

### **Embodied Artifacts: Coordinated Action as an Object-to-Think-With**

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#### 1. Let's Begin With the Design

Picture this. You're seated at a desk across from a tall vertically oriented screen. You're handed two devices, one for each hand. You are told: "Make the screen green."

What you are not told is that the height of each of your hands above the desk, say 3 inches on the left and 1 inch on the right, is being automatically measured (see Figure 1a). Furthermore, the ratio between these heights is being calculated (so, 3:1) and compared against a "correct" ratio that has been set on the interviewer's console, say 1:2.

The screen is red. You raise your right hand to 12" (so, 3" & 12", i.e., a 1:4 ratio), the screen turns yellow (see Figure 1b). Close! You raise your left hand (so 6" and 12", a 1:2 ratio), the screen turns green (see Figure 1c). Good! Wait, are you learning anything?

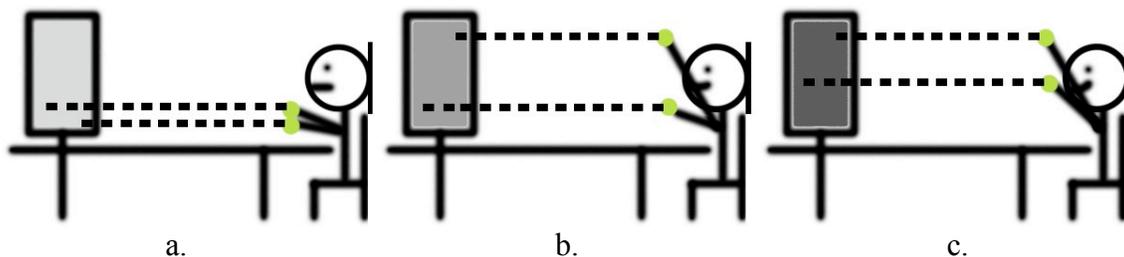


Figure 1. Searching for green: (a) incorrect performance; (b) almost correct performance; (c) correct performance (1:2 ratio between hand heights)

You're next asked to find green somewhere else. Now, if you're anything like our 4<sup>th</sup>-6<sup>th</sup> grade participants, you will "lock" the distance between your hands in a fixed unit and move them up or down. The screen will turn red (see transition from Figure 2a to 2b, below). In response, you'd relax the fixed distance between your hands, eventually striking green again (see Figure 2c, below). Fine. But are you learning anything yet?

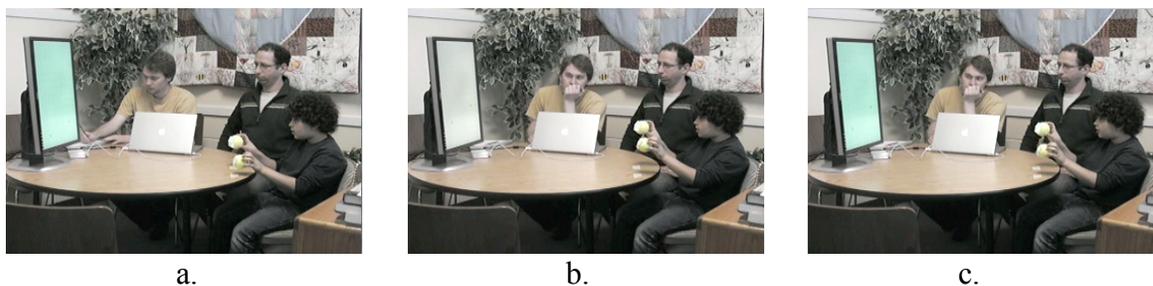
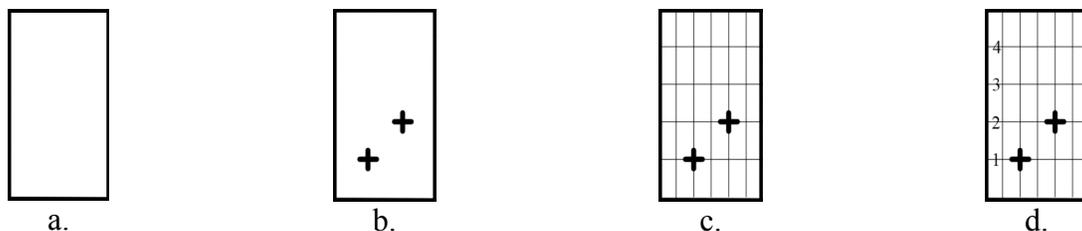


Figure 2. From fixed distance to variable distance: (a) Itamar finds his first "green pair"; (b) his fixed-distance motion upward turns the screen red; (c) he lowers his left hand and finds a new green position. Compare 3a and 3c, the two "green pairs." See that the hands (and therefore the crosshairs) are higher in 3c than in 3a, and their distance is greater.

<sup>1</sup>Design rationale and early build were first reported in Abrahamson and Howison (2008).

Up to now, you have been working in the continuous-space mode (see Figure 3a, below). Again, if you are remotely similar to our study participants, then by now you have probably determined and verbally articulated a rule to the effect that, “The higher you go on the screen, the greater the distance should be between your hands.” Next, two crosshairs appear on the screen (see Figure 3b), and you quickly recognize that they are mirroring your hands. They *are* your hands (Clinton, 2006; Merleau-Ponty, 1958/2005; Polanyi, 1967). You now have a better sense of the playing field—the spatial window wherein your gestures affect the color of the screen—and you become more acutely aware that your hands’ *locations* are governing the color. Next, a grid appears on the screen (see Figure 3c, below). The grid shifts your attention away from the distance “object” you have been carefully manipulating and monitoring and, instead, suggests that each hand location could be re-construed as a height above the datum line—a height that can be quantified in terms of discrete units (e.g., 1 and 2 units). You “snap to grid” and devise new strategies that use discrete-quantity relations, for example the recursive rule for transitioning from one green spot to the next: for every 1 box you raise your left hand, raise your right hand by 2 boxes. Next, numerals appear to the left of the grid (see Figure 3d, below)—you are thus spared the work of counting the boxes and instead refer to the numbers (this is helpful when the grid goes up to the 20’s and beyond). Moreover, specific “green” number pairs, such as 3 and 6, evoke basic arithmetic operations and “facts,” so that you recognize that the right hand should always be double the left hand. You might also state that the left hand is rising by 1’s while the right hand is rising (faster) by 2’s, and that their distance (difference) is growing by 1 each iteration. So now we are using numbers! Ok, but have you learned anything?



*Figure 3.* Four display configurations: (a) continuous-space mode; (b) continuous-space mode with crosshairs; (c) crosshairs with grid overlay; (d) crosshairs with grid overlay and y-axis numerals.

We believe you have indeed learned something. In fact, we tentatively claim that:

- students working with this interactive device, which we call a *Mathematics Image Trainer (MIT)*, occasion guided opportunities to develop the core ideas of proportionality “from scratch”; Moreover,
- students’ significant discoveries occur even before the quantification tools (the grid and numerals) have been overlain, specifically when they first realize that a fixed-distance transition away from green fails to sustain green, so that alternative strategies are needed; Our data suggest that
- students’ proportional strategies evolve from naïve additive reasoning;
- students’ strategies may transfer beyond the MIT to new situations, thus demonstrating the development of a utilization schema as a residual effect;
- students are building a dynamical image schema (Johnson, 1987; Martin, 2008;

Pirie & Kieren, 1994; Presmeg, 1992) for proportionality—an ambidextrous schema that mundane, extracurricular life experiences do not occasion for appropriation and simulation as reasoning tools (cf. Barsalou, 1999; diSessa, 1993), so that intervention is required to “induce” the image. The MIT may constitute a viable means of promoting and grounding students’ proportional reasoning by embodying a coordinated artifact as an object-to-think-with.

## 2. Background and Objectives of a Design-Based Research Study of Embodied Learning

Cognitive scientists are becoming increasingly aware of an imminent paradigm shift in the theorization of human reasoning. Evidence is growing to the effect that the mainstay of cognitive activity is not the processing of syntactical propositions but the simulations of multi-modal dynamical imagery drawn from lived experience (Barsalou, 1999, 2008; Glenberg, 1997). We have been intrigued by the broad resonance of these findings with dominant perspectives in cognitive psychology (Kosslyn, 2005; Schwartz & Black, 1999; Schwartz & Heiser, 2005; Varela, Thompson, & Rosch, 1991), cognitive development (Case & Okamoto, 1996; Piaget, 1968), neuro-psychology (Damasio, 2000) and neuro-cognitive development (Dehaene, 1997), cognitive linguistics (Lakoff, 1987; Lakoff & Johnson, 1980, 1999; Lakoff & Núñez, 2000), phenomenological philosophy (Dreyfus, 1990; Heidegger, 1962; Husserl, 1962, 1970, 2000; Jay, 1993; Levinas, 1998; Merleau-Ponty, 1958/2005), literacy studies (Zwaan, 2004), informatics (Dourish, 2001), mathematics and logic (Barwise & Etchemendy, 1991; P. J. Davis, 1993), and in particular mathematics education (Botzer & Yerushalmy, 2008; Freudenthal, 1986; Kieren, Pirie, & Gordon Calvert, 1999; Martin, 2008; Nemirovsky & Borba, 2004; Nemirovsky & Ferrara, 2009; Nemirovsky, Noble, Ramos-Oliveira, & DiMattia, 2003; Nemirovsky, Tierney, & Wright, 1998; Pirie & Kieren, 1994; Presmeg, 2006).

Oriented by the “grounded cognition” or “embodied cognition” paradigm, our research thus embarks from the premise that mathematical learning is the development of multimodal dynamical images and mathematical reasoning is the simulation thereof. We further assume that students who demonstrate difficulty with a particular mathematical concept nevertheless possess the cognitive wherewithal for leveraging imagery toward mathematical understanding—what these students lack is the initial requisite image that would constitute the cognitive substrate for that concept. Implicit in our approach is the view that effective instruction should create opportunities for students to develop a family of images that become resources for reasoning about a class of mathematical situations. Our general working conjecture is that *some mathematical concepts may be difficult to learn precisely because our everyday experience fails to provide adequate opportunities to develop the requisite body-based imagery underlying those specific concepts.*

Acting on the above rationale, we are conducting a study in which we provide students with a “ready-made” visual–kinesthetic basis for developing imagery pertaining to a difficult mathematical concept and then monitor the effects of such provision on their understanding of the targeted mathematical concept. This initially *amathematical* experience, which is induced through technological mediation, is to play a pedagogical role analogous to concrete techno–scientific artifacts, such as an abacus, pendulum, or dice, in terms of creating opportunities for guided reflection, mathematization, and reinvention of a cultural systems of knowledge (cf. Bartolini Bussi & Mariotti, 2008). Intrinsic to our design rationale is the principle of incrementally introducing

mathematical tools that empower students' goal-oriented actions even as they shift the students' perceptions and reasoning in nuanced yet conceptually critical ways (Abrahamson, 2009; Artigue, 2002; cf. Bamberger & diSessa, 2003). Our major analytic focus is on students' conceptual development at the tacit-cultural interface. There, embodied strategies are transformed even as they are signified by disciplinary forms—the strategies become “crystallized templates of action and schematized representations of ways of doing things as discovered in the history of human civilization” (Stetsenko, 2002, p. 129).

### 3. Focal Disciplinary Subject Matter: Mathematics—Proportional Progression

We chose the mathematical content domain of proportionality (e.g.,  $2:3 = 4:6$ ) both because the learning of rational numbers has historically been fraught with conceptual impediments (for a recent review, see Lamon, 2007) and because this content lends itself to challenging enactive embodiment, as follows.

Perhaps the most frequently demonstrated (and lamented) student error in the mathematics-education literature is the tendency to inappropriately apply strictly additive reasoning to multiplicative situations, for example reasoning that  $2/3 = 4/5$  because the difference is equivalent in both fractions (G. E. Davis, 2003; Post, Cramer, Behr, Lesh, & Harel, 1993). We conjecture that students' confusion stems from lack of suitable dynamical imagery in which to ground their understanding of proportionality. The Mathematics Image Trainer (see Section 1) is our attempt to create a technological mechanism that would potentially enable students to develop dynamical images for proportion. The current study is our first implementation of the technological design as a means of evaluating our conjecture through investigating student learning with the MIT.

We focused on the concept of *proportional progression*, that is, a sequence of equivalent ratios or fractions whose ordered pairs respectively increase by fixed amounts at each count (e.g., in the sequence  $2/3, 4/6, 6/9$ , etc., the numerator grows by 2 for every 3 the denominator grows). In an earlier pilot study, we demonstrated that students' conceptual difficulty in learning this progression appears to coincide with their physical inability to act out the progressions with their hands (Fuson & Abrahamson, 2005). In that study, we asked 5<sup>th</sup> graders to lay both hands on the desk and then raise one hand by 2 inches per beat while raising the other hand simultaneously by 3 inches per beat. Many students maintained a constant vertical distance between the levels of their rising hands.

To the extent that our “embodied Whorfian hypothesis” stands—that is, that students' imagistic “vocabulary” indeed determines and constrains their “mathematical perception”—students' challenge of executing a proportional progression *physically* may delimit the development of its *mental* simulacrum. Moreover, the sheer rareness of everyday chances to embody ambidextrous covariation suggests that it is unlikely that students construct the relevant imagery spontaneously, without intervention. Therefore, we decided to initiate opportunities for students to practice this physical action. The MIT is a computational “phenomenalization” (Pratt, Jones, & Prodromou, 2006) of proportion.

Because we wished to work with the two-hands gesture, we interpreted proportion as an *isomorphism of measure* (Vergnaud, 1983), wherein the  $x$  and  $y$  in the  $x:y$  ratio are experienced as separate magnitudes, such that there is no immediate meaning to a mix of these two measures. At the same time, in order to enable students to build an equivalence class of the hand-pair locations—that is, to somehow experience that  $2:3$  and  $4:6$  are “the

same”—we needed to provide the same feedback for different hand-pair positions. The green feedback (see Section 1) constitutes what Vergnaud (1983) named the *product of measure*, wherein  $x$  and  $y$  are integrated as  $x/y$ . Thus, students experienced identical sensory perceptions (green) whenever they generated the ratio. This unusual mix of isomorphism and product of measure was enabled by the computational environment.

The research questions that guided our inquiry were the following:

- Did students develop understandings for the mathematical concept of ratio and proportion and, if so, in what sense might we claim that they indeed learned?
- How did the task and tools contribute toward students' learning?

## 4. Methods

### *4.1 Purposes and Organization of a Collaborative Design-Based Research Study*

Our research project was conducted in accord with the design-based research approach (DBR), a relatively new approach to the study of education-related phenomena, in which instructional materials and theory are codeveloped simultaneously, interdependently, reciprocally, iteratively (Collins, 1992; Confrey, 2005; Edelson, 2002; Kelly, 2003; Sandoval & Bell, 2004). We chose the DBR approach to investigate proportional reasoning, because our research questions pertaining to this educational phenomenon were conjecture driven (Confrey, 1998). That is, we hypothesized a certain cognitive mechanism whose activity could potentially support conceptual learning, and so we were interested in investigating the nature of this mechanism. Yet, in order to carry out this investigation, we had first to devise and pilot empirical pedagogical situations that, we believed, would trigger this hypothetical mechanism into action so that we could study it. In particular, at the heart of DBR experimental design of empirical situations is an instructional design that is formed by theory and whose implementation, in turn, informs this theory. Thus, not unlike the M.C. Escher sketch of a hand drawing another hand even as that hand is drawing it, the theory and instructional content of DBR studies each depend on the other to gradually concretize. The design-based research approach offers frameworks for systematizing this interdependent development of theory and design through iterated cycles of design, implementation, analysis, and refinements.

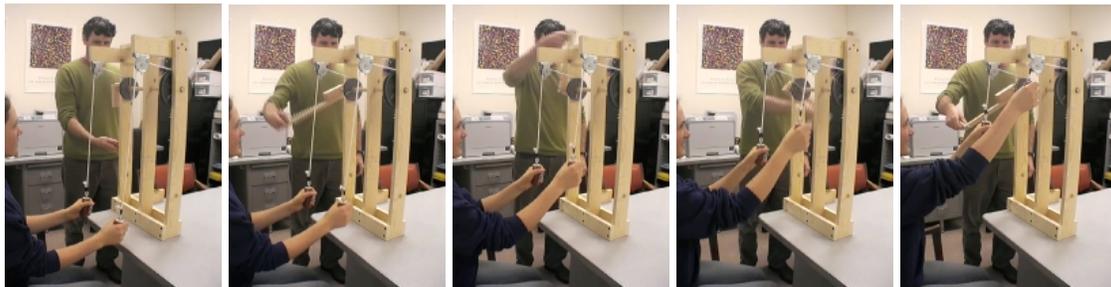
We perceived the initial phase of our project to be particularly explorative, and so we were interested in collecting empirical data that would enable the emergence and development of viable models of student reasoning, even as we were working iteratively on improving both the instructional and experimental designs. We thus elected to organize the DBR investigation in the form of generative case studies (Clement, 2000). Specifically, we devised and administered a task-based Piagetian semi-structured clinical interview (diSessa, 2007; diSessa & Cobb, 2004; Ginsburg, 1997; Goldin, 2000). We spread the implementation of these interviews thinly (no more than two per day), such that from day to day we would be able to introduce changes to the materials, activities, and protocol in light of the emergence and refinement of theoretical constructs. These rapid-prototyping changes—all motivated by the goal of optimizing the pedagogical quality and empirical utility of our subsequent interviews—were based on our field notes, preliminary analyses, verbal transcriptions, and minutes from our team's daily debrief meetings and electronic communications (all posted online and collaboratively edited day by day). Thus, both the interview protocol and the interactive affordances of the instructional materials evolved as we progressed through the pool of student volunteers.

Most notably, we gradually incorporated into the protocol activities and prompts that occurred to the researchers during the interview and appeared to elicit student reasoning apparently indicating the hypothetical cognitive construct we wished to study.

Into the research team's collaborative rapid-prototyping cycles of development, we infused discussion of selected theoretical and methodological readings from the Learning Science literature and beyond, whose relevance emerged in light of our coalescing frameworks and design concerns. Thus, as we were homing in on "interesting moments" and emerging behavioral patterns across our data corpus, a variety of research papers suggested potential ways of making sense of students' behaviors (e.g., Martin, 2008; Presmeg, 1998; Rivera, in press; Saenz-Ludlow, 2003), and a variety of design-oriented papers suggested how to improve our design in light of these observed behaviors (e.g., Figueiredo, van Galen, & Gravemeijer, 2009; Morgan, Mariotti, & Maffei, 2009; Pratt & Kapadia, 2009; Ruthven, Laborde, Leach, & Tiberghien, 2009). So, although the pacing of our research and development has been quite intense, our approach to data analysis has much of the trappings of grounded theory (Glaser & Strauss, 1967). At the center of our investigation is the drive to monitor for individual students' conceptual microgenesis as evident in their actions and utterances (Schoenfeld, Smith, & Arcavi, 1991; Siegler & Crowley, 1991) and to understand the manifold, nuanced ways that the researcher-interviewer steered this microgenesis (Mariotti, 2009; Stevens & Hall, 1998).

#### 4.2 Materials

*The MITs.* The original Mathematics Image Trainer, MIT1, was a mechanical device using pulley wheels with 4" and 6" diameters to effect a 2:3 ratio in movement of the attached ropes. The student holds handles attached at the ends of the ropes, and the researcher cranks a lever to raise and lower the ropes at steady rates (see Figure 4).

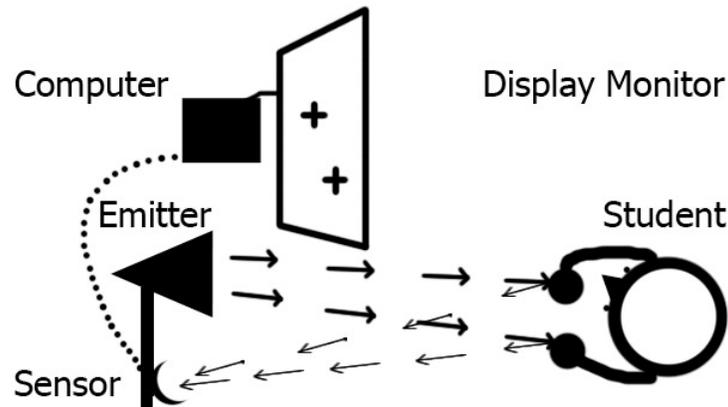


*Figure 4.* A proportions Mathematical Imagery Trainer (MIT) in action. This is our mechanical prototype for the computer-linked device. The student, seated, holds lightly onto two weights that are moving up at different rates, maintaining a 2:3 ratio between their respective heights above the table surface. Gradually, the student assumes agency, learning to mime this challenging ambidextrous performance. (The standing person is rotating a long handle, and this work is translated through two pulley wheels of different diameters, here relating as 2:3.) See <http://tinyurl.com/dor-mit> for a brief video clip.

MIT1 was limited, particularly insofar that its somewhat cumbersome mechanism allowed for the student to experience only a single ratio, a structural constraint liable to limit the potential of generalizing the phenomenon of proportion. These perceived limitations impelled us to design MIT2 in an attempt to increase the design's

mathematical flexibility, performance precision, electronic module interactivity, and dissemination potential.

In our second iteration, MIT2, we leveraged the high-resolution infrared camera available in the inexpensive Nintendo Wii remote to perform motion tracking of students' hands, similar to work by Johnny Lee (<http://johnnylee.net/projects/wii/>). The Wii remote is a standard Bluetooth device, with several open-source libraries available to access it through Java or C#. An array of 84 infrared (940nm) LEDs aligned with the camera provides out light (source), and 3M 3000X high-gain reflective tape attached to a tennis ball allows effective motion capture at distances as far as 12 feet. In use, infrared rays emanate from the MIT2, reflect off tape covering tennis balls held by the student, and are then sensed, interpreted, and visually represented on a large display in the form of two crosshair symbols (trackers). The display is calibrated so as to continuously position the crosshairs at the actual physical height of each hand in an attempt to enhance the embodied experience of virtual remote manipulation. Advanced MIT2 prototypes provide visual feedback of the student's performance on a green-red gradient, a design feature we discuss in the following section (see Figures 5; source and sensors are 3 feet left of the monitor).



*Figure 5.* A recent version of the Mathematics Image Trainer system: the student uses hand-held devices to remote-manipulate the locations of two crosshairs on a screen.

In a third revision, we experimented with a GameTrak controller, an HID USB device that enables three-dimensional input through thin, retractable cords that are attached to the student's hands. Unfortunately, the precision of the cord retraction proved insufficient to accurately measure the heights of the hands, leading us to indefinitely suspend research in this direction.

A variant on MIT2 is our current technological vanguard. Students hold and move infrared light beams aimed toward a Wii remote positioned on a tripod. The Wii remote bluetooths information on the hands' positions to a computer that processes the information further. The computer determines the coordinates of crosshairs that move vertically in accord with the hands' vertical position (horizontal position is ignored).

*Assessment.* To measure students' learning gains, we administered two tests both before and after the intervention. These two tests consisted of different yet comparable items. The items were designed to elicit students' conceptual schemes for ratio and proportion. Throughout the assessment, the interviewers attempted to gauge students' resources, strategies, reasoning, and vocabulary by asking them to explain their work.

In the first test, students received a set of white cards the size of regular playing cards (see Figure 6, below). On each card, a number pair was printed in large font along the landscape layout, such as, “2 3.” Prior to handing the cards to the participant, the interviewer shuffled the cards and stated that some of the cards in the set could be selected and arranged so as to form a sequence of four cards that “makes sense,” but that individual cards could possibly be used in more than a single sequence. The interviewer further imposed the constraint that the numbers “must keep going up.” If the student required further explanation, the interviewer began with general probes—“How are you looking at this?”, “What are you trying to do?”—and then possibly said that the cards should “tell a story, just like in a comics panel.” If the student still required more prompting, the interviewer laid down the first card of each of three proposed subsets and possibly built one of the sequences that was acting as a distracter. The number pairs were such that both “same difference” sequences (e.g., [2, 3] [4, 5] [6, 7] [8, 9]) and proportional sequences (e.g., [2, 3] [4, 6] [6, 9] [8, 12]) could be formed. The rationale of this assessment item was that students were not told which type of sequence to explore, so that a variety of solutions would be acceptable. In particular, the availability of a same-difference option was perceived as potentially enabling the participants to accomplish at least one possible solution to this task on the pretest, so that they would begin the interview with a sense of accomplishment rather than failure.

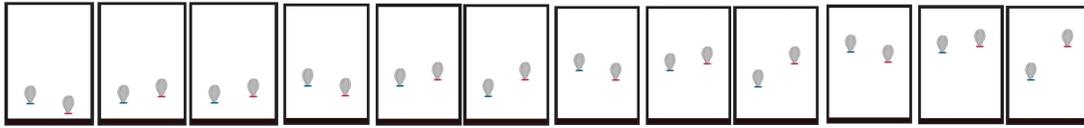


*Figure 6.* Some of the cards used for the pre/post-assessment “numbers” task. Students were asked to select and arrange one or more subsets in an order that “made sense” or “told a story” and explain this sense. All numbers were printed in black.

The second test was similar to the first in format and administration, however it consisted of cards each depicting a pair of hot-air balloons printed on the portrait layout of the cards (see Figure 7, below).<sup>2</sup> The idea for this particular setting as a means of probing students’ rate-based reasoning was borrowed from Alibali et al. (1999).

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<sup>2</sup> We were concerned that pre-intervention same-difference solutions would possibly prime the participants—both cognitively and pragmatically—to subsequently seek same-difference patterns in the remote manipulation activity and accordingly devise same-difference response schemes, both manual and numerical. That is, we were concerned that such priming would be even stronger than one might expect as based on the wide literature documenting young students’ “additive mindset.” So doing, we would run the risk of not being able to measure the learning gains that the activity may have actually fostered in the *absence* of this pre-test. We decided that the risk was reasonable, and that we would reconsider this pre-test based on the responses of our first three participants, “medium-level” students from each of the three grade levels. Similar, using these tasks as *post*-intervention assessments immediately after the proportions activities might strongly suggest to students that they should create a proportional progression. Nevertheless, we witnessed great variability in students’ performance on this task as well as in the quality and content of their explanations for the four-card sequences they assembled. Ultimately,



*Figure 7.* Some of the cards used for the pre/post-assessment “balloons” task. Students were asked to select and arrange one or more subsets in an order that “made sense” or “told a story” and explain this sense or narrative. In each card, the item on the left was blue and the item on the right was red.

### 4.3 Participants

The empirical data presented and analyzed in this paper were collected at a private K-8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students). In addition, we collaborated with the school principal, the head of general studies, and five mathematics teachers. Within each grade level, we grouped the pool of volunteering students according to three achievement levels as reported by their teachers (‘High,’ ‘Middle,’ and ‘Low’). Across these performance groups, we selected roughly equal numbers of students, balancing for gender. Also, we preferred working with students whom the teachers had indicated as typically more disposed to articulate and communicate their thoughts. These more verbose students were distributed almost uniformly across gender and achievement level. Whereas this selection as well as the demographics of the school may create methodological “blind spots,” this initial stage of the research required dense real-time verbal feedback from the students as they interact with the designed learning tools. Also, we wished to interview students with a command of multiplicative operations comparable to older students in underserved districts. To date, we have interviewed fourteen of twenty-three screened Grade 4-6 volunteers (one student turned out to be a special-education student—we conducted as much of the interview protocol as she was willing, but we decided to archive these data for a possible later study of special-education applications).

The research team included the first author—an experienced interviewer—as well as a technology expert with a Masters in mathematics-education research (the second author) and two novice researchers (Dan Reinholz and Dragan Trninic, graduate students in a STEM-education-research doctoral program). The research project thus occasioned training opportunities for graduate students as both researchers and emerging scholars.

### 4.4 Procedure

Interviews took place in a quiet room within the school facility. The first author led the earlier interviews, with one or two of the apprentice interviewers observing and occasionally interacting with the study participant. Most of the later interviews, however, were conducted by the apprentices, with the first author interpolating questions and sometimes “taking over” to experiment ad hoc with new directions that were difficult to communicate to the apprentice in real time let alone rehearse.

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this preliminary study was an opportunity to create and refine both instructional and experimental design. For the next study, we will create more sophisticated assessment.

Following the pre-assessment, the interviewer guided the participant through a sequence of activities by first explaining each activity and then monitoring the participant's performance and providing formative comments to ensure that the task was clear. As Section 1 suggested, the task description was simple: Make the screen green. At the completion of the MIT-related protocol, students again completed the "numbers" and "balloons" tasks as a post-assessment. The interview ended with an informal conversation, in which we explained to students the objectives of the study, helped them situate the activities within their school curriculum, both present and beyond into higher grades, and answered any questions they may have had, such as about our practice.

#### 4.5 Data Collected

Our data consist of fourteen videotaped interviews (mean interview duration 72 min., std = 23 min) as well as extensive fieldnotes, detailed written descriptions, and transcriptions of selected episodes. See Section 4.1 for our analytic approach.

### 5. Results and Discussion<sup>3</sup>

All participants succeeded in devising and articulating strategies for making the screen green, and these strategies were aligned with the mathematical content of proportionality. We observed some minor variation in individual participants' initial constructions of the situations as well as consequent variation in their subsequent trajectory through the protocol. However, by and large the students progressed through similar problem-solving stages, with the more mathematically competent students generating more strategies and coordinating more among the strategies.

We trace students' investigation into the MIT's interactive systematicity to the very first moment they were handed the devices and assigned the task of making the screen green. By virtue of devising solution strategies, we argue, students constructed bit by bit that which mathematicians call the concept of proportionality. Students' very first action was one of the following

- working with only one device at a time
- waving both hands up and down in opposite directions
- lifting both hands up at the same pace, possibly in abrupt gestures

The following list summarizes the sequence of students' insights:

- a) the actions of *both* hands are necessary to achieve green;
- b) green is achieved by positioning the hands at particular stable locations;
- c) the critical quality for achieving green is a type of *relation* between the hands' respective positions;
- d) these positions should be reinterpreted as magnitudes—the objects' heights above a common base line;
- e) the distance between the hands in correct pairs should not be constant—it will change between correct pairs;
- f) this distance should increase as the pair's height increases (and vice versa);
- g) moving from one correct position to another can be achieved by increasing the hands' heights differentially, for example, for every 2 units the left hand rises,

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<sup>3</sup> We are still in the midst of data collection (April, 2010), so we cannot yet offer extensive analyses of interviews and assessments.

- the right hand should rise 3 units (or the distance between the hands should grow by 1 unit)—a recursive rule for iterated transition;
- h) the multiplicative relation within each pair—for example at 4 and 6 units the right hand is 1.5 higher than the left hand—is also a constant;
  - i) one and the same number pair (e.g., 2 & 3) expresses three aspects of the interaction: for example 2 and 3 units are the lowest correct integer pair of heights, raising the left hand by 2 units for every 3 raised by the right hand will result in another correct location, and  $2/3$  or  $3/2$  is the constant within-pair multiplicative relation; and
  - j) when we transition from one green position to the next using the recursive rule (see Item g., above), each hand “skip counts” by a constant number, such as 2, 4, 6, 8 on the left vis-à-vis 3, 6, 9, 12 on the right.<sup>4</sup>

Along the way, a number of students also realized that there are infinitely many hand-position pairs, so that in fact one can *simultaneously* raise both hands, thus reinterpreting the solution as two hands moving at different speeds that could be characterized as, for example, 2 and 3 units per second. One participant said that the right hand is moving double as fast as the left hand. So doing, the students completed a U-shaped developmental trajectory (Stavy & Strauss, 1982), because they were once again moving their hands in continuous space but now with the knowledge developed through working with the grid.

In their post-assessments students showed a far greater propensity to arrange the cards proportionally as compared to additively. This activity and in particular the discussion of hot-air balloons created opportunities to discuss analogical cases of covariation in other contexts, such as the student riding a scooter while the parent drives a car. In these conversations, the new ambidextrous vertical gesture and its horizontal variant served as discursive supports that may have begun to evolve into a “model for” (van den Heuvel-Panhuizen, 2003) proportion. Students’ comments as they created proportional progressions with the cards, were revealing: the students would say that having worked with MIT they now knew that “it’s ok to be different,” meaning that they have now encountered a phenomenon that does not operate on a fixed-difference basis. We found it striking that students interpreted this short intervention as sanctioning the construction of proportional narratives.<sup>5</sup>

For the most recent interviews, we added a new task at the point in the protocol soon after both the grid and numerals had been introduced into the environment and incorporated by the students into their problem-solving strategies. Namely, once students had expressed an *a-for-b* strategy—whether *b* was the increment in the right hand or the distance between the hands—we removed the grid and numerals from the display and asked students to resume searching for green. Students’ gesture and verbal utterances strongly suggest that they continued operating as though a grid were present. However,

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<sup>4</sup> A fairly important design element that we chose not to elaborate upon in this paper is a “driver mode,” in which students input numbers into a table that then drives the crosshairs, turning the screen either red, yellow, or green in accord with the numbers vis-à-vis the ratio set on the experimenter’s console. This element created opportunities for students to notice further arithmetical properties of proportional progression.

<sup>5</sup> After Kundera, we named this phenomenon “the unbearable lightness of proportion.”

students' errors further suggested that their "mental grid" (cf. Hatano, Miyake, & Binks, 1977; Stigler, 1984) was functioning more like a scheme (Vergnaud, 2009) than as an image (Kosslyn, 2005). That is, the errors were not procedural but were expressed in imprecise magnitudes resulting in positional inaccuracies. As such, we may be witnessing the instrumental genesis of a utilization schema for proportionality (Rabardel, 1993; Vérillon & Rabardel, 1995). We wish to better understand the phenomenon of "equipping the mind" with mathematical instruments as the residual *effect-of* working with objects, because we view it as paradigmatic of mathematical learning (Norman, 1991; Salomon, Perkins, & Globerson, 1991; Vérillon & Rabardel, 1995; Vygotsky, 1930/1978).

## 6. Conclusions

Remote manipulation of virtual objects can be more than hand waving. We have presented the rationale and implementation of an innovative approach to the study of proportion. Our instructional design, the Mathematics Image Trainer (MIT), created opportunities for students to struggle with the core learning issues of proportion: through a succession of cognitive conflicts, and building on their fluency with addition and multiplication, the study participants each germinated an  $a:b$  conceptualization of proportion as the coordinated covariation of  $a$  and  $b$ . Our data also suggest that the students were able to use their invented strategies even when the supporting mathematical tools had been removed from the perceptual field, thus demonstrating a residual effect of working with the technology in the form of a utilization schema for proportionality (Artigue, 2002; Salomon, Perkins, & Globerson, 1991; Trouche, 2004; Vérillon & Rabardel, 1995). These achievements are impressive, given that the tutorial interview lasted only 72 minutes on average.

The researchers–interviewers played crucial facilitation roles: they guided students to develop their own emergent goal-oriented problem-solving strategies into mathematical meanings that were expressed in standard vocabulary and inscriptions, in accord with the designated didactical goals (Bartolini Bussi & Mariotti, 2008; Mariotti, 2009).

That said, it would be irresponsible to omit the following caveat: the pilot study reported in this paper was explorative, the protocol and assessment were flexible and evolved throughout the succession of interviews, and the number of participants to date has been quite small. Moreover, the day-to-day rapid development and prototyping of both materials (hardware, software, assessment items) and procedures (protocol, seating arrangements, etc.) resulted in some technological and logistical instability that sometimes challenged the investigative focus on substantive issues. It is therefore problematic at this point in our project to make any grand claims. Yet we hope to have stirred interest in what may be a promising variation on instructional technology design.

### *Future Directions*

In some future variant on the current design, we will explore a completely new direction: We will have students work in pairs, with each student holding one remote control. We anticipate that the students will struggle to coordinate their actions so as to produce green—they will engage and reinvent the "language game" (cf. Wittgenstein, 1953) of proportion. Students might thus develop and negotiate meanings even more so

than if they worked alone, because they would have to explicitize their reasoning in real time and not only a posteriori.<sup>6</sup>

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<sup>6</sup> In the words of W.-M. Roth (2009): “[I]n attempting to complete the requested task, the [paired] participants make available for one another what is required to remain aligned in solving the problem at hand” (p. 96); [B]ecause pairs of participants are asked to produce one interpretation, they need to ascertain that they are aligned with respect to what they are talking about, what they see or what dawns in their perception, so that they can talk about the feature or assist others in seeing the same” (p. 100).

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Figure Z. Our designed dynamical image schema is already practiced in high places.