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Fostering Hooks and Shifts: Tutorial Tactics for Guided Mathematical Discovery

**Dor Abrahamson · Jose Gutiérrez · Timothy Charoenying ·
Andrea Negrete · Engin Bumbacher**

Abstract How do instructors guide students to discover mathematical content? Are current explanatory models of pedagogical practice suitable to capture pragmatic essentials of discovery-based instruction? We examined videographed data from the implementation of a natural user interface design for proportions, so as to determine one constructivist tutor's methodology for fostering expert visualization of learning materials. Our analysis applied professional-perception cognitive–anthropological frameworks. However, several types of tutorial tactics we observed appeared to "fall between the cracks" of these frameworks, due to the discovery-based, physical, and semantically complex nature of our design. We tabulate and exemplify an expanded framework that accommodates the observed tactics. The study complements our earlier focus on students' agency in discovery (in Abrahamson et al., Technol Knowl Learn 16(1):55–85, 2011) by offering an empirically validated resource for researchers, instructors, and professional developers interested in preparing future teaching for future technology.

Keywords Cognition · Design-based research · Discovery · Education · Embodied interaction · Guided reinvention · Insight · Mathematics education · Natural user interface (NUI) · Proportion · Proportional reasoning · Remote control · Semiotic-cultural · Sociocultural · Symbolic artifact · Teaching · Technology · Virtual object

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D. Abrahamson · J. Gutiérrez · T. Charoenying · A. Negrete · E. Bumbacher
Embodied Design Research Laboratory, University of California, Berkeley, Berkeley, CA, USA

D. Abrahamson (✉)
Graduate School of Education, University of California, Berkeley, 4649 Tolman Hall,
Berkeley, CA 94720-1670, USA
e-mail: dor@berkeley.edu

1 Introduction

1.1 Background and Objectives

The Mathematical Imagery Trainer is a computer-based interactive pedagogical system that mediates grounded understanding of subject matter content. Students engage in a problem-solving task involving the remote-manipulation of virtual objects in an attempt to effect and sustain a designated goal state of the system, such as making a screen green. Through failure, discovery, and practice, students develop an effective perceptuomotor schema by which to move their hands in specific coordinated gestures that dynamically conserve the goal state. According to the designer's cognitive domain analysis, this operatory schema bears the pedagogical potential of constituting an embodied substrate of the targeted concept. Thus, once students have articulated the new schema, the instructor introduces into the problem space new symbolic and referential resources used in mathematical practice, such as a Cartesian grid that is layered onto the interactive objects. By and large, students recognize in these resources enactive or semiotic affordances for enhancing, explaining, or evaluating their interaction strategy. Yet when they engage or "hook" the resources, students implicitly distribute their operatory schema subgoals upon these resources' embedded affordances. As a result, their strategy "shifts"—it becomes newly instrumented, effectively reconfigured, and conceptually signified in ways that better resemble disciplinary practice. Thus students inadvertently bootstrap mathematical notions via appropriating available cultural tools. We have described this two-stepped discovery process as "hooks and shifts" (Abrahamson et al. 2011).

The study reported in the current paper attempted to complement our previous focus on learners' agency in the guided discovery process by investigating instructors' pedagogical practices that enable these discoveries. The earlier paper delineated a set of conditions predicting hooks and shifts. Essentially, these conditions unpack what a student should know and experience so as to visualize an interaction system sufficiently similar to the instructor. This paper focuses on the role of the instructor in satisfying these conditions. We thus seek to determine instructors' techniques for scaffolding students' interactions up to a point beyond which some students both hook and shift unguided and for further structuring this process as necessary for all other students.

As such, the empirical context of implementing the Mathematical Imagery Trainer continues to serve us as a laboratory both for improving the pedagogical artifacts and for developing relevant theoretical models. In particular, we are exploring relations among grounded-cognition and sociocultural theories so as to frame our designs and analyses of computer-based mathematics pedagogy (Botzer and Yerushalmy 2008; Trninic and Abrahamson in press). This research program sits squarely in the continued efforts of learning scientists to envision the roles of computational artifacts in expanding mathematical literacy (diSessa 2000; Noss and Hoyles 1996; Papert 1980). A better understanding of effective pedagogical methodology for scaffolding mathematical visualization should be of interest to the education manifold, including teachers, professional developers, designers, and researchers.

Extant literature does not appear to provide teachers and teacher developers with principles for the facilitation of discovery-based instruction. Ball et al. (2008) elaborate on Shulman's (1986) seminal construct of pedagogical content knowledge to articulate dimensions of content knowledge for teachers—what teachers should know both about the content per se and about how students approach the content (see also Lampert 2001). Drawing on these and other explanatory models of both teaching and learning, Sztajn et al. (2012) have proposed a framework based on the theory of learning trajectories. The framework offers a comprehensive macro-view of what teachers should know and do so as to support a classroom along curricular learning trajectories, all important information for designers of pre- and in-service professional-development courses. However, the framework is not grounded in a cognitive model of micro-learning processes and so is not geared to offer a pragmatic micro-view of how instructors should guide the collective of individual agents to engage in the micro-tasks that make up classroom activity. Closer to our interests, Ginsburg and Amit (2008) lay out twenty-seven general teacher practices in a reform-oriented early childhood mathematics classroom. Whereas we find in our data great resonance with many dimensions of specialized content knowledge cited in these and other previous studies, still those papers leave open the question of how to apply the models in practice. More specifically, existing literature is still moot on the finer granularity of facilitating student explorative interaction with learning materials such that they construct appropriate schema, visualize situations similar enough to the instructor, and discover latent mathematical principles embedded in, and emerging from their guided, goal-oriented interactions with learning materials. Our current study attempts to fill this gap in the literature—we attempt to portray the nitty-gritty of constructivist instruction.

We thus returned to our videotaped footage of the twenty-two 4th–6th graders who participated in our tutorial clinical interviews, and we set off categorizing the tutors' multimodal utterances in terms of what these utterances were apparently intended to achieve. It soon occurred to us that professional-perception frameworks from cognitive anthropology (Goodwin 1994; Stevens and Hall 1998) hold much promise in developing a systematic characterization of the tutor's "tactics," as we began to call these instructional moves. However, certain qualities of our pedagogical design—its discovery-based, embodied, and semantically complex nature—appeared to require elaborations on these methodologies. This paper reports on a set of tutorial tactics that have not been previously articulated yet we view as instrumental in fostering student discovery of mathematical notions. By embracing these proposed elaborations, professional-perception frameworks might bear greater methodological traction on guided, discovery-oriented activities, and in particular embodied-interaction activities with recent educational technology utilizing NUI (Natural User Interface). Given the increasing ubiquity of embodied-interaction technologies (see Marshall et al. in press), our proposed theoretical expansion is timely.

Section 1.2, below, builds context for this study by explaining the Mathematical Imagery Trainer as a design response to some enduring challenges of school mathematics. Section 1.3 focuses on particular qualities of our interactive design that appear to require expansions to the professional-perception frameworks. Section 2 explains our methods. Section 3 presents findings that we interpret as supporting the proposed framework expansions; we present these findings in the form of a table that summarizes and exemplifies a set of discovery-oriented tutorial tactics observed in our data, many of which are not easily captured in the existing frameworks. Finally, Sect. 4 offers conclusions, and Sect. 5 notes some limitations and implications.

1.2 Learning is Moving in New Ways: A Design for Proportion

In this section we introduce the Mathematical Imagery Trainer ("MIT"), a technological design engineered to support deep learning of mathematical content. In particular, we will discuss an MIT for proportional relations ("MIT-P"). This overview integrates earlier descriptions of the design rationale and findings from the empirical studies (Abrahamson et al. 2011; Howison et al. 2011; Reinholz et al. 2010).

The subject matter of ratio and proportion is didactically essential, because it underlies high-school STEM content and professional scientific reasoning. However, many students in elementary school and beyond experience difficulty in understanding and using the core notions of proportionality. In particular, students incur difficulty in developing a fluency with proportions that builds upon, yet is differentiated from, simpler non-multiplicative notions, notations, nomenclature, and procedures (Karplus et al. 1983; Lamon 2007).

We approached this design problem by analyzing the target content from an embodied-cognition perspective that models human reasoning as grounded in traces from spatio–dynamical experiences (Barsalou 2010). An appeal of this epistemological position for designers of reform-oriented mathematics education is in its categorical implication of mundane interaction as furnishing personal resources for learning and reasoning. The theory resonates with tenets of genetic epistemology, and in particular the constructivist thesis that conceptual activity is embodied in perceptuomotor schemas. For example, Piaget (1971) stated that, "Mathematics uses operations and transformations….which are still actions although they are carried out mentally" (p. 6). The grounded-cognition hypothesis further resonates with the growing consensus that mathematics is a situated, multimodal, multi-media, and multi-semiotic praxis (Bamberger 1999; Bautista and Roth 2012; Lemke 1998, 2003; Nemirovsky 2003; Núñez et al. 1999; Radford 2002; Rotman 2000; Skemp 1983).

Designers have historically sought to augment on mundane interactions so as foster new resources for learning content (Diénès 1971; Froebel 2005; Kamii and DeClark 1985). From a grounded-cognition perspective, we conjectured specifically that some mathematical concepts are difficult to learn because everyday experiences do not occasion opportunities to embody and rehearse perceptuomotor schemas underlying those concepts. In particular, we conjectured that students' canonically incorrect solutions for rational-number problems—"additive" solutions (e.g., "2:3 = 4:5" or "2/3 = 4/5"—cf. Behr et al. 1993)—indicate insufficient kinesthetic–visual action images to ground proportion-related concepts (cf. Fischer et al. 2011; Goldin 1987; Pirie and Kieren 1994). We thus sought to "phenomenalize" the concept of proportion in the form of an interactive device that would afford learners opportunities to develop and generalize the concept's principles (see Pratt et al. 2006, on the design principle of phenomenalization).

We engineered a computer-supported inquiry activity for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the Mathematical Imagery Trainer for Proportion (hence "MIT–P"; see Fig. 1), an embodied-interaction system designed to foster the development of perceptuomotor schemas for the "situated abstraction" of proportion (cf. Noss et al. 1997). Participants use both hands to remote-control a pair of virtual objects on a computer display monitor, one object per each hand, in attempts to "make

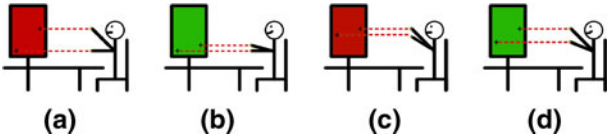

Fig. 1 The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the *right hand* needs to be twice as high along the monitor as the *left hand*. In a paradigmatic interaction sequence, the student: **a** positions the hands incorrectly (*red* feedback); **b** stumbles upon a correct position (*green*); **c** raises the hands *maintaining constant distance between* them (*red*); and **d** corrects the position (*green*). Compare 1b and 1d and note the different sized intervals between the cursors. (Color figure online)

the screen green." When students first engage in this activity, the screen is red (see Fig. 1a). Unknown to them, the screen turns green only when these two objects' respective heights above the bottom of the screen relate by a particular ratio that is set on the tutor's computer console. Students move their hands about until they happen to find a combination of cursor positions that corresponds with a green screen, as mediated via the code. If the ratio is set at 1:2, the screen is green only when the right-hand cursor is twice as high above the bottom of the screen as compared to the left-hand cursor (see Fig. 1b). Once the students discover a "green spot," we ask them to find another green spot. Invariably, they respond by moving their hands both up or both down, maintaining a *constant* spatial interval between the crosshairs, which causes the screen to turn red (see Fig. 1c), whereupon they correct back to a green screen by adjusting the distance (see Fig. 1d). Eventually, they establish a principled relation between the hands' elevation and interval, stating, for example, that, "The higher you go, the bigger the distance" (compare Fig. 1b, d).

We believe that students' attempt to sustain green by maintaining a fixed spatial interval between their hands (compare Fig. 1b, c) foreshadows their typical additive visualization of multiplicative symbolic expressions, such as equating 2/3 with 4/5. The MIT-P launches proportional reasoning by offering a new type of equivalence class, in which "different differences" might be conceived as "the same." It is thus that we attempt to engage rather than replace students' naïve conceptualizations (Borovcnik and Bentz 1991; Noss and Hoyles 1996; Smith et al. 1993).

The interview protocol then calls for the researcher to introduce a grid onto the screen (see Fig. 2a, c). Students typically respond by "hooking" the grid, such as engaging it with the intention of improving their qualitative strategy of "The higher you go, the bigger the distance." Yet in so doing they "shift," that is, they inadvertently modulate their strategy into "For every 1 that I go up on the left, I go up 2 on the right." We then introduce numerals upon the grid (see Fig. 2d), and students now shift to an explicit multiplicative scheme, for example, "The right one is always double the left one." This scheme enables them to predict and evaluate any green spots before enacting them. The interview continues with other ratios, such as 1:3, 2:3, 3:5, etc.[1]

Throughout the interview, the tutor poses tasks, clarifies instructions, and intervenes in the child's inquiry process with questions, hints, demonstrations, inference prompts, collaborative enactment, etc. For example, in Fig. 3 a tutor and child co-operate the two remote-control hand-held devices. It is precisely these types of rich interactions that this paper attempts to characterize and classify, because we believe they are essential to effective instruction, at least in the context of embodied-interaction designs. The inherent dynamical physicality of MIT inquiry-based activities bears the methodological advantage of making visible—for the student, instructor, and analyst—nuanced aspects of cognition and instruction that often cannot be seen but only surmised.

As a learning activity, the MIT-P task is dramatically different from traditional schoolwork, because the solution *method*, not only the solution per se, is unknown to the child. Moreover, this task is different from what mathematicians often do, because there is no theorem to prove (but see Thurston 1994, on mathematicians' "aimless" yet highly productive exploration). Rather, this task is closest to forms of inquiry that scientists

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[1] The activity protocol then concludes with a hands-on activity that we do not treat in this paper. Therein, the control mechanism is changed from manual to numeral: we introduce a ratio table that students need to fill in, and then the computer "plays out" the number inputs by moving the cursors automatically from one number pair to the next and giving the appropriate color feedback. We enable students to go back and forth between these interaction modes.

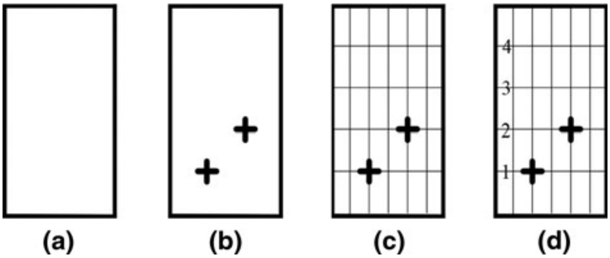

Fig. 2 MIT-P display schematics, beginning with **a** a *blank screen*, and then featuring a set of symbolic objects incrementally overlain onto the display: **b** *cursors*; **c** a *grid*; and **d** numerals along the *y*-axis of the *grid*. Not featured here is a ratio table

Fig. 3 Child and tutor co-remote-control two cursors on a computer monitor. The child is attempting to accomplish the designated task objective of making the screen *green*. The instructor is structuring the inquiry process by pacing the left hand's vertical motion as an independent variable, while the child searches for the complementary location of the right hand. (Color figure online)

engage, for example a botanist or entomologist who first encounter a specimen of an unknown species and are trying to understand its properties, or a chemist who has built a new compound and is attempting to determine its reactions to various agitations. But then again, scientists who discover an undocumented phenomenon, material, or star, etc., do not know a priori what specific behaviors they might witness and perhaps elicit, so that their interactions with the phenomenon are not initially oriented toward generating any specified goal state such as green. As such, the MIT task is positioned between closed- and open-ended inquiry-based learning activities.

Having described the activity design, outlined students' typical responses, and commented on the unique nature of the task, we will step back to highlight several dimensions along which MIT activities differ from naturalistic and vocational learning. These differences, we believe, explain the apparent shortcomings of professional-perception frameworks to capture aspects of MIT-based tutorial interaction. The empirical sections of this paper will then be offered as evidence supporting this thesis.

1.3 Methodological Challenges of Embodied-Interaction Discovery-Based Pedagogical Tasks for Science of Learning

The forms and media of the pedagogical design at the center of this study are uncommon. Plausibly, then, common methodological frameworks may not be quite geared to research these designs. Below, we explain the rationale of this type of design and then propose three uncommon design attributes that may present challenges for common methodological frameworks.

1.3.1 *In Search of a Methodology for Modeling the Mediation of Mathematical Visualization*

Our designs are of the NUI general type “embodied interaction.” In embodied-interaction activities, learners’ physical actions are integral elements of the information they gather as they engage in the completion of tasks, whether or not they are conscious that they are gathering these data (Abrahamson and Trninic 2011; Antle et al. 2009; Dourish 2001). For example, Fischer et al. (2011) demonstrated that young children enhance their fluency with the number line by participating in activities in which they hop to the right or left on a special mat in response to the visual display of numerals respectively greater or lesser than a target numeral.

Further, the structure, substance, framing, and facilitation of our designs are inspired by discovery-oriented pedagogical philosophy (Diénès 1971; Freudenthal 1971; Skemp 1993). Namely, we do not tell learners which specific actions result in the solution. Instead, we inform students of the desired goal state of the interactive technological system and then steer them toward exploring the embodied space in an attempt to effect this state. In so doing, the students’ physical actions convey to us their implicit reasoning, and we can observe as they adjust these perceptuomotor routines in tune with the emerging properties of the interactive system. For example, children raise their hands keeping a constant interval between them yet soon realize they should calibrate the interval correlative to the hands’ elevation.

As such, these activities, and hence the empirical data gathered at our implementation sites, differ from what one is likely to see in common settings where instruction takes place, such as vocational contexts. We believe that these differences should bear on how tutorial interactions such as these are researched. In particular, we submit, experimental learning activities may require a stretching of extant analytical frameworks so as to capture how a student is learning and how an instructor is guiding this learning. In practice, this should compel researchers to adjust how they attend to, parse, code, model, and describe aspects of interaction. Let us step back to describe two powerful frameworks for interpreting micro-ethnographic data of instructional interactions.

1.3.2 *Theories of Professional Perception*

Practitioners of technical fields such as mathematics and science “see” their domains of scrutiny differently from novices, a capacity Goodwin (1994) calls *professional vision*. Within Goodwin’s framework, the act of perception is not a transparent psychological process, but rather a socially situated activity that involves three discursive practices: coding, highlighting, and producing-and-articulating material representations. Through this multimodal discursive activity, the complexity of the perceptual field is transformed and categorized into objective, documented knowledge. Highlighting is the practice of making salient for the novice particular elements within the domain of scrutiny that are most relevant for drawing relevant information, thus creating a figure and ground. Then coding, or labeling, assigns professional nomenclature to these highlighted elements. In the third phase, these coded elements are elaborated in terms of the inferences they suggest within the context of larger activity structures. The framework elucidates how people coordinate and stabilize their initially disparate perspectives to regulate co-production (see also Hutchins 1995; Isaacs and Clark 1987; Sebanz and Knoblich 2009).

The Stevens and Hall (1998) framework, *disciplined perception*, takes a more nuanced perspective on what it means to see domains of scrutiny professionally and how this seeing is accomplished through guided interaction. As with Goodwin, they are interested in how interlocutors come to functional agreement over the domain-specific senses of shared referents in the visual field. Also similar to Goodwin, two central tenets of their framework are that vocational perception is socially constructed and that discourse involves more than just talking: speech and physical action, such as gesturing and pointing, mutually elaborate each other vis-à-vis available materials, so that a novice's visual orienting is mediated via the expert's multimodal and multi-media discursive actions. Their analyses of interlocutors' struggle for coordination focus on the construction of particular visual orientations, which are the various ways in which people can attend to material artifacts they perceive around them.

1.3.3 Limitations of Professional-Perception Theory

We have found the above frameworks very helpful in framing the initial analyses of our empirical data. However, we have also come to believe that the fairly unique pedagogical features of embodied-interaction discovery-based instruction bear on methodological choices we must make as we analyze our empirical data. Below we articulate three dimensions of uniqueness that, we posit, may require some elaborations on the current frameworks.

First, research on the development of professional vision has predominantly treated explicit instruction that might be characterized as "show and tell" (but see Mariotti 2009; Radford 2010). Yet as educators who believe in the cognitive, affective, and deutero advantage of discovery learning over explicit instruction (Bateson 1972; Freudenthal 1991), we are interested in understanding how instructors usher learners toward insight—a "show and *don't* tell" pedagogy, if you will. Therefore, in order to investigate guided inquiry, we submit that interactions could be studied as bi-directional and dialogic, rather than unilateral and expository. Namely, the instructor has to understand what the students are looking at and how they are looking at it just as much as, if not more than, the child has to understand what and how the instructor is observing (cf. Newman et al. 1989; Sherin et al. 2008; White 2008). As such, a study of students' emerging professional perception as mathematicians and a study of mathematics tutors' professional practice are in effect two tightly coupled and mutually informing sides of the same ethnography.

Second, research within the paradigm of instructional ethnography often models conceptual ontogenesis in terms of learners developing from not-knowing to knowing, such as in disciplining the novice eye to interpret brain images (Alač and Hutchins 2004). Yet we are also interested in how people learn by coordinating among a *set* of conceptually meaningful views on a situation, that is, by transitioning from knowing-one-way to knowing-another-way (cf. Abrahamson et al. submitted; Godino et al. 2011, p. 257). Instructors ushering this particular type of transition, it seems to us, do not orient as much as *re*orient learners' view toward the visual displays—a pedagogical objective that plausibly demands more nuanced forms of interaction and, accordingly, more nuanced models of interaction.

Third, professional-perception ethnographic research predominantly treats cases of passive perception, in the sense that learners are guided to make sense of preexisting perceptual stimuli in the domain of scrutiny, which they view and perhaps manipulate. In embodied-interaction designs, however, learners necessarily generate the data they analyze—their posture and motion are inextricable aspects of the emerging information. As such, the learners' physical actions are not only pragmatically subservient to the exploration, they are epistemically central to the field of scrutiny—they *are* the data (cf. Kirsh and Maglio 1994); embodied interaction is "hands *in*." We thus adhere to a conceptualization of perception and action as cognitively intertwined aspects of perceptuomotor schemas. Accordingly, we attend to what learners are doing as much as to what they are saying. Furthermore, we monitor how the learners are guided to develop these ways of doing, just as we monitor how they are guided to develop ways of seeing, especially given that our design fosters actions that only later become signified mathematically. As such, the description and documentation of actions are brought to the fore of our analyses, not only as supplementary gestural information that may disambiguate verbal utterance but as integral components of learning.

In sum, our study attempts to chart the tactics that tutors use in facilitating reform-oriented, problem-based, embodied-interaction mathematics learning activities. An emergent, concomitant goal of the study is to evaluate the fit of professional-perception frameworks as means of documenting tutorial practices in these particulars contexts.

2 Methods

The focal corpus of data consists of eighteen out of a total of twenty-two videographed sessions, in which Grade 4–6 students from a K-8 suburban school in the San Francisco East Bay participated voluntarily in either individual or paired task-based semi-structured clinical interviews (duration: mean 70 min; SD 20 min). In general, the students had not been exposed formally to the concepts of ratio and proportion, and where we administered pre-intervention assessments we found that students did not have proportion-related schemas as available means of organizing number pairs into meaningful sequences (Reinholz et al. 2010).

The main tutor—the first author—is a professor of mathematics education who teaches design-based research methodology. He has conducted hundreds of semi-structured task-oriented interviews for his research and has administered thousands of professional tutoring sessions in the private sector. The premise of this study is that one effective way of determining useful pedagogical practice is to examine the practice of an expert and attempt to model this practice along salient dimensions that appear pertinent to this effectiveness (see, e.g., Lampert 1990, for a notable precedent of an instructor–researcher as a reflective practitioner). As things stand, the recent novelty of embodied-interaction designs implies a scarcity of opportunities to study their facilitation, so that we would have been hard-pressed to find compatible research sites.

We analyzed the data both as individuals and collaboratively (Schoenfeld et al. 1991) with the objective of seeing, agreeing upon, naming, and sorting the tutor's tactics. We set off with only a broad working definition of tutorial tactics as "on-the-fly discursive and physical actions for productively mobilizing student inquiry during actual tutorial interactions."

We treated the tactics we found using principles from grounded theory (Strauss and Corbin 1990), by which categories emerge and are checked across a corpus of data. Yet in addition to this "bottom up" approach, our categories became progressively informed "top down" by professional-perception frameworks, as we realized the purchase of these frameworks on our data. For example, we created categories corresponding with "highlighting" and "coding" that are used in those frameworks, respectively to designate the showing and naming of features in the domain of scrutiny relevant to the practice. We were also informed by interaction parameters we had previously identified as governing these same students' inclination to adopt the mathematical instruments and bootstrap advanced strategies (Abrahamson et al. 2011).

Coding a tutorial tactic is a ticklish business, because one wishes to code the tutor's discursive actions, and yet the ad hoc meaning of each utterance is necessarily informed by the interaction leading up to it. We decided to code according to the tutor's presumed intention regarding future events, given both the history of the interview up to that moment and the next goal in the protocol. However, we did not code these utterances according to their observed effect on the student's behavior, because the student would not necessarily respond to the action per the tutor's intention or not respond at all. In these evaluations, we adopted a phenomenological, naturalist, hermeneutic–dialectic approach, in which participatory analysts first act as the measuring tools, then interpret and negotiate their subjective judgments (Guba and Lincoln 1982, 1989, 1998; Harris 1976).

Through the analysis process, it soon became evident that a great proportion of what the tutor was doing could be interpreted as domain-general discursive practice, such as dialogue maxims (Grice 1989) and means of regulating conversation (Schegloff 1996) or establishing common ground (Isaacs and Clark 1987). Furthermore, the tutor's actions could largely be interpreted as means of "entering the child's mind" (Ginsburg 1997), that is, as clinical techniques for probing the child's ongoing thoughts. This is to be expected in a task-based semi-structured clinical interview that was designed so as to evaluate the utility of a set of instructional materials and accompanying facilitation protocol (diSessa 2007; Goldin 2000). Yet as we became adept at seeing these many aspects of the tutor–student interaction, we could better discern against their background a set of tactics that appeared to be unique to discovery-oriented instructional methodology for hands-on inquiry learning. Importantly for our thesis, these tactics did not appear to lend themselves too well for coding via professional-perception categories.

Note that the effort in this particular exploratory study was to identify, name, and sort recurrent tactics. In Waismeyer and Abrahamson (submitted) we are attempting to determine the relative frequency of the various tactics and establish clusters and patterns of tactic sequencing across an entire interview, with an eye on tutor-vs.-student contribution source.

3 Results and Discussions

The objective of this study is to increase our understanding of instructional practices supporting student reinvention of mathematical concepts. Our data are a set of tutorial clinical interviews from a design-based research project that has been examining the potential utility of embodied-interaction technology for mathematics education. In the course of investigating the tutorial tactics observed in those data, we were obliged to expand on the professional-perception frameworks that were partially guiding our emerging taxonomy of these implicit instructional practices. These frameworks, which have proven to be powerful lenses on mainstream school-related and vocational instruction, did not appear to work well as lenses on our non-mainstream future-technology designs. Below, we cite additional resources from the literature that emerged as relevant to our taxonomy in the course of our analysis. Next, we present a table of all tutorial tactics identified across the data corpus. Then we highlight what we view as previously undocumented tactics and explain them in light of pedagogical and technological dimensions of our design. Finally, we revisit the tactics from the perspective of our earlier work on hooks and shifts in an attempt to evaluate how these tactics might serve in prompting hooks and shifts.

3.1 Theoretical Resources for a Grounded-Theory Analysis of Mathematics Tutoring

In the course of our data analysis, we drew on multiple resources from the learning sciences literature, some of which we first learned of through consultations with colleagues in workshops and conferences. In Table 1, below, the first categories capture domain-general discursive aspects of task-based interaction (Isaacs and Clark 1987; Schegloff 1996) that occur in clinical contexts (Ginsburg 1997), and the later categories treat aspects more specific to guided inquiry in embodied-interaction learning activities. The names and order of these later categories draw in part on the domain-general instructional interaction sequence identified by Goodwin (1994), by which experts enable novices to see and interpret elements in the perceptual field, with particular emphases borrowed from Stevens and Hall (1998) on *how* the novices see these elements. We were also helped by Mariotti (2009) in realizing how the tutor elicits from students their own past actions and statements and then focalizes them on particular elements therein conducive to steering the students toward mathematical generalizations. From Radford (2003) we borrowed elements of his semiotic–cultural theory of objectification. From Saxe (2004) we appropriated the idea that individuals co-opt forms to function as means serving personal goals in the context of solving collective problems. From Collins and Ferguson (1993) we borrowed the notion of an epistemic form, a cognitively ergonomic, domain-general cultural device we equip our students with for organizing inquiry. Also, we were informed by Bakker and Hoffmann (2005) in applying a Peircean lens on how students come to discern and name new patterns in perceptual information ("hypostatic abstraction," a form of diagrammatic reasoning). Finally, we were inspired by the social-anthropology work of Timothy Ingold (2011), who studies the guided mediation of skill in authentic hands-on practices, such as carpentry.

Granted, these and other resources originate in diverse epistemological perspectives, such as semiotics vs. sociocultural theory, so that one may legitimately worry about the prospects of conjoining them into a coherent framework. We can only submit that each of these resources appeared to bear well on our data. Perhaps this stitching may contribute to our field's enduring search for theoretical coherence (Artigue et al. 2009; Drijvers et al. in press). If this coherence is to be found anywhere, it makes sense that we find it in documentation of actual instructional practice (cf. Stetsenko 2002).

For example, Mariotti's notion of focalizing students on aspects of their own actions can be viewed as related to Goodwin's notion of highlighting elements in the perceptual field, only that Mariotti includes in the domain of scrutiny not only the objective stimuli but the learners' recollections of their interaction with these stimuli. In this example, we see how a tentative structural alignment of perspectives from otherwise disparate traditions—neo-Vygotskian sociocultural theory and cognitive anthropology—may help researchers draw on the wealth of the learning sciences literature.

Similarly, we read into Radford's and Saxe's neo-Vygotskian theses a certain convergence with constructionist views from Pratt and Noss (2010), namely the idea of learners spontaneously recognizing in the learning environment utilities that serve their situated objectives: Radford discusses semiotic utilities of symbolic artifacts as means of objectifying presymbolic notions; Saxe discusses emergent pragmatic utilities of cultural–historical artifacts for participating in social practices in flux; Pratt and Noss discuss enactive

Table 1 Tutorial interaction tactics supporting mathematical learning via hands-on problem-solving activity

Tactic	Description	Example
Context general		
A. Para-content framing	Administrative, logistical, and procedural actions to organize, monitor, and optimize interlocutor's comfort and engagement	Adjusting S's grip of electronic device. Fixing room light, heat
B. Clinical-interview techniques	General methodology that overlaps with many tutorial tactics, e.g., eliciting S's vocabulary via direct questions or indirect discursive ploys, echoing S speech, probing S reasoning, etc.	T: "So how could we call your other theory, then?"
C. Discursive practices	Amid ambiguous expression, interlocutors initiate measures to interrogate and establish meanings. The agent with superior contextual positioning (T) suggests signs and elicits confirmation	T: "What do you mean by 'the same way'?"
Tutorial		
1. Establish joint problem space*	Establishes the perceptual field, significant elements, mode of physical interaction, available resources, and task. Typically occurring at the beginning of the intervention, T might later re-evoke this information for clarification and/or introduce new tasks. These introductory communications are explicit and direct as compared to implicit and hinted cues during S's guided inquiry	Setting up the MIT-P activity by introducing and naming situation elements (tr, camera, monitor, laptop), interaction mode (lifting/lowering tr), and task (making the screen green)
2. Elicit orientation of view, strategy, reasoning, vocabulary, and knowledge	Performs actions intended to glean information about S's perception, planning, acting, and prior learning that are not self evident. T evaluates this information in terms of its sources, rationale, consistency, and coherence as these relate to the design's targeted views and strategies. Whereas these solicitations may stimulate reflection (see 4., below), they are initiated primarily so as to inform T of that which cannot be seen or heard	*Ex 1.* T: "How are you counting? Do that same thing for me, please." *Ex 2.* T: "What do you think Dan was doing?" *Ex 3.* T: "Have you learned multiplication?"

Table 1 continued

Tactic	Description	Example
3. Orient S's perceptuomotor interaction toward phenomenon's critical features	Uses discursive means to steer learner toward sharing similar ways of perceiving, planning, and acting in relation to completing tasks in the domain of scrutiny. Based on S's manifest/elicited orientation of view, T evaluates that S is prone toward an impasse with respect to discovering properties critically pertinent for productive engagement with the problem. So T ushers S to fertile data-gathering locations, orientations, or aspects of the situation	
3.1 Highlight	Takes multimodal measures to make salient features of domain	
3.1.1 Feature	Draws S's attention toward an object or aspect of the domain, in a way that modifies S's perceptual construction of the domain, such as by suggesting directionality and introducing fictive motion	T: "Ok so what do you think of the numbers going down this way?"
3.1.2 Restructure	Introduces conceptual or semiotic model from another domain, so as to steer S to mentally reframe a set of data, such as by "rewiring" correlations along two dimensions of interaction	S analogizes color gradients to a rainbow, T suggests traffic light instead
3.1.3 Objectify	Responds to S's description of an action or relation in the joint domain of scrutiny by using a noun phrase, which might be a cognate noun (possibly a homonym), to staple a new referent as a semiotic resource for further discussion	S: "I think they have to be diagonal [*adj.*] from each other." T: "What about up high—would it be the same diagonal [*n.*]?"
3.2 Code	Initiates, negotiates, and establishes consensual means of reference:	
3.2.1 Code aspects	…to specified aspects of the domain (elements, properties, allusions, etc.);	T: "Same kind of 'game' but different… 'rule,' I guess"
3.2.2 Code actions	…to possible actions in relation to these aspects (e.g., actions with or upon objects, manipulation strategies, principles, etc.)	T: "Ah….as if you're holding a ladder?"
3.2.3 Re-code**	Substitutes S's situated term with a more general mathematical term	S uses "distance," T quotes him as if he had said "difference"

Table 1 continued

Tactic	Description	Example
3.3 Customize interaction parameters	Introduces a new case or level, so that S attempt by induction or refinement to apply a previous method. T selects or modifies these dimensions strategically, e.g., to foster cognitive conflict	
3.3.1 Select qualitative case	Refers to an interaction dimension and uses qualitative descriptors to specify a degree/extent along this dimension	T asks S to try "higher" on the screen
3.3.2 Select quantitative case	Refers to an interaction dimension and uses quantitative descriptors to specify a particular value along this dimension	T: "So if your LH is at half, where should your RH be?"
3.3.3 Customize margin of error	Modifies S's task demands by tightening or loosening the interaction margin of error in response to manifest evidence of S's apparent (physical) capacity	T adjusts the "tolerance" value on the console to help S enact a green screen
3.4 Coach interaction	Distributes, orchestrates, guides, (co-) performs, (co-) simulates, and monitors physical interaction operations. [overlaps/intersects with other categories]	
3.4.1 Demonstrate	Enacts an optimal perceptuomotor interaction, possibly accompanied by explanation that highlights features and subgoals of this performance as well as their relation to data and principles, so that S will then imitate	T shows that it is possible to sustain a constant green screen while moving hands
3.4.2 Guide	Takes turns in performing solo interaction, while other person guides the performance physically and/or verbally	T holds both S's forearms and guides them up and down
3.4.3 Distribute	Coordinates and paces manipulation. In appropriate contexts, T might establish their own actions as the ad hoc independent variable, thus structuring the data gathering yet enabling S to discover each dependent-variable datum	T holds LH tr, S holds RH tr. T paces S along a sequence of paired-hand positions that may prompt discovery
4. Scaffold reflection and elaboration	Invites S to seek coherence within prior data and inference: highlights S's utterances; prompts recollection of pertinent data; and provides epistemic forms to organize data, identify conflicts, and formulate inference	

Table 1 continued

Tactic	Description	Example
4.1 Encourage evaluation	Asks S to attempt to confirm their theory with empirical data. Suitable when S does not appreciate a problem or conflict in their own inference	T: "Do you want to try and show me what you mean?"
4.2 Ask for summary	Asks S to relate previous activity and conversation succinctly. Enables S to select, chunk, structure, represent, and generalize; fosters metacognition; and enables T to assess S	T: "So… if you wanted to summarize what you've seen so far?"
4.3 Support summary	Performs actions that help S recall, organize, and condense the data into a briefer report, which foregrounds patterns	T holds LH tr to reproduce with S the sequence 1-2, 2-4, 3-6,…
4.4 (Re-)organize data	Makes implicit patterns in the data more salient via using multimodal and material semiotic resources for the production of implied or actual inscriptions or via rearranging objects in the physical problem space. Re-encodes data in forms that help S recruit associated meanings and production rules	*Ex 1*. T tabulates S's reported data on a board to help S notice latent numerical patterns. *Ex 2*. T asks S to re-order ratio utterance RH:LH as LH:RH
4.5 Recount	Reminds S what s/he had said and done earlier by re-evoking/reenacting a previous episode, including actions, discussion, inferences. Yet so doing, T explicitly or implicitly introduces supplementary structure	Returning S from the 3:2 challenging numerical case to the previous, easier 1:2 case, T recaps findings in this case.
4.6 Problematize	Prompts S reasoning by restating findings or inferences as juxtaposed one to the other. Underscores logical tensions	T: "And so, is that like 'doubling' or not?"
4.7 Generalify	Frames S's utterance/action as a rule-based case bearing broader validity. The utterance/action thus shifts epistemological status	
4.7.1 Echo	Repeats S's factual statement or a fragment thereof in a manner that connotes an opportunity for inference, for generalization. T thus highlights but does not code aspects of data S had detected	S: "…about…I donno… three squares higher." T: "*Three* squares higher."
4.7.2 Launch	Builds upon S's own words a sequence, pattern, or incomplete proposition that cues or implies a need for closure	T: "2 ahead, then 3, so what'll it be now?" S: "I think 4"

Table 1 continued

Tactic	Description	Example
5. Valorize	Uses explicit speech acts and/or affective inflection to communicate positive or negative judgment pertaining to prospective utility of S action or notion; informs S of the quality of their ideas or performance; cues to focus/drop particular efforts	
5.1 Positive valorizing	Praises, encourages, exclaims admiration, etc. Positively valorizing may encourage a challenged or frustrated S to perseverate amid difficulty	T: "So, you're getting pretty good at this!"
5.2 Negative valorizing	Marks S action as imperfect, usually by hems and haws, etc. Negative valorizing is liable to disempower S. Uses sparingly, sensitively, mostly to draw attention	T: "Let's make three… Woops… Is that working?"
6. Pre-orient: frame new semiotic resource as S's prospective means of better enacting, explaining, or evaluating interaction strategy	Affects uncertainty as to: (a) details, quality, or validity of S interaction strategy; or (b) referent of S speech/gesture. Then states that a new symbolic element is about to be introduced into the working area and specifies its purpose as bearing rhetorical utility in clarifying uncertainly. T thus implicitly problematizes S's work by suggesting it was inadequate. T both creates discursive need for repair and offers pragmatic means for repair	T: [numerals on] "So what we're gonna do now… Can you see the little numbers that appeared on the left?…. I wonder if those numbers in any way can help you explain to us the rule"

T the tutor, *S* the student, *Ex* example, *LH* left hand, *RH* right hand, *tr* hand-held electronic tracker device

*Tactic 1 has significant overlap with Tactic 3, and yet we wish to mark milestone events in typical tutorial activity sequences

**The analysis has revealed some rare cases of "re-code" that seem to be inadvertent tutor actions, that is, they apparently do not result from deliberate tactics but from the tutor unwittingly expressing his way of seeing the objects in echoing the student

utilities of representations as control devices. In gluing together these strange theoretical bed fellows, we were helped by instrumental genesis (Vérillon and Rabardel 1995), by which conceptual change is the residual effect of learning to wield new tools in accomplishing objectives (see also Salomon et al. 1991; Zhang and Norman 1994). Such wielding of new tools may include mutual adaptations between the cognitive agent and the material (or virtual) resources in the workspace (Kirsh 1996; Martin and Schwartz 2005; Zhang and Patel 2006).

3.2 Unique Dimensions of Embodied-Interaction Discovery-Based Mathematics Tutoring

A critical reading of Table 1 reveals tutorial practices rarely if ever treated in the literature on mathematics-education research.

3.2.1 Learning by Developing Perceptuomotor Skill

The tutor oriented the student's perceptuo*motor*—not just perceptual—orientation toward the activity; he customized the student's engagement parameters; and he coached via demonstration and hands-on guidance (see Items 3.4). This cluster of tactics reflects the design's roots in a grounded-cognition epistemology: we attempt to steer students toward developing particular ways of moving as much as we are steering them to see the domain of scrutiny in new ways. This steering may take on the guise of distributing across the tutor and student the two manual actions necessary to achieve the task objective.

Assisted performance may better enable a student to notice aspects of the action relevant to the target concept. This increased salience, we explain below, is due both to subjective and objective processes resulting from the assistance. First, when the instructor leads the action, students have more cognitive resources at their disposal for noticing patterns in the information. For example, when the tutor literally lifted a student's hands simultaneously, the student need not have monitored the precise position of each virtual object on the screen relative to the gridlines and was thus better inclined to notice relations *between* the objects. Second, the tutor's more rapid, smooth manipulation may enhance the detectability of the action's latent properties. For example, when some of the students were physically guided to move their hands faster within the "green zone" than they were able to do alone, they began commenting on the relative velocity of the hands and tying this observation to the hands' measured rate of motion.

3.2.2 Learning by Discovering Interaction Principles

The tutor did not explain to the student any logical or mathematical principles but instead shaped and steered the student's reflection and inference (see Items 4 and 6). It is perhaps surprising how well the students functioned in this "strange" discursive genre, where the tutor clues but does not disclose information, sometimes even affecting surprise at each new finding. For example, the tutor would ask the student to summarize, compare, and contrast certain observations and assertions that the student had uttered shortly before, but the tutor would not reveal the embedded generalization therein. Whereas it is difficult to evaluate to what extent the students were attuned to the tacit pragmatic message of the tutor's behavior—whether they "saw the wizard behind the curtain"—we note that this particular student cohort engaged fluidly in this adidactical situation (see Brousseau 1997). Only through future research with diverse populations will we be able to evaluate the commonality of this discursive tradition among school students.

3.2.3 Learning by Reconciling Latent Polysemy

We have noted the phenomenon of semantic complexity inherent to our domain of scrutiny and have identified Tutorial Tactic 4.6, Problematizing, as a practice that prompts and supports students in transitioning adroitly among different meanings of stimuli in the perceptual field (see also Items in 3.1–3.3). In particular, the MIT-P bimanual solution action is conceptually polysemous: it can be visualized either as two smooth motions with a constantly changing interval between the hands or as co-iterations of two different composite units (e.g., 1 and 2). These and other visualizations are logico-mathematically inter-derivative, and articulating their relations carries much of the design's learning objectives. That is, each of these strategies in and of itself captures a partial meaning or conceptual aspect of proportional equivalence, yet we see great potential in students understanding how these superficially distinct strategies are deeply related. For example, by reconciling multiplicative and additive visualizations of the gesture, students can ground multiplicative views of proportion in additive productions of equivalent ratios (Abrahamson et al. submitted).

The productive role of polysemy in mathematical discourse—and more generally, ambiguity, vagueness, or looseness—is becoming increasingly apparent (Foster 2011; Mamolo 2010; Newman et al. 1989; Rowland 1999). Polysemy has been used explicitly in pedagogical design (Abrahamson and Wilensky 2007). As such, it should be worthwhile to expand professional-perception frameworks so as to articulate how experts manage and utilize productive polysemy. Our future research will focus on methodology for encouraging and supporting students in exploring and integrating meanings of polysemous manipulation.

3.3 Fostering Hooks and Shifts

Having explained the tutorial tactics, we can now finally appraise their roles in fostering student discovery via guided, mediated interaction. In Abrahamson et al. (2011), we listed interaction criteria predicting hooks and shifts, and now we can juxtapose the tutorial tactics and discovery criteria. As the juxtaposition will demonstrate, the current study has retrospectively sharpened our classification of which discovery criteria fall more under the purview of the student as compared to the tutor. The current study of tutorial tactics thus characterizes the instructor's charge as equipping students for discovery by enabling them to fulfill prerequisite engagement criteria within their purview.

Prior to hooking and shifting with new objects, students have first to determine an interaction strategy ("Content," student purview), confirm its effectiveness ("Validation," student purview), and entertain it mentally at the moment new resources are introduced ("Priming," tutor purview). Just how the tutor introduces these objects is critical in positioning them for the student as potential resources with embedded utilities pertinent to the task at hand ("Framing," tutor's purview). Yet the student must be minimally familiar with these objects so as to recognize and avail of their contextual affordances ("Fluency," student prior knowledge). Furthermore, the particular way students have been visualizing the problem space has to be aligned in nuanced ways with the tutor's informed visualization ("Compatibility"), and these refined calibrations may emerge only later in the discourse, after the new objects have been introduced. Finally, the duration of inquiry may differ widely among students, so that the tutor should be guided by individual needs as monitored in real time ("Facilitation," tutor purview).

Tutorial Tactics 1–5 address the criteria under the student's purview by supporting their development and validation of effective interaction strategies. Tutorial Tactic 4.1, "Encourage Evaluation," specifically serves the criterion of Validation. Looking at criteria that fall under the tutor's purview, Priming and Framing are served by Tactic 6, "Pre-Orient." The Facilitation criterion is served by Context General practices of monitoring asymmetrical clinical discourse. Therein the pedagogical therapist does not directly suggest information or ideas but rather probes whether the patient can arrive at desirable insights through engaging in a semi-controlled, designed interaction.

Tutors use multimodal coaching so as to orient and organize students' engagement with, and reflection on the dynamical domain of scrutiny and thus enable them to determine and confirm latent patterns therein. Where a student arrives at an activity with insufficient Fluency, the tutor must orient the student to the artifact's embedded affordances and thus re-present it as bearing potential utilities pertinent to the student's current interaction strategy. Having to foster a student's professional vision of a new artifact on the fly is suboptimal for the constructivist agenda, because in so doing the student is deprived the empowering experience of drawing spontaneously on prior knowledge so as better to accomplish a task. This is yet another reason why a student's prior knowledge is critical to an effective learning progression (see also Fyfe et al. in press). Seeing as a student's prior mathematical fluency is important for the pedagogical efficacy of the activity, the criterion of Fluency, too, should fall under the instructor's purview, which underscores the pivotal role of assessment.

4 Conclusion

Theories of professional perception (Goodwin 1994; Stevens and Hall 1998) offer powerful constructs for modeling instructional interaction. Yet, we assert, for these theories to be durable and effective, they should be updated and expanded vis-à-vis evolving perspectives on learning (embodied cognition), technological advances driving innovative learning environments (embodied interaction), and pedagogical frameworks informing the implementation of emerging designs (guided inquiry).

As evidence supporting the above assertion, we have presented findings from the analysis of one constructivist tutor's instructional tactics. During our analysis, we attempted to work with categories and constructs from professional-perception frameworks. However, these frameworks did not readily fit our empirical data, so that we were compelled both to qualify some of the categories and to build new categories. As a result, we now have a tabulated taxonomy of tutorial tactics, which we have presented as our findings from this study.

As embodied-interaction technology becomes increasingly accessible, our field might do well to pay renewed theoretical attention to the very constitution of domains of scrutiny during instruction, and not only to how an instructor shapes student seeing of an existing domain. In particular, future design such as the Mathematical Imagery Trainer may increasingly engage children's perceptuo*motor* activity, a naturalistic mode of learning. That is, when students need to physically generate the domain even as they are investigating it, an instructor may need to coach this activity by hinting, highlighting, and coding aspects of *physical* and *spatial–temporal*, not only perceptual orientation (e.g., Gerofsky 2011).

Moreover, an embodied-cognition view on teaching and learning underscores the centrality of perceptuomotor activity regardless of the particular media supporting the instructional activity. Thus all mathematics-education researchers and practitioners could avail from greater attention to physical dimensions of students' multimodal utterance.

Finally, our work has highlighted and coded nuanced discursive actions tutors perform in order to scaffold students' mathematical inquiry. We evaluate these tactics as transcending medium and task, and so we hope that documenting these tactics could inform the practice of any teachers who are not already using these tactics intuitively.

If professional-perception frameworks were to assimilate dimensions of instructional interaction suggested by our analysis, they could stand better to inform pedagogy of indirect instruction. We view this as an important objective, seeing as indirect learning may better help students develop 21st Century skills, which differ from traditional vocational conceptualizations of what it means to be a professional.

Our findings bear relevance also to scholarly discussions around pedagogical methodology for fostering effective computer-based science inquiry practice (e.g., "structuring and problematizing," see Edelson and Reiser 2006; Reiser 2004). Notably, some instructional designers of scripted inquiry-based learning environments are informed by the belief that scientific methodology can be scaffolded by having children log and share their observations, articulate hypotheses, conduct controlled experiments, reflect on their findings, etc. (Slotta and Linn 2011). Whereas we share the objective that children develop powerful inquiry practices, we wish to see more learning environments that foster continuity from natural inquiry to disciplinary practice (Gopnik et al. 1999; Karmiloff-Smith 1988; Lakatos and Feyerabend 1999).[2] One intriguing line of research would be to design an artificial tutor that proceduralizes the tactics we have discovered. The greatest challenges of such a project, we foresee, would be in determining *what* elements on the screen children are attending to and, moreover, *how* they are attending to those elements and what *meanings* these elements evoke.

5 Limitations and Implications

We have evaluated the utility of professional-perception frameworks to shed light on non-standard expert–novice instructional sessions. Based on our findings, we have asserted that these frameworks require elaboration along several dimensions, which we have specified. Our assertions have been contingent on the assumption that our empirical data are representative of the phenomena in question. However, that may not be the case.

One obvious limitation of our study is a sampling problem. Namely, we have looked at the practice of but one tutor, whose professional practice might be idiosyncratic, thus mitigating the external validity of our findings. That is, we should exercise caution in making claims that our findings of this tutor's tactics generalize to, and might even inform the practice of the greater population of instructors.

A second problem is that this tutor operated in a particularly constrained setting. Namely, the tutorial session implemented a task-based semi-structured clinical-interview protocol serving a design-based research study, so that perhaps the tutor was after data as much as after learning, and these objectives may not always have converged.

Although these limitations all point to a need to gather further data, the particular settings of our study might actually bear some opportunity for internal validation. Namely, a possible methodology for triangulating our findings using the same corpus of data would be to compare the practice of expert and novice tutors who participated in the study. In fact, data from the current project have offered us some preliminary evaluation of our findings, because in addition to the PI, several researchers-in-training participated in conducting the interviews. Consequently, we have had opportunities to witness several occasions of novice tutors exercising tactics that appeared less effective. These observations, in turn, were very instrumental in "opening our eyes" to what the more experienced

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[2] That said, the particular embodied-interaction problems that we have implemented so far in the Mathematical Imagery Trainer perhaps do not demand of students to manage as much information as do inquiry tasks in biology, and, more generally, it is not unproblematic to compare design frameworks across STEM disciplines.

interviewer was doing "transparently." Namely, that which appeared "natural" or seamless in the experienced interviewer's practice thus became reified as a deliberate tactic. Whereas these observations are anecdotal, a brief example in the Appendix may serve two ends. We hope to demonstrate: (a) the viability of characterizing localized tactics as a means of studying professional instructional practice; and (b) the potential of this line of research to inform the training of researcher–interviewers as well as tutors and teachers.

The phenomenon of hooks and shifts, which we had studied primarily from the perspective of the student (Abrahamson et al. 2011), has now been complemented with a study of the tutor's actions that foster these discoveries. Semi-spontaneous discovery is not unique to embodied-interaction designs. Rather, it should be observed in any design genre wherein students are encouraged to use artifacts in the learning environment as means of accomplishing emergent objectives, whether enactive, expressive, or epistemic. For example, Abrahamson (2009a, b) has presented cases of students who made conceptual progress by using a compound event space—a collection of paper elements on the desk—as a means of expressing their intuitive predictions for the outcome distributions of a probability experiment. It is thus that designers seek to build for learners experiential continuity from culturally embedded to culture transforming practice (Noss and Hoyles 1996).

What is unique to embodied-interaction design, and hence for the student and teacher's interaction with materials and each other, is the role that physical action plays in inquiry—whether it is merely pragmatic or also epistemic (cf. Kirsh and Maglio 1994). Although manipulatory actions inherent to mathematical reasoning may be covert ("in the head"), they may be designed as embodied in overtly physical actions. Such is the case with our Mathematical Imagery Trainer, in which overt physical actions mark culturally specified embodied artifacts—visual–kinesthetic image schema bearing the semiotic potential of becoming conceptual performances via signification and contextualization into mathematical activity structures (Trninic and Abrahamson in press). That is why, as in pottery, dance, and martial arts, the embodied-interaction coach must closely monitor and correct student performance. Whether the student is willing to follow the classroom's multimodal leading discourse, however, may depend on the student's sense of identification (Heyd-Metzuyanim and Sfard 2012).

We end with a reflection on the multiplicity of theoretical perspectives we have drawn on for this study. As pedagogical designers with both a pragmatic leaning toward "making things work" and an analytic eye on "how things work," we find that this mélange of diverse models creates tightly textured arabesques rather than motley quilts. For us, this confluence of theory in the learning-sciences laboratory is testimony to the transformative potential of the design-based research approach, which is a constructionist empirical approach. Design-based research studies are both Petri dish and litmus test for what *theory* works. Yet as we continue to develop both pedagogical artifacts and theoretical models of teaching and learning, so we hope our design-based research will continue to contribute to emerging theories of design (Abrahamson submitted).

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Appendix

Novice Tutorial Tactics

Charlie (pseudonym), a novice tutor, was facilitating his first session ever, as part of his training as a graduate student. The elementary-school study participant had been manipulating the two cursors on the screen and had determined a demonstrably effective strategy for moving her hands while keeping the screen green. Charlie had thus reached the point along the protocol where he was to overlay a virtual grid upon the screen. He said to the student that he was about to bring up something on the screen and that she should see *whether that changes anything*. He then lit up the grid. The child picked up the tracker devices that had been lying on the desk. Just as before, she located a "green spot" and then lifted her hands further up, maintaining a green screen in accord with her existing strategy. No, she reported, nothing has changed. She laid down the trackers and did not proceed to avail of the grid.

Of the total of a near two-dozen students who participated in this study, this student was the only one who responded thus. Other students tended to appropriate the grid as a means of better enacting, explaining, or evaluating their strategy. In retrospective analysis, we realized that how the tutor frames the introduction of a new artifact partially predicts whether or not the student engages it as a useful instrument (Gutiérrez et al. 2011). Thereafter, Charlie learned to frame the introduction of new symbolic artifacts as potentially promoting the interaction, and the research group amended the protocol to provide the clinical interviewers with appropriate guidance.

Incidences such as this, which we have been archiving for training purposes, are essential in the preparation of interviewers, because they occasion opportunities for supervisors to flesh out implicit dimensions of their own practice. As such, for a PI charged with training graduate students as much as with conducting research, videotaped documentations of "interview bloopers" are vital for building a laboratory's organizational knowledge and capacity. Yet for the particular methodological needs of the current study, these incidences accentuate the rationale of our approach. Namely, professional tutors exercise a repertory of specific tactics that affect the nature (if not quality) of students' engagement in learning activities.

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