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Rethinking Intensive Quantities via Guided Mediated Abduction

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Some intensive quantities, such as slope, velocity, or likelihood, are perceptually privileged in the sense that they are experienced as holistic, irreducible sensations. However, the formal expression of these quantities uses a/b analytic metrics; for example, the slope of a line is the quotient of its rise and run. Thus, whereas students' sensation of an intensive quantity could serve as a powerful resource for grounding its formal expression, accepting the mathematical form requires students to align the sensation with a new way of reasoning about the phenomenon. I offer a case analysis of a middle school student who successfully came to understand the intensive quantity of likelihood. The analysis highlights a form of reasoning called *abduction* and suggests that sociocognitive processes can guide and mediate students' abductive reasoning. Interpreting the child's and tutor's multimodal action through the lens of abductive inference, I demonstrate the emergence of a proportional concept as guided mediated objectification of tacit perception. This "gestalt first" process is contrasted with traditional "elements first" approaches to building proportional concepts, and I speculate on epistemic and cognitive implications of this contrast for the design and instruction of these important concepts. In particular, my approach highlights an important source of epistemic difficulty for students as they learn intensive quantities: the difficulty in shifting from intuitive perceptual conviction to mediated disciplinary analysis. My proposed conceptualization of learning can serve as an effective synthesis of traditional and reform-based mathematics instruction.

This paper offers a new perspective on the pedagogy of intensive quantities, which are central yet persistently challenging topics in both science and mathematics curricula (Howe, Nunes, & Bryant, 2011; Nunes, Desil, & Bell, 2003;

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J. L. Schwartz, 1988). In the physical sciences, an intensive quantity is a scale-invariant property of a system—it does not depend on the system's size or the amount of material in the system. For example, the temperature of a system is the same as the temperature of any part of it. Density, too, is an intensive property of a substance, because it does not change in accordance with the particular amount of substance under consideration. By contrast, mass and volume, which are measures of the amount of a substance, are extensive properties. I am particularly interested in what I call *perceptually privileged intensive quantities* (PPIQ) as they relate to mathematics learning.

PPIQ such as slope, velocity, or likelihood present central dilemmas for research on mathematics instruction. On the one hand, students arrive to the classroom with rich intuitive understandings of these quantities. Granted, these are naïve, holistic, and unarticulated understandings that are mostly manifest in everyday activity and colloquial expression, such as situated dealings with the steepness of a hill, the speed of a vehicle, or the likelihood of an event. Yet these naïve understandings enable students to share a fundamental sense of what it is they are studying with their teachers even before the teachers have introduced new vocabulary, notation, and procedures. Moreover, students' particular quantitative sensations, such as "very steep," can guide their evaluation of the information inherent in the formal models, for example in comparing two slopes. On the other hand, students learning PPIQ are expected to adopt radically new ways of seeing and speaking about these quantities; in particular they are required to use quotient (i.e., a/b) metrics, for example in rethinking steepness as rise over run.

What is not clear to researchers is how students accomplish the rethinking of holistic sensation in analytic form such as a quotient. How do students align the commonsense meaning of an intensive quantity and the meaning of its related quotient metric? A better theoretical understanding of this process may inform better design and instruction of these important concepts. Moreover, I believe that determining how students rethink PPIQ would inform the theorization of mathematics learning more generally, because the naïve-versus-analytic polarization of views typical of PPIQ learning brings into relief for scrutiny how instructors and students negotiate contested views of situations and their models. Looking in particular at intensive quantities, I consider the intriguing notion that instruction should begin from phenomenal gestalts rather than from the a and b of the a/b concepts. That is, students should first experience the sensation that an intensive quantity evokes and only then learn to analyze and describe the phenomenon as a measured quotient.

The conjecture that both naïve perceptions and mathematical analyses play vital roles in PPIQ learning suggests the *dialectical approach to conceptual change* (diSessa, 2008) as appropriate for studying this learning process, because this approach draws explicitly on both cognitive and sociocultural theories to explain the microgenesis of students' understanding. The dialectical approach is still nascent, and researchers are currently developing and exploring a variety of constructs and methodologies to describe the complementarity of individual and social factors in content learning (cf. diSessa, Philip, Saxe, Cole, & Cobb, 2010; Greeno & van de Sande, 2007; Halldén, Scheja, & Haglund, 2008). The approach proposed in this paper is to view mathematics learning as *guided mediated abduction,* my dialectical expansion of the logical inferential mechanism first suggested by Peirce (1931–1958). To my knowledge, this is the first attempt to carefully describe formal inferential structures and processes as inherently situated, embodied, distributed, and interpersonal. More broadly, I thus pursue a view of mathematical learning as subjective cognitive achievement designed and steered by cultural forces (see also Enyedy, 2005; Saxe, 2004).

Elsewhere I have sketched several cases of students using abductive reasoning to make sense of cultural models for PPIQ (Abrahamson, 2008). The present paper presents a dialectical case analysis of one middle school student's guided mediated abduction of likelihood. In what follows I unpack my theoretical structure and then apply it to the case analysis.

THEORETICAL BACKGROUND AND EMPIRICAL CONTEXT

I propose a view of logical inference as socioculturally situated. Specifically within educational contexts, I argue, logical inference can be construed as guided mediated abduction, that is, as the appropriation of cultural forms through a heavily scaffolded (yet) creative reasoning process. This view, I maintain, gives both constructivist and sociocultural theorists footholds to conceptualize guided reinvention as a desirable pedagogy. I begin to build my thesis by elaborating on the motivation for selecting PPIQ as a phenomenal class for the study of conceptual change. Next I situate and elaborate on my thesis in the context of a particular design for probability. Finally I explain abduction, demonstrate it by applying it to learning processes supported by my design, and suggest its epistemic and affective entailments.

Rethinking Appropriation: Toward an Empirically Based Dialectical View

The conjecture that humans perceive some intensive quantities holistically is supported by the empiricism of cognitive science (Gelman, 1998). For example, Suzuki and Cavanagh (1998) identified a cognitive faculty for discriminating shapes with similar or dissimilar aspect ratios, Resnick (1992) discussed everyday proportional reasoning, Gigerenzer (1998) argued for the ecological roots of humans' sense for frequency, Xu and Vashti (2008) demonstrated infants' capacity to gauge the representativeness of statistical samples, and Thacker (2010) showed middle school students' sensitivity to relative steepness of line segments. Yet what role might this tacit perceptual capacity play in understanding the analytic reinterpretation of intensive quantities?

The relation between tacit and analytic constructions of quantities is nuanced, and so are the pedagogical implications of this relation. Piaget and Inhelder (1969) argued that the relation between innate perceptual capacity and the natural development of relevant logico-mathematical structurations is not direct but rather is mediated through individuals' reflective abstraction on their goal-oriented spatial–temporal interactions. When these interactions are designed to support mathematical learning, however, students' intuitive perceptions may present educators with a double-edged sword because of the incompatibility of inferences from intuitive and formal views on situations (Cobb, 1989; Fischbein, 1987; Mack, 1990). In general, students are likely to resist formal analyses of sensory information when the analyses re-parse their perception by dimensions and gradations they had not attended to (Bamberger & diSessa, 2003). Granted, designers can mitigate the clash of tacit and analytic views of phenomena by creating representations of quantitative information that accommodate our species' perceptual inclinations (Abrahamson & Wilensky, 2007; Gigerenzer, Hell, & Blank, 1988), yet the pedagogical challenge for students to re-see phenomena analytically remains. And although teachers regularly help students re-see phenomena (Alibali, Phillips, & Fischer, 2009; Goodwin, 1994; Stevens & Hall, 1998), the question of how and why students might accept analytic construction as aligned with intuitive judgment remains. An example might be useful at this point.

Although this paper ultimately focuses on an empirical case pertaining to the intensive quantity of likelihood, I use for this example the intensive quantity of slope. The rhetorical advantage of elaborating my thesis via slope rather than likelihood is that sensory perception of diagonal orientation is encoded directly on the cortex (Hubel & Wiesel, 1962), whereas perception of likelihood is more epistemically complex—it is mediated via contextual mental construction, wherein the property of chance is attributed as an apparent propensity of a situation (M. Borovcnik, personal communication, August 15, 2011). Thus, slope and chance are structurally akin for the purposes of discussing mediated abduction, yet slope affords a simpler analysis.

Consider a child who has compared two inclined planes with respect to their steepness and has indicated the steeper one. This child may resist an instructor's prompt to do the following: (a) attend to, discern, and mark the rise and run inherent in each plane's incline, possibly by inscribing a diagram of the plane lying upon perpendicular axes of a Cartesian space; (b) measure these two extensive quantities of rise and run in each plane using appropriate standard units (e.g., centimeters); (c) calculate the quotient of these two pairs' measured values (e.g., 9 cm/15 cm and 14 cm/20 cm); (d) accept these two numbers (i.e., 0.6 and 0.7) as meaning the planes' respective reified property of slope; and (e) infer that one plane is steeper than the other *because* its slope value is greater (Thacker, 2010).

(Later I return to the issue of causality inherent in the final phase, which I see as implicating an emotionally challenging yet epistemically necessary rethinking of naïve perceptions not as a priori synthetic truths but as tenuous sensations warranting proof.)

Sociocultural theorists (Newman, Griffin, & Cole, 1989) explicitly perceive no theoretical difficulty in students replacing their naïve view of a phenomenon with the normative analytic view of the phenomenon. For these theorists, student appropriation of normative views through participation in the social enactment of cultural practice is the prevalent modus operandi of knowledge mediation. In fact, this perspective posits that participation in social activity precedes an understanding of the activity's rationale:

> It is not the case that the child first carries out the task because she/he shares the adult's definition of situation. It is precisely the reverse: she/he comes to share the adult's definition of situation because she/he carries out the task (through other-regulation). (Wertsch, 1979, p. 20)[1]

Endorsing the Vygotskian perspective, I view mathematics learning as the appropriation of disciplinary practice, and I examine how this practice is mediated in the context of participating in socially enacted problem-solving activities utilizing cultural–historical artifacts. At the same time, however, I have been concerned that these models of learning undertheorize learners' tacit perceptual capacity and the cognitive operations upon which appropriation is predicated (cf. Gelman, 1998; Karmiloff-Smith, 1988). That is, I accept that students can be guided to enact cultural–historical solution procedures for a particular class of mathematical problems prior to understanding the rationale of these procedures, but I persist in asking how and why the students eventually arrive at understanding these rationales. That is, I do not see doing-before-understanding as a "crime" (Freudenthal, 1971, p. 424; von Glasersfeld, 1987) just as long as students can infer how the teacher's structures cohere with and empower their earlier understandings (Cazden, 1981; Lobato, Clarke, & Ellis, 2005; D. L. Schwartz & Bransford, 1998).

[1]Vygotsky's conception of social mediation has inspired many theorists. It has been elaborated, for example, as: (a) students becoming fluent in the "leading discourse" concerning situated objects (Sfard, 2002, 2007); (b) teachers' "semiotic mediation" of mathematical re-descriptions for the situated task-based manipulation of pedagogical artifacts (Bartolini Bussi & Mariotti, 2008; Mariotti, 2009); (c) learners implicitly "instrumenting" themselves with conceptual systems inherent to objects they "instrumentalize" to attain their objectives (Artigue, 2002; Vérillon & Rabardel, 1995); (d) members of a community of practice routinizing the use of available cultural forms as the means of accomplishing personal goals for the solution of collective problems (Saxe, 2004); and (e) students appropriating available objects and inscriptions in the learning environment as semiotic means of objectifying their presymbolic notions regarding quantitative properties in situated problems (Radford, 2003).

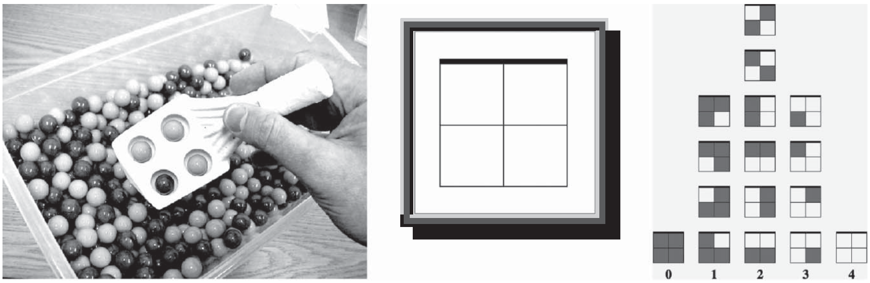
In Abrahamson (2009b) I focused on the objects that students are guided to generate in these instructional activities—objects that instructors view as constituting analytic models of some phenomenon under scrutiny yet that students do not yet see as such. Building on my interpretations of a case analysis, I suggested that students perform heuristic semiotic leaps by which they determine a way of seeing the objects as signifying their own naïve sensation regarding the properties in question. Once they see an artifact as a viable model of their tacit judgment, students retroactively validate as trustworthy and meaningful the procedure leading to the construction of the model.

In this paper I return to the case analysis as a means of elaborating on my proposed construct of guided mediated abduction. By way of setting the grounds for an explanation of mediated abduction, I now introduce the empirical context of the case.

A Design for Probability as a Context for Studying Guided Mediated Abduction

The empirical data I draw on for this study were collected during an interview-based pilot implementation of an experimental mathematics mini-unit on probability. The PPIQ in this unit was "likelihood" (Xu & Vashti, 2008), and the situation was modeled mathematically as the quotient of "number of favorable events in the event space / total number of events in the event space" ($p = .5$). For more comprehensive treatments of the design project, including a grounding of its rationale in research on probabilistic reasoning, see Abrahamson (2009b, 2009c, 2011).

In this study, participants were asked to examine an urn-type random generator: a box containing a mixture of marbles of two colors with equal numbers of green and blue marbles as well as a utensil for drawing out of the box samples of exactly four marbles (see Figure 1a). Unlike literally all designs for probability involving random generators, such as coins, dice, or spinners, at this point we did *not* let the participant experiment with the random generator so that no samples were drawn. Instead, once the participants understood how the device worked, we asked them to predict outcomes of running this experiment. Generally speaking, participants predicted correctly that the most likely sample would have two green and two blue marbles (hence "2g2b"), and asked to support this response they alluded to the equal numbers of green and blue marbles in the box. Next we guided the participants to color in template cards (see Figure 1b) so as to construct the experiment's event space through an algorithmic process formally known as *combinatorial analysis* (see Figure 1c). Namely, participants were to find all of the possible combinations (0g4b, 1g3b, 2g2b, 3g1b, 4g0b) and, for each combination, find all distinguishable four-element permutations (1, 4, 6, 4, and 1 permutations,



(a) (b) (c)

FIGURE 1 Materials used in a design for probability: (a) the marbles scooper, a random generator for drawing out ordered samples from a box full of marbles of two colors; (b) a card for constructing the experiment's event space (a stack of such cards was provided, as were two crayons, and students colored in all possible outcomes); and (c) a combinations tower, a distributed event space of the marbles-scooping experiment anticipating the conventional histogram representation of actual outcome distributions ($p = .5$).

respectively, for the five classes). We then asked them to examine what they had built and reflect on their earlier prediction.

A mathematically knowledgeable view of the event space triangulates the naïve prediction of a 2g2b plurality, because there are more elemental events with exactly two shaded cells as compared to other columns. However, the participants did not initially share this view. They found the analysis surprising, because it parsed the phenomenon along dimensions that they had not attended to and therefore included information that appeared to them as irrelevant to the problem as stated or as referring to different objects. Specifically, the event space displayed permutations and not only combinations, whereas the participants had not been attending to the internal order of green/blue marbles in the scooper but only to overall numbers by color (cf. English, 2005). The intervention thus generated interactions wherein participants were encouraged to perceive an analytic model of a phenomenon as indexing their gestalt sensation of the phenomenon. Eventually the students accepted the artifact they had built as a representation of the phenomenon's focal quantitative properties. In particular, they indicated the plurality of 2g2b elements in the event space as implying a plurality of 2g2b experiment outcomes.

In the next subsections, and later in the case analysis, I scrutinize the interaction dynamics between the researcher and participant as they negotiated competing perceptions of the phenomenon and its model. I attempt to identify "cognitive" and "sociocultural" factors that apparently contributed to the students' successful alignments. The rhetorical structure is as follows. I first apply Peirce's construct of abduction to plot the participants' deliberations as a triadic inferential reasoning process. Then I cast upon this triadic structure a Vygotskian perspective that foregrounds heavy-handed social mediation inherent to the inferential process. In so doing, I hope to demonstrate how the instructor's multimodal utterance framed the material resources both perceptually and socioepistemically and, in turn, how these actions stimulated and guided the student to figure out the mathematical principle targeted by the pedagogical design. That is, I believe that abductive inference does not transpire in an unsituated void. Rather, from a dialectical perspective I interpret abduction as crucially drawing on cultural resources.

Thus, in the remainder of this section I explain Peirce's construct of abduction that frames my dialectical inquiry of guided reinvention. This general theoretical presentation of guided mediated abduction foreshadows the empirical section of the paper, in which I demonstrate mediated abductive inference in a case analysis featuring a child who is guided to reinvent the PPIQ of likelihood.

Abduction and Artifacts

This subsection is composed of three parts. I begin with a general explanation concerning the nature of abductive reasoning. I then build on this introduction to elaborate a dialectical view of guided reinvention as mediated abduction. Finally I support the plausibility of my view by modeling the probability activity, which I outlined earlier, along the lines of mediated abduction. These structures are implemented in a later section, where I offer a case analysis of guided reinvention.

What abduction is. Abduction, a form of inferential reasoning most closely associated with the philosopher C. S. Peirce (1931–1958), can be explained through comparison with the more familiar constructs of deduction and induction. Formally speaking, deduction, induction, and abduction are logical inference sequences, and each is modeled as a triadic structure leading from a premise through to a proposition whose truth is necessary (deduction), apparent (induction), or possible (abduction). All three types of inference are predicated on the epistemological framing that within any situation under inquiry there is some general *case* that necessarily implies a local *result* due to some *rule.* The rule assigns to all members of some class of entities a particular property, the case states that a specific entity is a member of the class, and the result specifies that the particular entity (therefore) bears the property in question. An individual person entering a situation may have only partial knowledge of the totality of these three notions. Which of these three notions the individual initially knows dictates the type of inference required in order to obtain more complete knowledge of the situation.

The bean situation and its variants are often used in treatises on inferential reasoning comparing deduction, induction, and abduction. The situation involves a large sack of beans on the floor and a small pile of white beans on the table.

Note in particular how each of the three inference mechanisms can be expressed as a different permutation of the case–rule–result inference triad:

- *Deduction*. If I know that all of the beans in a sack are white (the rule) and that a pile of beans is from this sack (the case), then I can confidently deduce that the pile of beans is exclusively white (the result).
- *Induction.* If I know that a pile of beans is all from a sack (case) and that the pile is all white (result), then I have grounds to induce that perhaps all of the beans in the sack are white (rule).
- *Abduction.* If I know that all of the beans in a sack are white (rule) and that a pile of beans is all white (result), I may abduce that the pile is from the sack (case).

This event of finding white beans on a table, mundane as it seems, should be taken as a parable for the discovery of any unexplained phenomenon, such as in scientific inquiry, when the findings are compelling but initially cannot be accounted for.

Following Shank (1987, 1998), I use a modified presentation of the abductive triad: Instead of rule–result–case I use result–rule–case. This modified version lends the inferential mechanism greater phenomenological and cognitive viability. Namely, although the person entering the bean situation implicitly knows the rule prior to the inquiry process, the rule is not on the fore of the person's mind; consequently, recalling, selecting, and applying this rule occurs only after the person has encountered the puzzling result and begun to reason about it. Also following Shank, I interpolate into my narrative hypothetical inner speech that may render the paradigmatic bean situation more reminiscent of what a scientist or student may subjectively experience:

> I have just entered a room and am surprised to find upon a table a pile of white beans (result). I do not know for a fact where these beans came from. Yet as I consider this question, it occurs to me that a nearby sack contains exclusively white beans (rule). Now, if only it were true that the pile of beans is from the bag (case), it would make sense that all of these beans are white (result).

Again, what is unique about abduction, as portrayed in the bean example, is that elements in the perceptual field that ultimately bore critical information for the solution of the conundrum were not initially prominent aspects of the situation. Rather, only upon facing the perplexing presence of a pile of white beans did the individual engage in explorative reasoning through which she first detected the sack and implicated it as implying a candidate rule attributing the beans' color to their plausible origin. In the present case study, too, the child is offered a "case" as a means of rethinking an intensive quantity "result," however the child initially does not consider some elements of this case as pertinent to the situation. The pertinence of the entire case emerges even as the child determines a rule justifying this pertinence in light of the evident result.[2]

Note also the crucial yet tentative attribution of "truth" to the case. The guided mediated abduction conjecture problematizes this attribution by recognizing its socioepistemic construction. Namely, when an instructor proposes a cultural artifact as constituting a model of a situation under inquiry, it is socially incumbent upon the student to comply with this proposal and adopt the artifact as such. Yet I am interested in whether and how the student abduces the artifact's historical logic (see Harel, in press, on social vs. intellectual needs). Moreover, I am interested in how the syllogistic architecture of inferential processes mediates a transition of conviction and causality from a phenomenon to its model.

Abduction as guided reinvention. Since Athenian antiquity, logical inference has been positioned as a rarified, insulated cognitive process that ties premises through to conclusions using the "power of the mind" alone. Syllogistic reasoning and, moreover, its reification and formalization were regarded as a great epistemic achievement of the emerging Western psyche, so much so that the capacity to reason thus was evaluated as marking advanced civilizations. This appeal for the transcendent nature of logical inference was revisited and bolstered from the Enlightenment through to the 20th century by some philosophers of science. Notably, Hans Reichenbach (1956) attempted to purge inferential reasoning of any semblance of anthropomorphism by conceptualizing intuitive reasoning as a component of induction.[3] This historical, popular, and formal framing of logical processes as an exercise of sheer intelligence, however, is liable to preclude a researcher's attention to the necessarily mediated, situated, and collaborative nature of intellective competence (cf. Bamberger, 2011; Lee, 2006; Sawyer, 2007). Furthermore, any notion of inferential reasoning as a solipsistic intellective activity becomes absurd in the context of educational research into mathematical learning, wherein psychological trajectories toward reinvention are paved out a priori for students to traverse under an instructor's attentive tutelage.

In sum, when people use logical inference to solve problems, they do so within interactive cultural–historical contexts of inquiry that are populated with humans

[2]Although there has been a sustained interest among philosophers of science, cognitive scientists, and science-pedagogy researchers in Peirce's treatises on the logics of discovery (McKaughan, 2008; Prawat, 1999; Thagard, 1981), recent decades have seen an application of this work to research on mathematical learning as well. Often these publications present Peirce's constructs of abductive reasoning and hypostatic abstraction as tentatively challenging Kantian constructivism (Norton, 2008, 2009; Rivera, 2008) or offering a response to the "learning paradox," which traces back to Plato's *Meno* (Bakker, 2007; cf. Bakker & Hoffmann, 2005; Bereiter, 1985; Hoffmann, 2003).

[3]For a revealing discussion of historical relations between intuition and induction in the philosophy of science, see Braude (2012).

and artifacts. Moreover, when these contexts are patently framed as instructional settings, an instructor likely presides and facilitates student inquiry. I now examine one such context—my probability design—to explore abductive reasoning in guided reinvention.

An abduction-based framing of a design for probability. As a broad class of content domains, PPIQ are particularly conducive for modeling mathematical learning via abduction. The polarization of tacit and professional perceptions of PPIQ brings into relief educators' efforts in support of students' struggle to triangulate these perceptions. As I now explain, the tacit–analytic polarization also reveals the necessity of epistemic reorganization: For learners to achieve coherence, they must reposition a taken-as-true perceptual gestalt as the *result* of applying an emerging *rule* to a particular *case.*

Recall the general structure of perception-based designs for PPIQ. Each design involves two focal objects: (a) a phenomenon under inquiry that affords "correct" perceptual judgment for a property in question and (b) an artifact that the instructor frames as the conventional mathematical means of modeling the same properties of the phenomenon. The guided learning process culminates in a "semiotic leap" by which the student figures out how to see the mathematical model as cohering with the perceptual judgment. I now turn to my design for probability as a context for interpreting this guided reinvention protocol as engendering mediated abduction (see Table 1).

At first, students perform a naïve perceptual judgment to determine the most likely event "2g2b" (the *result*). Yet, subsequently confronted with the event space, wherein event classes vary by number of permutations (the *case*), they initially do not see it as obtaining on their earlier judgment. They resolve the conflict by reasoning that the event space would be a viable model of the phenomenon on the condition that "events with more permutations have greater likelihood" (the *rule*). In so doing, students apply a domain-general heuristic ("More A—More B"; Tirosh & Stavy, 1999). Thus, by resolving cognitive conflict, students are guided to reinvent the classicist Law of Ratio.[4]

However, these three inferential roles—result, case, rule—are not preassigned to their respective notions. Rather, I submit, these roles emerge as a syllogistic triad only through pragmatics of interpersonal discursive interaction. Namely, when the instructor positions the mathematical artifact as an apparent model

[4]The particular mathematical domain of classicist probability appears to be uniquely geared to the study of abductive inferential reasoning, because structural properties of random generators are therein viewed as a priori mechanical "causes" that are temporally antecedent to the predicted experimental "effects." Thus, when students are asked to make sense of the event space with respect to the random generator, they may be drawing on domain-general cause–effect schemes inherent to classicist probability and thus may tend to accept the analysis as explanatory of their own prediction.

TABLE 1
Coming to Understand a Mathematical Rule via Guided Mediated Abduction: Reinventing the Law of Ratio

Inferential Element	*Notion*	*Elaboration*
Result	$l_{2g2b} > l_{3g1b}$	When I see the apparently equal number of green and blue marbles in the box and consider the structure and operation of the marbles scooper, I sense that a scoop with exactly two green marbles is more likely than a scoop with exactly three. [Perceptual judgment of source phenomenon]
Case	$\#_{2g2b} > \#_{3g1b}$	When I look at the combinations tower, the alleged mathematical model of the situation, I note that there are more unique cards with exactly two green squares as compared to cards with exactly three. [Perceptual judgment of model]
Heuristic	More A—More B	A known inequality between two objects along some dimension implies a corresponding unknown inequality along some other dimension. [Intuitive domain-general heuristic]
Rule[a]	$\#_{2g2b} > \#_{3g1b} \Rightarrow l_{2g2b} > l_{3g1b}$	If it were true that a greater number of cards of one type as compared to another implies a greater likelihood of drawing that type of card from the box, then my intuitive prediction would be confirmed by the mathematical model; the model itself would thus be validated as a sensible means of analyzing the situation. [Abductive inference]

Note. l = likelihood; g = green; b = blue; # = number.
[a]In the interest of clarity, a specific numerical proposition is presented here rather than the general rule.

of the phenomenon, the students are expected to reposition their earlier intuitive judgment, which they had tacitly accepted as an a priori fact, as a pending proposition to be inferred from analysis. I view this epistemological relegation of intuitive judgment from certain to suspect as implicating an essential demand of constructivist design. In turn, I posit, investigating how instructors and students resolve this challenge hones dialectical research on guided reinvention.

This section explained the mechanisms of abduction and hypothesized its unique instantiation in educational settings. Using my design for probability as illustrative context, I argued for the inherent situatedness of inferential reasoning by demonstrating its social construction in tutorial interaction. In particular, I interpreted students' acceptance of mathematical analysis as contingent on an epistemological shift in the status of their intuitive judgment from what is known to what needs to be proven. Namely, introducing mathematical machinery into a domain of inquiry may prompt students to revisit and temporarily doubt their own perceptual convictions for phenomena in question, such that the convictions become volatile impressions warranting inferential support; the mathematical machinery that spurs this doubting and repositioning of intuitive judgment in turn takes on the role of supporting the judgment analytically. This abduction-based interpretation of guided reinvention may hone critical-pedagogy inquiry into vital issues of authority, identity, and trust pertaining to students' inclination to consider an unfamiliar artifact as a useful model of subjective perceptual experience.

With the discussion of mediated abduction, I conclude the theoretical section. The next section presents a dialectical case analysis of mediated abductive inference drawn from my design-based research studies of mathematical cognition and instruction.

CASE ANALYSIS

Li, an 11-year-old male Grade 6 student, was one of 28 Grade 4–6 students who participated in a set of individually administered semistructured clinical interviews (Abrahamson & Cendak, 2006). He had not experienced any formal introduction to probability prior to the study. Though rated by his teachers as a high achiever in mathematics, the essence of Li's responses did not differ from those of his fellow participants, and yet his persistent argumentation is particularly demonstrative of the cognitive conflicts and resolutions all students appear to have experienced. The data analysis here focuses on Li's wavering interpretations of the two objects at the center of the activity—the source phenomenon (the marbles urn) and its mathematical model (the event space)—with respect to the anticipated distribution of experimental outcomes. I interpret the tutorial interaction between the interviewer and the student as a process of guided reinvention, because the interviewer facilitated an activity in which Li experienced the subjective discovery of the classicist Law of Ratio, by which chance is rethought as the quotient of favorable events and all events in the event space. I approach Li's inferential reasoning dialectically; that is, I expand on traditional presentations of inferential reasoning as individual achievement by highlighting the intrinsic roles of the designed objects as well as the interviewer's actions that frame the objects perceptually and position them socioepistemically. In particular, I view the episode as exemplifying guided mediated abduction.

An Annotated Transcription of Li's Episode

At the outset of the interview Li predicted 2g2b scoops as the most likely outcomes, and he supported this prediction by citing the equal numbers of green and blue marbles in the box. The interviewer then asked Li to enumerate all of the different possible events, and Li listed five possible events—4b, 1g3b, 2g2b, 3g1b, and 4g. The interviewer handed Li the stack of empty cards as well as blue and green crayons and asked him to show all possible events. Li created only five cards, one for each event category, rather than all of the 16 possible outcomes that included permutations on these event categories (see the bottom row of the structure in Figure 1c). Moreover, once he had completed creating these five cards, Li immediately declared that he was changing his mind about the expected distribution, stating that there was a 1-in-5 chance of getting 2g2b (a flat distribution).

I have attempted to explain Li's abrupt change of mind by implicating the unfamiliar procedures and media he was guided to engage—specifically combinatorial analysis and blank cards—as deforming Li's tacit notion of frequency distribution. That is, Li colored in five cards so as to indicate *which* outcomes he anticipated in the hypothetical experiment. However, he did not yet know how to use these cards so as to indicate *how often* he expected these outcomes to occur. That is, he could use the cards to show only the five objects but not their expected frequencies. Then, once he had constructed these five cards that did not inscribe their respective felt frequencies, Li "read" the cards as implying a flat distribution even though this interpretation contradicted his initial guess. Granted, the literature documents cases of students apparently harboring an a priori "equiprobability bias" toward anticipated statistical distributions (Falk & Lann, 2008; LeCoutre, 1992). However, I do not know of literature documenting a *change of mind* from a heteroprobable to equiprobable distribution (Abrahamson, 2009c). In any case, I do not know of literature that explains such a bias or change as tacitly shaped by the inherent forms of disciplinary media introduced into the learning environment (cf. Bamberger & diSessa, 2003, on "ontological imperialism"). Thus, the episode supports my view of learners' subjective inference from tacit judgment as epistemically volatile amid sociocultural reframing. Careful design and facilitation are thus required if learners are to both sustain and coordinate these dialectical resources.

The interviewer then asked Li to use additional blank cards from the stack so as to create all of the possible permutations of the five events. Li consented to perform this assignment. Once he had completed the construction of the entire event space and its assembly in the form of the combinations tower (see Figure 1c), Li was asked to interpret the four cards bearing 1g3b (see column 1 in Figure 1c). In response, Li stated explicitly that the three cards above the bottom card "don't really matter," explaining that the initial task was to determine the likelihood of events, not to determine individual spatial configurations of these events that ostensibly do not obtain on the question of likelihood. Li, similar to his fellow participants, was thus reluctant to endorse order-based events as meaningful components of the event space—he apparently found it peculiar that the property of order in the random generator could possibly bear on its frequency distribution. For Li to view the *entire* event space as a viable model of the scooping experiment, he was to adjust his five-objects view to a five-*sets*-of-objects view. Indeed, the interview subsequently transpired as a negotiation of these two views (see below).

The interviewer asked Li yet again to state his expectation for the outcome frequency distribution. The following conversation ensued, in which Li again changed his mind at least twice (see the key statements that are underlined; the 3.5-min video clip can be viewed at http://tinyurl.com/Li-Dor-mov):

Li: /5 sec/ Well, actually . . . /3 sec/ <u>yeah [one out of five]</u>!

Dor: Ok.

Li: /2 sec/ Actually, /7 sec/ <u>it kinda seems like it would be six out of sixteen</u>.

Dor: Huh! Ok, so what . . . so . . . "One out of five" now went to "six out of sixteen." What . . . . how . . . //

Li: Well, it's like . . .

Dor: //That's quite a difference!

Li: Yeah. It . . . /10 sec/ Well, there are sixteen . . . /4 sec/ Well, actually . . . /10 sec/ No, it's still—<u>I think it still would be one out of five.</u>

In this interaction, Li first held firm to his view of the combinations tower as comprising 5 cards that "really matter" (the bottom row) and another 11 cards above them that were apparently irrelevant to predicting the experimental outcome distribution. Then, after some pause, Li apparently viewed the entire collection of 16 cards in accord with the mathematical concept of an event space. Yet then he returned to the 1-in-5 construal of the event space. In response, the interviewer drew Li's attention back to the marbles-scooping experiment in an attempt to create for Li an opportunity to reexperience the intuitive expectation of randomly scooping a plurality of 2g2b. To hone this conflict between the initial and later assertions, the interviewer spelled out implications of Li's 1-in-5 expectation for the experiment, namely that outcomes would be flatly distributed across the five possible events. Li reacted as follows:

> Li: /5 sec/ Actually, no. I would . . . I'm going back to . . . there's, out of all the possibilities you could get, six out of sixteen are two-and-two [6/16 of the cards are of type 2g2b], and these [indicates the 0g and the 4g cards] are only one out of sixteen, so . . . Like, what I was saying—"one-out-of-five chance"—that would mean . . . /6 sec/ 'Cause, [vehemently] *you'll get* these [hand sweeps up and down the 2g2b column] more than these [holds up the single 4g card], 'cause there's six of these and there's only one of these. So *that* [the "one-out-of-five-chance" statement] would mean that you would get about 20 percent of . . . Uhh, you would get 20 percent of the four-greens and four-blues . . . But now I'm realizing that's not true, because . . . [indicates that the 2g2b column is taller than the other columns]

Li was led to re-notice that the middle column of the combinations tower (2g2b) was taller than other columns. He inferred that the middle-column event was thus more frequent than the other events, an inference that cohered with his perceptual judgment of the situation.

The data excerpt demonstrates one student's guided struggle to coordinate holistic and analytic views. Li's initial perceptual judgment for the phenomenon's property in question was mathematically valid, yet through attempting to model the phenomenon formally he came to repudiate this evanescent notion in favor of a mathematically incorrect inference. In particular, when Li looked at the phenomenon, he did not attend to the order of the elemental events within samples, and yet event spaces require analysis, generation, classification, and enumeration of these elemental events. Operating naïvely, Li was thwarted in his attempt to articulate his correct notion using the available media. Moreover, the misfit between Li's naïve view and the analytic construction materials resulted in Li building a representation that, in turn, caused him to modify his own conclusion regarding the phenomenal properties in question. In response, the interviewer reoriented Li to the source phenomenon, and Li consequently rebooted his initial perceptual judgment. Turning back to the mathematical model, Li succeeded in seeing it as meaning his renewed intuitive judgment of the source phenomenon. Thus, Li matched two views: an intuitive view of a quantitative property in a source phenomenon and a professional view of its mathematical representation in a disciplinary model.

Subsequent to this episode, the interviewer and Li continued to discuss the likelihoods of various elemental and aggregate events represented in the event space and then worked with computer-based simulations of the experiment and discussed their relations to the marbles random generator and the event space. Qualitative analysis of the interaction suggested that Li had learned to differentiate stably between a view of the space as 16 equiprobable elemental events and as

5 heteroprobable aggregate events, at least throughout the remaining duration of the interview.

Epistemological Shifts in Guided Mediated Abduction

Li's initial judgment (2g2b) did not engender any further inferential reasoning because once the situation was framed by a property in question (likelihood), he perceived this PPIQ as a holistic magnitude in his visual field. Subsequently, though, the artifacts Li produced as he performed the construction task (the event space) took on the epistemological role of new information that Li was prompted to consider as an accepted cultural structure bearing on the initial judgment task.

When this tutorial interaction is viewed as a case of guided mediated abduction, certain epistemological issues that may be critical to reinvention are revealed that were only implicit to the earlier microgenesis. Namely, the artifact Li built was socioepistemically positioned by the interviewer as a *case* to be reckoned with, and, as such, Li's initial tacit judgment was devalued as a mere *result* still warranting verification. This relegation in the epistemological status of Li's initial judgment from something that is compelling to something that ought to be explained was structured by the available artifacts and steered by the discursive interaction. As Sfard (2007) wrote, "The mediating discursive routine drives a wedge between the formerly undistinguishable acts of recognition and naming and, as a result, the act of identification is now split into a series of autonomous steps" (p. 603). In response, Li searched for a *rule* that, if applied to the case of the entire event space, would imply his initial perceptual judgment. In sum, Li's abductive inference was mediated by the tutor's discursive framing of the artifacts both perceptually and socioepistemically. Abduction can be mediated.

CONCLUSION

In this paper I expanded on the work of C. S. Peirce to model how one student reinvented a mathematical concept as a case of guided mediated abduction, that is, as socially orchestrated creative logical inference. I conducted my investigation within the empirical context of implementing a design for likelihood. Likelihood, similar to slope, density, and velocity, is a PPIQ, and this conceptual class, I reckon, bears unique affordances both for mathematics learning and for research on this learning.

For learners, PPIQ evoke a holistic sensation of magnitude; then, through appropriate instructional process, these presymbolic sensations ground the quotient-based disciplinary metrics (e.g., the sensation of steepness is denoted by rise/run slope). This semiotic emergence—coming to view formal models as signifying holistic sensation—is not trivial and requires careful design and facilitation in the form of both material and discursive mediation. Namely, instruction of PPIQ calls for: (a) the availability of both the source phenomenon and suitable media for modeling the phenomenon as well as (b) a socioepistemic framing of the mathematical model as explanatory of the holistic judgment.

To elaborate, my study sketched a heuristic design framework for mathematics teachers to foster conceptual change via guided mediated abduction, at least for intensive quantities. I focused on a critical moment in learning in which the student struggles to align two sources of information concerning a property of an actual intensive quantity: (a) the naked-eye view of the source phenomenon and (b) the disciplined-eye view of its mathematical model. To match these views, the student in this case study determined a principle for interpreting the model as signifying the sensation, and this principle was precisely the pedagogically targeted mathematical content of the instructional design. Namely, he was guided to determine the Law of Ratio to explain how an event space signifies likelihood. It thus appears important that students have opportunities to judge the properties of PPIQ before participating in the enactment of formal analysis procedures. These initial sensations evoke the fundamental meaning of a mathematical notion. Learning writ large is a quest for symbol grounding, a struggle to determine a principle by which a proposed model signifies a sensation.

In terms of the abduction nomenclature, when students are asked to judge and infer some target property of a situation, they initially construe this property as a "case" bearing no further justification. Yet once they are guided to construct an artifact that the teacher frames as a mathematical model of the situation, the students experience a social need to support their judgment not with the situation but with the model. Consequently, now the *model* becomes the "case" and students seek a "rule" for seeing the model such that it implies their initial judgment of the situated property as a "result." In a sense, the progression respects students' naked-eye perception yet dresses it up.

My tentative claim for the pedagogical power of guided mediated abduction comes with the caveat that the claim demands evaluation via experimental designs that explicitly compare student learning under direct and abduction-oriented instruction. I would recommend conducting such studies in the context of intensive quantities.

The guided mediated abduction hypothesis has implications for mathematics teachers' practice, in particular for how teachers might think about the goal of deep understanding (at least with respect to the curricular content of PPIQ). Guided mediated abduction is a form of instruction that attends closely to students' intuitive notions while at the same time structuring their sense making of mathematical analyses. Teachers can play pivotal roles in making mathematics meaningful for their students by guiding the students both to draw on their intuitions for situated phenomena and to rethink these sound intuitions in light of formal models. At its broadest, learning as guided mediated abduction implies that educators should (a) create situations that enable students to form mathematically correct judgments pertaining to a specified property of some targeted quantity, even if these judgments are naïvely formed and only qualitative; (b) walk students through analytic procedures that result in constructing the mathematical model for the property in question; and (c) guide students back and forth between the phenomenon and its model, highlighting aspects of the model that afford its heuristic perception as signifying the initial judgment. As a result of participating in these lessons, students should be able to accept the analysis process and rehearse it meaningfully.

Guided mediated abduction offers an effective synthesis of traditional and reform-based mathematics instruction. Namely, the approach respects teachers' mandated charge to ensure student acculturation into mathematical practice while respecting the children's intellectual need and agency to make sense of these artifacts of their mathematical heritage. I do not view this synthesis as a compromise engineered to appease vying camps in the policy debates. Rather, I view guided mediated abduction as offering an empirically based, synthetic sociocognitive conceptualization of reinvention that could inform design and administration of institutionalized mathematics education.

Also, guided mediated abduction is certainly not instructors' only alternative pedagogical framework for supporting student reinvention of mathematical concepts. For example, D. L. Schwartz (1995), Asterhan and Schwarz (2009), and White and Pea (2011) have all demonstrated cases of student dyads/groups developing and abstracting knowledge structures via unguided collaboration and reflection on instructional tasks. In my own recent work with embodied interaction technology I have documented cases of students bootstrapping principles of proportional reasoning via appropriating available mathematical instruments to better enact, explain, or evaluate their interaction strategies (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011; Howison, Trninic, Reinholz, & Abrahamson, 2011).

Yet conceptual change via guided mediated abduction has theoretical implications, too. Specifically, from neo-Piagetian perspectives, it may appear peculiar that students accept analyses of phenomenal properties through making heuristic semiotic leaps rather than by building conceptual schemes piecemeal (cf. Abrahamson & Cigan, 2003; Kalchman, Moss, & Case, 2000; Lamon, 2007). I thus join Norton (2009) in viewing Peircean analysis of mathematics learning as posing questions for constructivists. At the same time, I recognize that sociocultural theorists still need to account for students' perceptual primitives and inferential reasoning mechanisms that are at play in the development of disciplinary skills. Ultimately, though, I view these mutual challenges across the cognitive–sociocultural divide as honing the work of educational theorists engaged in constructing dialectical perspectives on learning.

In their dialectical investigations, educational theorists whose empirical work consists of studying mathematical learning could avail themselves of frameworks for designing instructional materials that leverage the unique methodological affordances of PPIQ (see Abrahamson, 2009a; Abrahamson et al., 2011; Abrahamson & Wilensky, 2007). Instructional designs emanating from these studies could then be tested and incorporated into mainstream curricula. As researchers develop these frameworks, a constant concern involves the question of students' general dispositions toward the mathematical analyses they are required to adopt. On the one hand, "[students] have no other option than to engage in the leading discourse even before having a clear sense of its inner logic and of its advantages" (Sfard, 2007, p. 607). Yet on the other hand, engaging in this leading discourse demands of both teachers and students the trust and courage to see the world as others do (Abrahamson, Gutiérrez, & Baddorf, in press).

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REFERENCES

Abrahamson, D. (2008, December). *The abduction of Peirce: The missing link between perceptual judgment and mathematical reasoning?* Paper presented at the Townsend Working Group in Neuroscience and Philosophy, University of California, Berkeley. Available at http://www.archive.org/details/ucb_neurophilosophy_2008_12_09_Dor_Abrahamson

Abrahamson, D. (2009a). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics*, *70*(1), 27–47.

Abrahamson, D. (2009b). Orchestrating semiotic leaps from tacit to cultural quantitative reasoning—The case of anticipating experimental outcomes of a quasi-binomial random generator. *Cognition and Instruction*, *27*, 175–224.

Abrahamson, D. (2009c). A student's synthesis of tacit and mathematical knowledge as a researcher's lens on bridging learning theory. *International Electronic Journal of Mathematics Education*, *4*(3), 195–226.

Abrahamson, D. (2011). Towards instructional design for grounded mathematics learning: The case of the binomial. In N. L. Stein & S. Raudenbush (Eds.), *Developmental cognitive science goes to school* (pp. 267–281). New York, NY: Routledge.

Abrahamson, D., & Cendak, R. M. (2006). The odds of understanding the law of large numbers: A design for grounding intuitive probability in combinatorial analysis. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the Thirtieth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 1–8). Prague, Czech Republic: PME.

Abrahamson, D., & Cigan, C. (2003). A design for ratio and proportion. *Mathematics Teaching in the Middle School*, *8*, 493–501.

Abrahamson, D., Gutiérrez, J. F., & Baddorf, A. K. (in press). Try to see it my way: The discursive function of idiosyncratic mathematical metaphor. *Mathematical Thinking and Learning*, *14*(1).

Abrahamson, D., Trninic, D., Gutiérrez, J. F., Huth, J., & Lee, R. G. (2011). Hooks and shifts: A dialectical study of mediated discovery. *Technology, Knowledge, and Learning*, *16*(1), 55–85.

Abrahamson, D., & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning*, *12*(1), 23–55.

Alibali, M. W., Phillips, K. M. O., & Fischer, A. D. (2009). Learning new problem-solving strategies leads to changes in problem representation. *Cognitive Development*, *24*(2), 89–101.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245–274.

Asterhan, C. S. C., & Schwarz, B. B. (2009). The role of argumentation and explanation in conceptual change: Indications from protocol analyses of peer-to-peer dialogue. *Cognitive Science*, *33*, 373–399.

Bakker, A. (2007). Diagrammatic reasoning and hypostatic abstraction in statistics education. *Semiotica*, *164*(1/4), 9–29.

Bakker, A., & Hoffmann, M. H. G. (2005). Diagrammatic reasoning as the basis for developing concepts: A semiotic analysis of students' learning about statistical distribution. *Educational Studies in Mathematics*, *60*(3), 333–358.

Bamberger, J. (2011). The collaborative invention of meaning: A short history of evolving ideas. *Psychology of Music*, *39*(1), 82–101.

Bamberger, J., & diSessa, A.A. (2003). Music as embodied mathematics: A study of a mutually informing affinity. *International Journal of Computers for Mathematical Learning*, *8*(2), 123–160.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. D. English, M. G. Bartolini Bussi, G. A. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (2nd rev. ed., pp. 720–749). Mahwah, NJ: Erlbaum.

Bereiter, C. (1985). Towards the solution of the learning paradox. *Review of Educational Research*, *55*(2), 210–226.

Braude, H. D. (2012). *Intuition in medicine: A philosophical defense of clinical reasoning*. Chicago, IL: University of Chicago Press.

Cazden, C. B. (1981). Performance before competence: Assistance to child discourse in the zone of proximal development. *Quarterly Newsletter of the Laboratory of Comparative Human Cognition*, *3*(1), 5–8.

Cobb, P. (1989). A double-edged sword [Review of the book *Intuition in Science and Mathematics*]. *Journal for Research in Mathematics Education*, *20*(2), 213–218.

diSessa, A. A. (2008). A note from the editor. *Cognition and Instruction*, *26*, 427–429.

diSessa, A. A., Philip, T. M., Saxe, G. B., Cole, M., & Cobb, P. (2010, April/May). *Dialectical approaches to cognition*. Symposium conducted at the annual meeting of American Educational Research Association, Denver, CO.

English, L. D. (2005). Combinatorics and the development of children's combinatorial reasoning. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 121–141). Dordrecht, The Netherlands: Kluwer.

Enyedy, N. (2005). Inventing mapping: Creating cultural forms to solve collective problems.*Cognition and Instruction*, *23*, 427–466.

Falk, R., & Lann, A. (2008). The allure of equality: Uniformity in probabilistic and statistical judgment. *Cognitive Psychology*, *57*, 293–334.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, The Netherlands: D. Reidel.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*, 413–435.

Gelman, R. (1998). Domain specificity in cognitive development: Universals and nonuniversals. In M. Sabourin, F. Craik, & M. Robert (Eds.), *Advances in psychological science*: *Vol. 2*. *Biological and cognitive aspects* (pp. 50–63). Hove, England: Psychology Press.

Gigerenzer, G. (1998). Ecological intelligence: An adaptation for frequencies. In D. D. Cummins & C. Allen (Eds.), *The evolution of mind* (pp. 9–29). Oxford, England: Oxford University Press.

Gigerenzer, G., Hell, W., & Blank, H. (1988). Presentation and content: The use of base rates as a continuous variable. *Journal of Experimental Psychology: Human Perception and Performance*, *14*, 513–525.

Goodwin, C. (1994). Professional vision. *American Anthropologist*, *96*, 603–633.

Greeno, J. G., & van de Sande, C. (2007). Perspectival understanding of conceptions and conceptual growth in interaction. *Educational Psychologist*, *42*(1), 9–23.

Halldén, O., Scheja, M., & Haglund, L. (2008). The contextuality of knowledge: An intentional approach to meaning making and conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 509–532). New York, NY: Routledge.

Harel, G. (in press). Intellectual need. In K. Leatham (Ed.), *Vital directions for mathematics education research*. New York, NY: Springer.

Hoffmann, M. H. G. (2003). Peirce's "diagrammatic reasoning" as a solution of the learning paradox. In G. Debrock (Ed.), *Process pragmatism: Essays on a quiet philosophical revolution* (pp. 121–143). Amsterdam, The Netherlands: Rodopi.

Howe, C., Nunes, T., & Bryant, P. (2011). Rational number and proportional reasoning: Using intensive quantities to promote achievement in mathematics and science. *International Journal of Science and Mathematics Education*, *9*, 391–417.

Howison, M., Trninic, D., Reinholz, D., & Abrahamson, D. (2011). The Mathematical Imagery Trainer: From embodied interaction to conceptual learning [CD-ROM]. In G. Fitzpatrick, C. Gutwin, B. Begole, W. A. Kellogg, & D. Tan (Eds.), *Proceedings of the annual meeting of the Association for Computer Machinery special interest group on Computer Human Interaction (CHI 2011), Vancouver, May 7–12, 2011* (Vol. "Full Papers," pp. 1989–1998). New York, NY: ACM Press.

Hubel, D. H., & Wiesel, T. N. (1962). Receptive fields, binocular interaction and functional architecture in the cat's visual cortex. *Journal of Physiology (London)*, *160*, 106–154.

Kalchman, M., Moss, J., & Case, R. (2000). Psychological models for the development of mathematical understanding: Rational numbers and functions. In S. Carver & D. Klahr (Eds.), *Cognition and instruction: 25 years of progress* (pp. 1–38). Mahwah, NJ: Erlbaum.

Karmiloff-Smith, A. (1988). The child is a theoretician, not an inductivist. *Mind & Language*, *3*(3), 183–195.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte, NC: Information Age.

LeCoutre, V. P. (1992). Cognitive models and problem spaces in "purely random" situations. *Educational Studies in Mathematics*, *23*, 557–568.

Lee, C. D. (2006). The teaching and learning of intellective competence. In E. W. Gordon & B. L. Bridglall (Eds.), *Affirmative development: Cultivating academic ability* (pp. 155–188). New York, NY: Rowman & Littlefield.

Lobato, J., Clarke, D., & Ellis, A. B. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal for Research in Mathematics Education*, *36*(2), 101–136.

Mack, N. K. (1990). Long-term effects of building on informal knowledge in a complex content domain: The case of multiplication of fractions. *Journal of Mathematical Behavior*, *19*(3), 307–332.

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: The role of the teacher. *ZDM: The International Journal on Mathematics Education*, *41*, 427–440.

McKaughan, D. J. (2008). From ugly duckling to swan: C. S. Peirce, abduction, and the pursuit of scientific theories. *Transactions of the Charles S. Peirce Society*, *44*(3), 446–468.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: Working for cognitive change in school*. New York, NY: Cambridge University Press.

Norton, A. (2008). Josh's operational conjectures: Abductions of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education*, *39*, 401–430.

Norton, A. (2009). Re-solving the learning paradox: Epistemological and ontological questions for radical constructivism. *For the Learning of Mathematics*, *29*(2), 2–7.

Nunes, T., Desil, D., & Bell, D. (2003). The development of children's understanding of intensive quantities. *International Journal of Educational Research*, *39*, 651–675.

Peirce, C. S. (1931–1958). *Collected papers of Charles Sanders Peirce* (C. Hartshorne & P. Weiss, Eds.). Cambridge, MA: Harvard University Press.

Piaget, J., & Inhelder, B. (1969). *The psychology of the child* (H. Weaver, Trans.). New York, NY: Basic Books. (Original work published 1966)

Prawat, R. S. (1999). Dewey, Peirce, and the learning paradox. *American Educational Research Journal*, *36*, 47–76.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, *5*(1), 37–70.

Reichenbach, H. (1956). *The direction of time*. Berkeley: University of California Press.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373–429). Hillsdale, NJ: Erlbaum.

Rivera, F. D. (2008). On the pitfalls of abduction: Complicities and complexities in patterning activity. *For the Learning of Mathematics*, *28*(1), 17–25.

Sawyer, R. K. (2007). *Group genius: The creative power of collaboration*. New York, NY: Perseus Books Group.

Saxe, G. B. (2004). Practices of quantification from a sociocultural perspective. In K. A. Demetriou & A. Raftopoulos (Eds.), *Developmental change: Theories, models, and measurement* (pp. 241–263). New York, NY: Cambridge University Press.

Schwartz, D. L. (1995). The emergence of abstract representations in dyad problem solving. *Journal of the Learning Sciences*, *4*, 321–354.

Schwartz, D. L., & Bransford, J. D. (1998). A time for telling. *Cognition and Instruction*, *16*, 475–522.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41–52). Hillsdale, NJ: Erlbaum.

Sfard, A. (2002). The interplay of intimations and implementations: Generating new discourse with new symbolic tools. *Journal of the Learning Sciences*, *11*, 319–357.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you—Making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences*, *16*, 567–615.

Shank, G. (1987). Abductive strategies in educational research. *American Journal of Semiotics*, *5*, 275–290.

Shank, G. (1998). The extraordinary ordinary powers of abductive reasoning. *Theory & Psychology*, *8*, 841–860.

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107–149). New York, NY: Cambridge University Press.

Suzuki, S., & Cavanagh, P. (1998). A shape-contrast effect for briefly presented stimuli. *Journal of Experimental Psychology: Human Perception and Performance*, *24*, 1315–1341.

Thacker, I. E. (2010). *Not too slippery a slope: Fostering student grounding of the mathematics of slope in perceptions of steepness*. Unpublished master's thesis, University of California, Berkeley.

Thagard, P. (1981). Peirce on hypothesis and abduction. In K. L. Ketner (Ed.), *Proceedings of the C. S. Peirce Bicentennial International Congress* (pp. 271–274). Lubbock: Texas Tech University Press.

Tirosh, D., & Stavy, R. (1999). Intuitive rules: A way to explain and predict students' reasoning. *Educational Studies in Mathematics*, *38*, 51–66.

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education*, *10*(1), 77–101.

von Glasersfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 3–18). Hillsdale, NJ: Erlbaum.

Wertsch, J. V. (1979). From social interaction to higher psychological processes: A clarification and application of Vygotsky's theory. *Human Development*, *22*(1), 1–22.

White, T., & Pea, R. (2011). Distributed by design: On the promises and pitfalls of collaborative learning with multiple representations. *Journal of the Learning Sciences*, *20*, 489–547.

Xu, F., & Vashti, G. (2008). Intuitive statistics by 8-month-old infants. *Proceedings of the National Academy of Sciences*, *105*, 5012–5015.