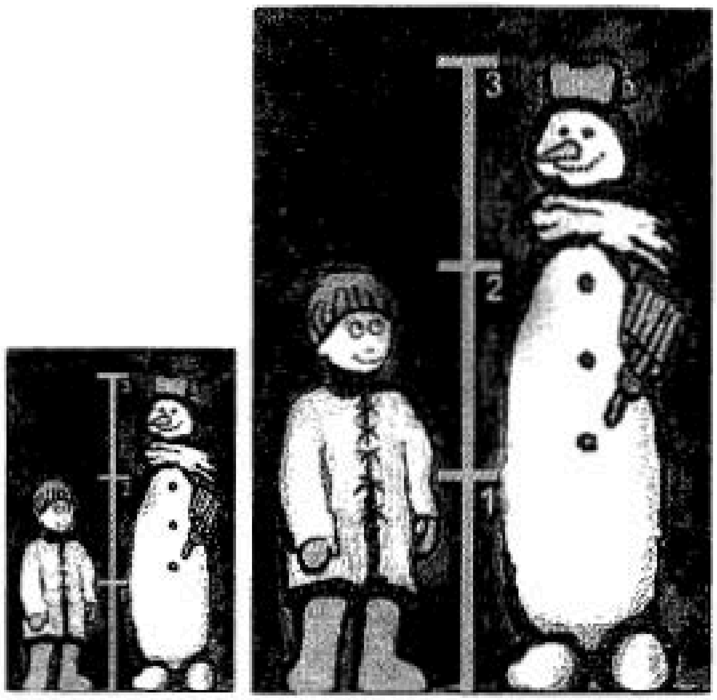

Figure 1a. Cards A and B as seen laid flat on the table.

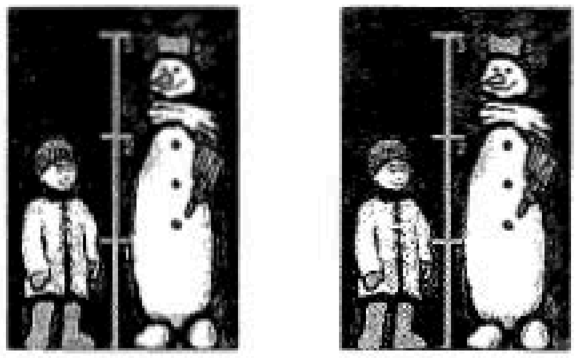

Figure 1b. Cards A and B as seen through the eye-trick illusion.

scopic vision. Holding both pictures, they then move the larger picture away from the smaller picture (farther from their eye) until they find a point where the two pictures produce images of the same size (Figure 1b). This proportion is then examined numerically by attending to the embedded ruler in each image (2&3 units in both pictures), and through measurement, using a stretchable rubber ruler. The unstretched units of this ruler correspond to 2 and 3 units of height in Card A and to 4 and 6 units of height in Card B, but the stretched units of the ruler correspond to 2 & 3 units in Card B. The entire set of materials included a total of five cards per ratio set (e.g., 2:3, 4:6, 6:9, 8:12, 10:15) as well as additional sets of cards (a 3:4 set and a 3:5 set) bearing different images of object pairs.

We have employed the eye-trick tasks both in whole-class design studies (Abrahamson & Fuson, in preparation), and in clinical interviews, of which Aliya's (8.5-year-old) interview was typical. I worked with and video-taped Aliya over three 1-hour periods spanning 15 days.

Results

Aliya (a) *saw* that two cards of different size appeared "the same"; (b) measured these cards with the stretchable ruler and tabulated these data (2&3, 4&6, in Figure 2a); (c) claimed these data were mathematically nonsensical since 3-2=1 but 6-4=2 (the differences are different); (d) sought an alternative numerical pattern to explain what she saw, wondering aloud whether the differences of 1 and 2 units, respectively, could possibly signify a trend of 1, 2, 3, etc., which would predict a difference of 3 in an additional card; (e) explored and verified her hypothesis by using a card in which the relevant dimensions were 6cm:9cm (6 & 9, Figure 2a); (f) compared these data to a case of head-start equal-rate growth (Figure 2b, Bob and Joe were born exactly one year apart); (g) discussed the viability of each table as a mathematical descriptor of some real-world class of situations; (h) practiced using her hands to simulate and differentiate equal-rate and different-rate growths: starting from holding her hands 2 and 3 "units" above the table, respectively, she raised her hands whilst either maintaining a fixed difference between them or by gradually increasing the difference; (i) re-interpreted the proportion table as modeling "unit-splitting", e.g., 3 "becomes" 6 because each 1-unit became 2 smaller units but the visible total remained the same size (Fig. 3, compare to 2* 3 as 3+3 where the total visibly doubles in size); (j) came to accept proportional equivalence as the numerical phenomenon corresponding to the stretch/shrink or "change unit" classes of real-world situations.

Ratio Change	
Danny	Snowman
2	3
4	6
6	9

Figure 2a. Tabulated measurements.

Head-Start Equal Rate	
Bob	Joe
2	3
4	5
6	7

Figure 2b. Sibling ages.

Conclusions

The eye-trick provides a powerful sensory support for understanding proportional equivalence. This visual support was successful in overriding the well-documented "additive frame", by which 2:3 cannot equal 4:6 because 3 is 1 more than 2 but 6 is 2 more than 4. It enabled Aliya to build an additive-multiplicative frame for proportion situations, initially as additive increasing-difference situations (within the ratio-table rows), and then as a multiplicative interpretation of unit splitting within both the eye-trick pictures and the ratio table (between its rows).

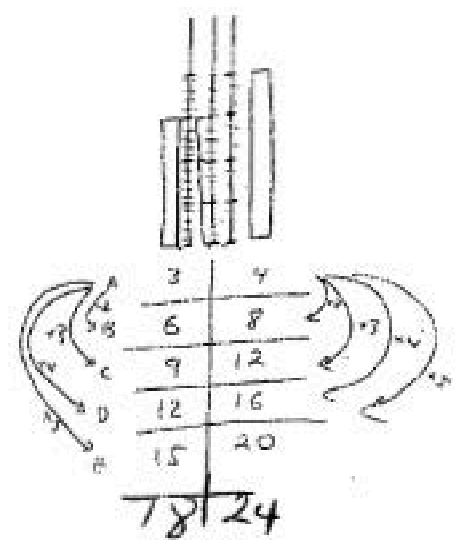

Figure 3. Explaining proportional equivalence as coming from unit-splitting interpreted multiplicatively.

Acknowledgement

This research was supported in part by the National Science Foundation under Grant No. REC-9806020. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of NSF.

References

Abrahamson, D. (2002). *Learning to see and to trust multiplication in proportion problems*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Abrahamson, D., & Cigan, C. (2002). *Awakening the sleeping beauty: The multiplication table as conceptual support for the solution of ratio and proportion situated word problems*. Unpublished manuscript.

Abrahamson, D., & Fuson, K. C. (in preparation). *Domain analysis of ratio and proportion.*

Cobb, P., & Steffe, L. (1998). Multiplicative and divisional schemes. *Focus on Learning Problems in Mathematics, 20(1)*, 45-61.

Confrey, J. (1998). Building mathematical structure within a conjecture driven teaching experiment on splitting. *Proceedings of the twentieth annual conference of the North American Chapter of the Psychology of Mathematics Education* (pp. 39-48). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Freudenthal, H. (1981). Major problems of mathematics education. *Educational Studies in Mathematics, 12*, 133-150.

Gelman, R. (1993). A rational-constructivist account of early learning about numbers and objects. *Learning and Motivation*, *30*, 61-96.

Kaput, J., & West, M. M. (1994). Missing-value porportional reasoning problems. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 237-287). Albany, NY: SUNY.

Piaget, J., & Inhelder, B. (1946). *The child's conception of space.* London: Routlage & Kegan.

Piaget J., & Inhelder, B. (1969). *The psychology of the child.* New York: Basic Books.

Singer, J. A., & Resnick, L. B. (1992). Representations of proportional relationships: Are children part-part or part-whole reasoners? *Educational Studies in Mathematics*, *23*, 231-246.

Suzuki, S., & Cavanagh, P. (1998). A shape-contrast effect for briefly presented stimuli. *Journal of Experimental Psychology: Human Perception and Performance, 24(5)*, 1315-1341.

Thinus-Blanc, C. (1988). Animal spatial cognition. In L. Weiskrantz (Ed.), *Thought without language* (pp. 371-395). Oxford: Oxford Science Publications.