Tech Know Learn (2011) 16:55–85
DOI 10.1007/s10758-011-9177-y

# Hooks and Shifts: A Dialectical Study of Mediated Discovery

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Published online: 16 June 2011


**Abstract** Radical constructivists advocate discovery-based pedagogical regimes that enable students to incrementally and continuously adapt their cognitive structures to the instrumented cultural environment. Some sociocultural theorists, however, maintain that learning implies discontinuity in conceptual development, because novices must appropriate expert analyses that are schematically incommensurate with their naive views. Adopting a conciliatory, dialectical perspective, we concur that naive and analytic schemes are operationally distinct and that cultural–historical artifacts are instrumental in schematic reconfiguration yet argue that students can be steered to bootstrap this reconfiguration in situ; moreover, students can do so without any direct modeling from persons fluent in the situated use of the artifacts. To support the plausibility of this mediated-discovery hypothesis, we present and analyze vignettes selected from empirical data gathered in a conjecture-driven design-based research study investigating the microgenesis of proportional reasoning through guided engagement in technology-based embodied interaction. 22 Grade 4–6 students participated in individual or paired semi-structured tutorial clinical

Abrahamson's coauthors are all members of the Embodied Design Research Laboratory in the Graduate School of Education at the University of California, Berkeley. The construct of "hook and shift" first emerged during a debriefing interaction between Abrahamson and Trninic. Thanks to Mark Howison for his technological development work on the MIT, Daniel Reinholz for earlier contributions to the project and the art in the figures, and Brian Waismeyer and Lucie Vosicka for thinking with us and reviewing earlier drafts. The Kinemathics project was sponsored by a UC Berkeley Committee on Research Junior Faculty Research Grant (Abrahamson, PI) and IES pre-doctoral training grant R305B090026 (Gutiérrez, Trninic). Project findings were first presented to the Gesture Study Group at University of California, Berkeley (Abrahamson and Howison 2008). This paper builds on AERA presentations (Abrahamson and Howison 2010a; Abrahamson et al. 2011; Gutiérrez et al. 2011; Trninic et al. 2011) and conference proceedings of CHI (Howison et al. 2011) and PME-NA (Reinholz et al. 2010; Trninic et al. 2010). We wish to thank Andy diSessa, Maria Alessandra Mariotti, Luis Radford, Geoff Saxe, Jeanne Bamberger, and Katharine G. Young for illuminating conversations around our data analyses and theoretical modeling and Tobin White for his comments on a draft. We further thank Lulu Healy and TKL anonymous reviewers who offered very constructive critiques. Live and learn.

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interviews, in which they were tasked to remote-control the location of virtual objects on a computer display monitor so as to elicit a target feedback of making the screen green. The screen would be green only when the objects were manipulated on the screen in accord with a "mystery" rule. Once the participants had developed and articulated a successful manipulation strategy, we interpolated various symbolic artifacts onto the problem space, such as a Cartesian grid. Participants appropriated the artifacts as strategic or discursive means of accomplishing their goals. Yet, so doing, they found themselves attending to and engaging certain other embedded affordances in these artifacts that they had not initially noticed yet were supporting performance subgoals. Consequently, their operation schemas were surreptitiously modulated or reconfigured—they saw the situation anew and, moreover, acknowledged their emergent strategies as enabling advantageous interaction. We propose to characterize this two-step guided re-invention process as: (a) *hooking*—engaging an artifact as an enabling, enactive, enhancing, evaluative, or explanatory means of effecting and elaborating a current strategy; and (b) *shifting*—tacitly reconfiguring current strategy in response to the hooked artifact's emergent affordances that are disclosed only through actively engaging the artifact. Looking closely at two cases and surveying others, we delineate mediated interaction factors enabling or impeding hook-and-shift learning. The apparent cognitive–pedagogical utility of these behaviors suggests that this ontological innovation could inform the development of a heuristic design principle for deliberately fostering similar learning experiences.

**Keywords** Additive reasoning · Cognition · Conceptual change · Design-based research · Discovery · Embodied interaction · Functional extension · Guided reinvention · Mathematics education · Proportion · Proportional reasoning · Remote control · Sociocultural · Symbolic artifact · Virtual object

*Given a perpetually new natural and historical situation to control, the perceiving subject undergoes a continued birth; at each instant it is something new.* (Merleau-Ponty 1964, p. 6)

## 1 Introduction and Objectives

Consider an apprentice carpenter who, tasked by his Master to drive a screw into a solid wood plank, elects to apply a hammer. He sedulously pounds the screw with great might but minor success, occasionally striking his thumb. The Master carpenter, alarmed to witness this travesty, hastily proffers the apprentice a screwdriver. However, being a radical-constructivist Master carpenter, she merely places the screwdriver within the apprentice's visual field. The apprentice responds to the cue: he lifts the screwdriver, inserts its tip into the screw-head groove, lifts the hammer again and… pounds the screwdriver's handle butt, a larger and thus more convenient and safer surface. So doing, though, his clenched fist that holds the screwdriver in place inadvertently rotates it and, with it, the screw. Ah, observes the apprentice, this is a better way of handling things. He lays down the hammer and thereafter applies the screwdriver masterfully.

A learning scientist who happened to be on premises notes that the normative uses of the hammer and screwdriver are markedly distinct in terms of their sensorimotor engagement, kinesthetics, and perceptuomotor feedback loops. Thus, reasons the scientist, we might analyze the vignette as demonstrating discontinuity between the hammer and screwdriver action plans for driving a metallic artifice into a wooden surface. And yet, the scholar muses, the apprentice's subjective experience was not discontinuous but a phenomenological flow of doing, noticing, evaluating, and adjusting. Moreover, the normative use of the screwdriver was never modeled for the apprentice, so we can hardly say he appropriated the use through imitation and emulation. For example, the particular perceptual features of the screwdriver relevant to its normative use were never made salient to the apprentice through explicit highlighting. Rather, initially responding to the Master's pragmatic directive to use the screwdriver, the apprentice recognized how the screwdriver may better enable the execution of the hammer schema. Then, performing the newly instrumented schema, he stumbled upon the screwdriver's more expedient application. We might say that the apprentice initially "hooked" the screwdriver (appropriated the artifact) as a means of amplifying the implementation of his hammer schema, and yet in the course of enacting this schema with the screwdriver he "shifted" (reappropriated the artifact) to a new strategic horizon that elicited and availed of interactive potential embedded in this cultural–historical object.

This paper is about such hooks and shifts in mathematics education. As in the allegorical case, above, we will witness: (a) instructors performing actions, utterances, and pragmatic cues in introducing artifacts into a problem space as potential means of accomplishing assigned tasks; and (b) students initially using these artifacts to extend their naïve schemas yet, through doing so, recognizing the artifacts' normative disciplinary utility even in the absence of any direct demonstration. However, there are at least two relevant differences between our allegorical and empirical contexts.

First, unlike the allegory, our study deals not with directly graspable objects applied to other media but with symbolic artifacts—virtual objects instantiated in computer-based media—whose grasp is mediated by handling remote-control electronic operating devices and whose effect is mediated by computational procedures. From a phenomenological perspective, though, we do not view symbolic and physical artifacts as ontologically distinct, because they are equally apparent as objects-to-work-with in the perceptual field. However, whereas the most skilled carpenters still require actual hammers or hammer-like objects to pound nails, mathematicians who have sufficiently rehearsed and reflected on distributed interactions with new symbolic artifacts sometimes cognitively instantiate these forms so as ultimately to obviate any need to recreate them in inscriptional media—the new forms become part of the utilization schemas the mathematicians are equipped to apply in modeling and solving newly encountered problem situations (Collins and Ferguson 1993; Schoenfeld 1998; Stigler 1984; van den Heuvel-Panhuizen 2003; Vérillon and Rabardel 1995).[1]

Second, the instructors and students in our empirical data engaged in discourse, thus supplementing and coordinating the "doing" task (enacting) with a "showing" task (explaining). Explaining one's own behavior turns out to be a complex semiotic endeavor demanding of students to reify and proceduralize tacit aspects of unreflective, instrumented embodied interaction (Bamberger 1999; diSessa et al. 1991; Kuchinsky et al. 2011; Papert 1980; Shreyar et al. 2010). In line with the Vygotskian thesis on the formative role of discourse in human reasoning and development, representation requirements emerging in discursive tasks may thus create opportunity for conceptual reconfiguration beyond the localized manipulation task per se.

This paper argues for the plausibility of the hooks-and-shifts analysis of mediated mathematical discovery and the utility of this analysis for research on, and practice of

[1] We recognize that this distinction between carpenters' and mathematicians' practice is not clear-cut. Carpenters can internalize some instruments to a degree (e.g., measuring tape), and many mathematical forms still must be inscribed to provide optimal use.

mathematics education. In particular, as we elaborate below, the proposed perspective may bear on research efforts to build theoretical models that account dialectically for both endogenous and cultural forces at play in the microgenesis of mathematics learning. In turn, a research project to develop such dialectical theory of mediated mathematical discovery is important also for practice and policy, inasmuch as reinvention-based learning is viewed as a pedagogically desirable outcome of formative intervention (Engeström 2008; Freudenthal 1968, 1971; Gravemeijer 1999; von Glasersfeld 1992).

Note that the objective of this paper is not so much to propose a new theoretical approach to relations between artifacts and mathematical cognition and practice as much as to use common interpretive methodology to describe and characterize a form of interactive learning that appears to be under-documented and under-theorized yet pedagogically desirable. Namely, we are referring to situations where learners working with mathematical artifacts stumble upon and appropriate cultural forms of reasoning that have not been directly modeled for them yet qualitatively shift their operational strategy in the service of their contextual objectives.

Let us stress where we see the limitation of existing approaches in explaining these particular features of our data. A range of distinctive research programs broadly characterizeable as sociocultural or neo-Vygotskian have offered nuanced theoretical models that clearly articulate how learners "graft" disciplinary and practical knowledge onto their naïve action plans (Bartolini Bussi and Mariotti 2008; Radford 2003; Saxe 2004; Sfard 2002; Vérillon and Rabardel 1995). These models spell out microphases of learners' ontogenetic conceptual development via their mediated appropriation of artifacts in the context of participating in organized instructional and communal activities. In all these models, more experienced persons play critical roles: they support and steer novices in appreciating the added value of adopting new ways of seeing and acting upon the world in pursuit of collective or negotiated objectives.

We have found these theoretical models powerfully illuminating of many interesting moments in our data, and in particular for interpreting students' strategic uptake of the artifacts we interpolated into their learning environment. We thus view these models ultimately as explaining what we are calling the "hook"—when problem solvers are encouraged through pragmatic and discursive cues to consider the contextual utility of a particular symbolical artifact placed in their working space. What these models do not appear to explain, however, are instances when problem solvers working with an artifact spontaneously re-instrumentalize it in a manner that has not been modeled for them, what we are calling the "shift." Indeed, we were initially alerted to these moments in our empirical data precisely because in so doing, students anticipated particular sophisticated uses of the artifacts that we had planned to demonstrate only later on in the interview.

## 2 Theoretical Background and Deliberations

How does instructional intervention foster conceptual change? This very general research problem has spawned vast educational scholarship, which has both oriented the construction of theoretical models of situated cognitive dynamics and, in parallel, informed the design of materials, activities, and principles for teaching and learning targeted subject matter content (diSessa 2005). Working within the specific disciplinary domain of mathematics, our research, too, is concerned with understanding the phenomenon of conceptual microgenesis. Thus, informed by the interdisciplinary learning sciences, we generate and build from empirical data to model the emergence of activity structures consisting of human agents expanding their skills through goal-oriented uptake of pedagogical artifacts (Abrahamson 2009a, b, c).

Our particular context of inquiry into the proposed dual construct of hook-and-shift was an ongoing analysis of a corpus of empirical data that we gathered in a conjecture-driven design-based research study, in which we investigated the microgenesis of mathematical forms of reasoning that students develop through participating in embodied-interaction designs. In particular, we report here on the effects of interpolating objects into children's problem-solving space on their reasoning and learning. Prior to having developed the construct through intensive data analyses, we had not understood how student–interviewer interactions sometimes resulted in students using new interaction strategies nor why at other times they did not. Our emergent construct was instrumental in modeling these challenging data episodes. In particular, the construct enabled us to identify within student–interviewer interactions a set of factors that appear to determine or at least characterize the nature and ultimate pedagogical quality and effect of these embodied-interaction instructional activities. In the current paper, we explain and demonstrate the construct in detail and highlight several of these interaction factors in our case analyses. Elsewhere, we elaborate on these interaction dimensions, demonstrate in our empirical data success and failure vignettes for each of the dimensions, and offer conclusions from summative quantitative displays of our findings (see Gutiérrez et al. 2011, and see the Conclusions section of this paper for a summary and overview of all the interaction dimensions).

## 2.1 Study Rationale: Making Sense of Hooks and Shifts

Our case analyses suggest that students experiencing pedagogical perturbation are likely to undergo conceptual change that manifests as substituting one working theory for another. Such major schematic reconfiguration, it has been argued by some Vygotskian theorists, marks the inherent incommensurability of naïve and scientific views (Newman et al. 1989). Whereas we concur that students' skills may change dramatically as they appropriate cultural artifacts, *such dramatic change need not imply subjective discontinuity*, if the students initiate the strategy reconfiguration. As such, learning mathematical forms of reasoning through problem solving can be modeled as a concatenation of inflection points, when one operational schema modulates into another. In particular, in the course of deploying schemas for accomplishing assigned tasks new affordances emerge into consciousness only through *post facto* articulation and evaluation of one's own action and utterance. In some sense, this action-before-concept interpretation of conceptual change maintains the sociocultural notion of participatory activity as preceding internalization, only that participants' micro-actions that precede their evaluative reflection need not be emulative of common practice but can emerge semi-spontaneously through the process of engaging received cultural objects as problem-solving instruments.

A conjecture of subjective continuity across schematic modulation would be supported by evidence of student agency in applying objects whose use is never modeled by an instructor, because such appropriation would mark that the students recognized the inherent potential of the objects with respect to the problem space as they perceived it. Our case analyses will present examples of student agency in not only appropriating artifacts whose disciplinary use was never demonstrated to them but also, through engaging these artifacts, discovering more advanced strategies these artifacts enable.

## 2.2 Hooks and Shifts in Light of Learning Sciences Theory

Instead of hammers and screwdrivers, the analogical "objects" at the center of our investigation are symbolic artifacts. We assume the epistemological position that symbolic artifacts bear mediating, formative, and constitutive roles in the enactment, appropriation, and reenactment of disciplinary practice (Hutchins 1995; Norman 1991; Stetsenko 2002; Vygotsky 1934/1962). That is, we recognize that engaging external representations, such as a notation system, may shape and re-shape a problem space for individuals (Bamberger 2010; Meira 2002; Noss et al. 1997; Zhang and Norman 1994) or even re-structurate entire disciplines (Brock and Price 1980; Goody 1977; McLuhan 1964; Olson 1994; Wilensky and Papert 2010).

Looking closely at the *process* of reasoning with cultural forms, we concur with Saxe (2004) that individuals engaged in solving a collective problem will wield available forms, such as symbolic artifacts, if they identify these ad hoc as advantageous means of accomplishing a goal (cf. Radford 2003; Sfard 2002). Also, we recognize the pivotal roles of more knowledgeable members of the community in helping novices come to see the world anew by directly disciplining or domesticating their professional eye (Goodwin 1994; Radford 2010; Stevens and Hall 1998). Indeed, interpolating symbolic artifacts into the problem-solving process should certainly be viewed as implicit invitations to see the situation anew through the lens of the artifacts' emergent affordances. In particular, we further maintain that by utilizing these proposed symbolic artifacts, students' goals may become implicitly modified by structure information they perceive in these artifacts as potential horizons of engagement (cf. Gelman and Williams 1998; Schön 1992). Consequently, students who *hook* onto some symbolic artifact as an apparent means of enhancing and/or talking through their problem-solving strategy may gravitate toward, and ultimately *shift* into a different, unanticipated strategy, which they may then evaluate as functionally superior.

The two-step sequential structure of the hook-and-shift construct bespeaks our emphasis on reasoning as a process, in which each mirco-interaction with an artifact frames the emergence of the next cognitive operation (McNeill and Duncan 2000; Slobin 1996). This distributed problem-solving process, in which students iteratively "see" then "move" then "see" anew, marks the guided reinvention of mathematical concepts as a form of conversation with materials (Bamberger and Schön 1983, 1991). Students' inventions lead them from one stable cognitive structure to another, and although these structures may be very different, the passage between them is experienced as continuous.

In our qualitative analyses, we searched for theoretical accounts of learning by which to model the shifts, yet existing literature took us only so far in making sense of these data. A strong candidate was the Instrumented Activity Situations model (Vérillon and Rabardel 1995). This model presents learning as reflexive dialectics within a triadic structure of subject (mind), artifact (matter), and objective (task). In particular, the model details individual learning as a process of constructing and accommodating cognitive schemas through attempting to accomplish objectives using external artifacts. The model elegantly explains the development and practice of new fluencies through appropriating cultural tools that have evolved historically as useful means of accomplishing similar objectives. Yet the model has not been elaborated to account for individuals' discovery of qualitatively new ways of using these tools, that is, on-the-fly, idiosyncratic, inadvertent re-instrumentalizations of the tools, by which the tools' embedded potentialities (affordances) emerge for the user's utilization only in the course of using the tools, from prehension to comprehension.

Another candidate model was Peirce's construct of "hypostatic abstraction," which describes the discovery of meaningful patterns in information arrays as a form of diagrammatic reasoning (Hoffmann 2003). Yet this groundbreaking pragmatist notion does not readily apply to the case of action-based practices utilizing cultural artifacts.

We closely studied the work of Sfard (2002, 2007) to evaluate its bearing on our data. Indeed, her constructs "intimations and implementations" elegantly explain nuances in students' adoption and evaluation of symbolical displays, by which their intuitive, pre-articulated forms of reasoning are reshaped. Yet, again, this model addresses the "hook" but not the "shift." The same issue holds for the semiotic–cultural theory of objectification (Radford 2003), which powerfully explicates what we are calling the "hook" but does not yet illuminate the "shift." Similar, the theory of semiotic mediation posits the teacher's formative role in steering students toward developing mathematical signs from their "artifact signs" (Bartolini Bussi and Mariotti 2008; Mariotti 2009). Whereas the model insightfully analyzes the evolution of signs from the semiotic potential of pedagogical instruments and through to their contextual uptake in problem-solving tasks, the model is not particularly geared to illuminate serendipitous aspects of this evolution. The form–function model (Saxe 2004) does describe and theorize the sociogenesis of a community re-instrumentalizing available forms so as to serve new functions in response to emerging collective problems. Moreover, whereas the phenomena of interest in Saxe's earlier work involved changing historical contexts of indigenous commercial practice, his research program has evolved to encompass mathematics classrooms (Saxe et al. 2009). Yet classroom cases of spontaneous microgenesis are difficult to document as they enfold, more so when the nature of the particular instructional tasks does not lend itself ideally to shifting strategy.

Because sociocultural models were helping us explain the hooks but not the shifts, we turned to what we view as complementary theoretical models focusing on individuals' cognitive agency in problem solving. Indeed, as stated by Karmiloff–Smith (1988, p. 184), "Although plunged from the start into a social context, the child is also an individual cognitive organism and much of her theory building is endogenously provoked rather than socially mediated." Namely, we looked at empirically supported theories of innate or very early perceptual inclination (Gelman 1998; Xu and Denison 2009) and naturalistic inferential mechanisms (Abrahamson 2008; Bakker and Derry 2011; Gigerenzer and Brighton 2009; Shank 1998; Thagard 2010; Tirosh and Stavy 1999).

Working within these complementary traditions, we thus offer the notion of artifact-mediated discovery as a potentially viable, uncompromising dialectical synthesis of pedagogical perspectives stemming from the respective epistemological positions of radical constructivism (von Glasersfeld 1992) and socioculturalism (Newman et al. 1989). In so doing, we attempt to resolve some of the apparent theoretical tensions inherent in scholars' characterization of education as learners either re-inventing tools or appropriating the mediated use of ready-made tools. Ultimately, accounting for the dialectical roles of "nature" and "nurture" in human problem solving enables us both to make sense of student behavior that cannot be modeled independently by either perspective and to design better activities for mediated discovery of mathematical forms.

In sum, a particular appeal of the proposed construct "hook and shift," which may render it of broad use beyond the localized contexts in which it was discovered, is the implication that students can bootstrap cultural forms and principles recognizable by experts as generative for grounding mathematical concepts. We posit that a hooks-and-shifts perspective on conceptual change may hone scholarly efforts to frame and clarify historically parallel theoretical positions on the nature and import of students' cognitive agency in developing mathematically relevant ways of seeing and doing. As such, the proposed construct may equip us better to evaluate and avail of pedagogical philosophies derived from these divergent epistemological perspectives. An ideal outcome of this study would be to promote the dialectical exploration of possible complementarities among these traditionally vying views and concomitant practices.

### 2.3 Proposed Contribution

We believe that we are witnessing multiple cases where, by virtue of engaging instruments introduced into a problem space—namely symbolic artifacts that an instructor layered onto a computer-based microworld and presented as possibly helpful for the students—the students' problem-solving strategies transformed in conceptually important ways in line with our didactical objectives. It is this unanticipated transformation—a goal-oriented, artifact-catalyzed conceptual reconfiguration—that we wish to understand. Granted, we do not as yet have the requisite empirical cargo for quantifying a relation between this phenomenon and measured learning gains, however we believe the data episodes and interaction dimensions we shall present speak for themselves in terms of demonstrating the pedagogical scope and bearing of our findings.

We hope to contribute a proposal for how students bootstrap themselves to higher levels of mathematical reasoning by virtue of engaging symbolic artifacts as problem-solving tools (cf. Hall 2001; Neuman 2001). Specifically, we propose that when students engage artifacts that are framed as bearing problem-solving utility, contextually salient affordances of these artifacts re-orient students' naïve actions, such that the students find themselves employing new, potentially more sophisticated forms. We deliberately use here the vague colloquial idiom "find themselves," because we conjecture that students experience conceptual change somewhat inadvertently and *often realize vital aspects of their discovery only after they have modulated their schema* to assimilate emergent features in the environment. That is, students initially recognize the new artifact either as an auspicious means of enhancing their control over the interaction space or as a discursive means of explaining and evaluating their strategy, in possible accord with the instructor's pragmatic prompt (i.e. they "hook"). Yet as they engage the artifact, embedded meanings of its features present themselves as more powerful operative–discursive grips on the interactive situation, so that the original strategy becomes reconfigured (i.e. they "shift").

Studying mathematics learning through the lens of hooks and shifts, we maintain, is valuable to educational research and practice. If supported, the construct could inform the development of the "dialectical" approach (diSessa 2008) that negotiates cognitivist and sociocultural perspectives on mathematics and science learning processes (Abrahamson 2009c; Cole and Wertsch 1996; diSessa et al. 2010; Greeno and van de Sande 2007; Halldén et al. 2008; Vérillon and Rabardel 1995). On the one hand, the proposed hook-and-shift mechanism casts human learning as deeply dependent on cultural artifacts, but on the other hand, the construct also suggests that conceptual reconfiguration is not as discontinuous as characterized in the sociocultural literature. As such, students can remain *connected* (Wilensky 1997) to mathematical ideas, even as they radically reconfigure their operation schemes toward expert practice. The construct of hooks and shifts could also inform the development of pedagogical frameworks for instructional design and classroom regimes. Namely, the hook-and-shift perspective envisions a compatibility of ostensibly orthogonal stances, by which: (1) students need to "discover" mathematical knowledge (von Glasersfeld 1987); and (2) teachers need to "funnel" student inquiry (Voigt 1995).

## 3 Data Source: A Conjecture-Driven Design-Based Research Study of the Emergence of Proportional Reasoning from Guided Embodied-Interaction Problem-Solving Activity

This paper reports on findings from an investigation into the nature of situated problem solving. The investigation was conducted in the form of collaborative, intensive microgenetic analyses (Schoenfeld et al. 1991). In these analyses, we applied general principles of grounded theory (Glaser and Strauss 1967) as a means of identifying patterns observed in the behavior of mathematics learners. The investigation yielded the identification of an unfamiliar behavioral pattern, and we coined the phrase “hooks and shifts” to name the pattern. The “hooks and shifts” construct emerging from these analyses is offered as an ontological innovation that may inform the practice of fellow researchers and practitioners of mathematical cognition and instruction (cf. diSessa and Cobb 2004).

Our data corpus consists of videotapes and fieldnotes gathered during 20 clinical interviews with 4th–6th grade students who participated voluntarily in the implementation of an experimental design for proportion (Abrahamson and Howison 2010a; Reinholz et al. 2010). This section explains the rationales of the instructional and experimental designs created for, and used in our study. An expanded introduction of these source data—the design’s materials, activity, and protocol-based procedure—is necessary so as to prepare the reader for the data analyses.

Our research project was conducted in accord with the design-based research approach (DBR), a relatively new approach to the study of education-related phenomena, in which instructional theory and materials are codeveloped simultaneously, interdependently, reciprocally, and iteratively (Collins 1992; Confrey 2005; Edelson 2002; Engeström 2008; Kelly 2003; Sandoval and Bell 2004).[2] The initial phase of our project was explorative: we collected empirical data that would enable the emergence and development of viable models of student reasoning, even as we were working iteratively on improving both the instructional and experimental designs. We thus elected to organize this DBR investigation in the form of generative case studies (Clement 2000). Specifically, we gathered empirical data for a set of case studies by devising and administering a task-based Piagetian semi-structured clinical interview (diSessa 2007; Ginsburg 1997; Goldin 2000). We spread the implementation of these interviews, conducting no more than two per day, such that from day to day we would be able to introduce changes to the materials, activities, and protocol in light of the emergence and refinement of theoretical constructs. These rapid-prototyping changes were based on preliminary analyses of accumulating data consisting of fieldnotes, video recordings, verbal transcriptions, and minutes from our team’s daily debrief meetings and electronic communications (all posted online and collaboratively edited day by day). Thus, both the interview protocol and the interactive affordances of the instructional materials evolved as we progressed through the pool of student volunteers. Most notably, we gradually incorporated into the protocol activities and prompts that the researchers initiated spontaneously during the interview and appeared to elicit student reasoning in line with the hypothetical cognitive construct we began developing.

[2] We view the semi-structured nature of our task-based clinical interview as well as the iterative modification of the protocol from one interview to the next as affording viable responses to Engeström’s thoughtful critique of some genres of design-based research as being non-responsive. In fact, we view students’ shifts as manifest evidence that our interventions are formative, not linear; and that our practice accords with the Vygotskian principle of “double stimulation” (Engeström 2008, p. 5).

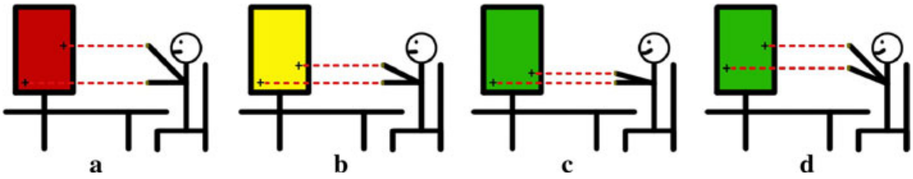

**Fig. 1** The Mathematical Imagery Trainer (MIT) set at a 1:2 ratio, so that the right hand needs to be twice as high along the monitor as the left hand: **a** incorrect performance (*red* feedback); **b** almost correct performance (*yellow* feedback); **c** correct performance (*green* feedback); and **d** another instance of correct performance. (Color figure online)

That is, as is typical of DBR studies, our experimental design was driven by a conjecture respecting a specific cognitive capacity that is usually dormant in the instruction of some targeted content yet that, given appropriate pedagogical settings, could potentially be elicited and leveraged as a powerful means of grounding mathematical concepts and solution procedures (Confrey 1998). As we explain below, the dormant cognitive capacity that our design targeted relates to the intrinsically embodied nature of mathematical concepts, reasoning, and learning.

Our conjecture, which drew its inspiration from the embodied/enactive approach (Barsalou 1999; Lakoff and Núñez 2000; Nemirovsky 2003; Núñez et al. 1999), was that some mathematical concepts are difficult to learn due to a resource constraint of mundane life. Namely, everyday being does not occasion students with opportunities to embody and rehearse the particular dynamic schemes that would form requisite cognitive substrate for meaningfully appropriating the target concepts' disciplinary analysis of situated phenomena. Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—"additive" solutions (e.g., "2/3 = (2 + 2)/(3 + 2) = 4/5"—cf. Behr et al. 1993)—indicate their lack of multimodal kinesthetic–visual action images with which to model and solve situations bearing proportional relations (Goldin 1987; Pirie and Kieren 1994).

Accordingly, we engineered an embodied-interaction computer-supported inquiry activity for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the *Mathematical Imagery Trainer* (MIT; see Fig. 1).

The MIT device measures the heights of the users' hands above the desk. When these heights (e.g., 10″ and 20″) relate in accord with the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display at an appropriate rate (e.g., raising her hands by 5″ and 10″, respectively, resulting in 15″ and 30″), the screen will remain green; otherwise, such as if she maintains a fixed distance between her hands while moving them up (e.g., raising both hands 5″, resulting in 15″ and 25″), the screen will turn red. Study participants were tasked first to make the screen green and then, once they had done so, to maintain a green screen even as they moved their hands.[3]

[3] Interviews consisted primarily of working with the MIT. In designing the MIT, we leveraged the high-resolution infrared camera available in the Nintendo Wii remote to perform motion tracking of students' hands, similar to work by Johnny Lee (2008). The Wii remote is a standard Bluetooth device, with several open-source libraries available to access it through Java or C#. In an earlier version of the MIT, an array of 84 infrared (940 nm) LEDs aligned with the camera provides out light (source), and 3 M 3000X high-gain reflective tape attached to a tennis ball enables effective motion capture at distances as far as 12 feet. In use, infrared rays emanate from the MIT, reflect off tape covering tennis balls held by the student, and are then

In line with our design conjecture respecting the embodied nature of mathematical reasoning, we expected that once students found a first pair of hand locations resulting in a green screen, they would search for another bimanual configuration by moving both hands up or down while *maintaining a fixed distance between them* rather than changing the distance; consequently, the screen would turn red. We viewed such hypothesized behavior as marking legitimate interpretation of the interactive inquiry task (Borovcnik and Bentz 1991; Smith et al. 1993). Namely, students initially have no information to suggest that a fixed-distance expectation would prove incorrect for the particular context they are still exploring—students thus default to their simplest available schema for generating invariance, that is, a motor action plan for maintaining identity (actual equivalence) rather than similarity (proportional equivalence).

We conceptualized such inductive reasoning about a spatial–dynamical, non-numerical phenomenon to underlie and anticipate these students' prospective mathematical errors. That is, naïvely expecting to generate invariance by enacting absolute rather than proportional equivalence between spatial extensions demonstrates and explains students' typical "additive errors" in solving rational-number problems, such as in reasoning that "1/2 = 2/3." Thus, we view fixed-distance *physical* actions as external manifestations anticipating inappropriate fixed-difference *symbolic* solutions. Again, our conjecture was that fixed-distance reasoning indicates a limited personal history of interacting with changing-distance situations of proportional covariation.[4]

Our design rationale was to "phenomenalize" the conceptual system of proportion (cf. Pratt and Kapadia 2009) in the form of an initially asymbolic, immersive microworld. Therein, our participants would experience an opportunity to encounter and ultimately construct a new, embodied ontology of variant physical action (changing distance) effecting invariant sensory feedback (green). Finally, the protocol included instructions for the interviewer to incrementally layer onto the microworld supplementary mathematical instruments, such as a Cartesian grid. Our intent was that the participants could be guided to appropriate these symbolic artifacts as means of expressing their discovery of variant-interval equivalence and, so doing, would re-invent mathematical principles of proportional equivalence. We thus hoped to scaffold the development of proportional schemas as emerging yet gradually differentiating from additive schemas (see Confrey 1998, for an alternative embodied approach to the mathematics of proportion; see Fuson and Abrahamson 2005).

At first, the condition for green was set as a 1:2 ratio, and no feedback other than the background color was given (see Fig. 2a for a system overview; see Fig. 2b—this challenging condition was used only in the last six interviews). Then, crosshairs were

Footnote 3 continued
sensed, interpreted, and visually represented on a large display in the form of two crosshair symbols (trackers; see Fig. 2a). In the current version of the MIT, students point LED beams directly at the special camera. The display is calibrated so as to continuously position the crosshairs at the actual physical height of its controlling hand in an attempt to enhance the embodied experience of remote manipulation.

[4] The mathematical concept of proportion is one of several interrelated concepts in the multiplicative conceptual field, which also includes rational numbers and other intensive quantities (Vergnaud 1983, 2009). In order to enable students to construct proportion as a new equivalence class, we needed a technological contrivance that associated pairs of ontologically independent left and right hand locations—a token of what Vergnaud (1983) named *isomorphism of measure*—in the form of a single epistemic entity, a *product of measure*, wherein *x* and *y* are ontically integrated as a single intensive quantity *x*/*y*. In so doing, we hoped, students would experience the ordered pairs [2 3] and [4 6] as "the same." Thus, the MIT links isomorphism-of-measure input with product-of-measure sensory feedback. As such, the MIT resembles designs used by Yerushalmy and collaborators (Botzer and Yerushalmy 2008; Yerushalmy 1997).

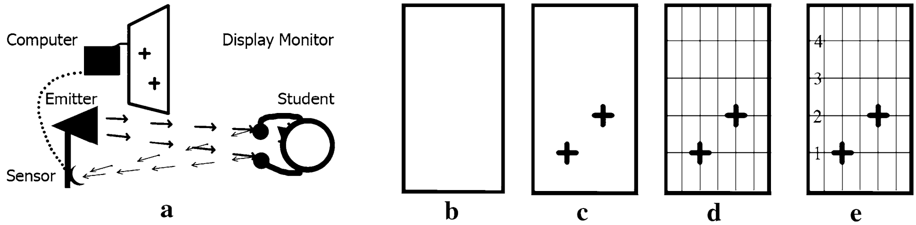


**Fig. 2** The Mathematical Imagery Trainer: **a** *top view* of the system featuring the earlier MIT version, in which students held tennis balls with reflective tape. **b**–**e** are schematic representations of different display configurations, beginning with (**b**) a blank screen, and then featuring a set of symbolical objects incrementally overlain onto the display: **c** crosshairs; **d** a grid; and **e** numerals along the *y*-axis of the grid

introduced that "mirrored" the location of participants' hands (see Fig. 2c). Next, a grid was overlain on the display monitor to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions (see Fig. 2d). In time, the numerical labels "1, 2, 3,…" were overlain on the grid's vertical axis on the left of the screen to help students construct further meanings by more readily recruiting arithmetic knowledge and skills and more efficiently distributing the problem-solving task (see Fig. 2e). Not treated in this paper is yet another structure layered onto the screen, namely a ratio table with interactive affordances (see in Reinholz et al. 2010).

Participants included 22 students from a private K − 8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students; one student participated twice). Students participated either individually (17 of the 20 interviews) or paired (the last 3 interviews) in a semi-structured interview (duration: mean 70 min; SD 20 min). For this a posteriori study, we initially drew on the last 15 interviews, wherein our protocol had stabilized (for a total of 18 participants—see Gutiérrez et al. 2011). Yet for this particular journal report, which is more expansive on theory, we were obliged to focus only on two paradigmatic case studies (but see the Conclusions section for an outline of findings from the full analysis).

In other publications emerging from this study, we reported on students' apparent learning trajectories through the interview (Abrahamson and Howison 2010b; Reinholz et al. 2010). Those publications focused on the range of mathematical meanings that students generated as they engaged the problem-solving inquiry activity. In the current study, we moved on from the "whether" to the "how" and asked:

> What heuristic, semiotic, discursive, and pragmatic mechanisms facilitated and modulated the participants' strategic change in this artifact-mediated embodied-interaction design?

Our proposed construct of "hook and shift" is offered as a partial answer to this question.

## 4 Case Analyses: Progressive Mathematization Through Hooks and Shifts

The semi-structured clinical interview protocol used in this study guided the interviewer to sequentially introduce the following set of symbolic artifacts into the learning environment so as to support the participants' problem solving and, so doing, foster their mathematical growth: (1) a pair of crosshairs mirroring the users' hand positions; (2) a Cartesian grid of perpendicular vertical and horizontal lines; and (3) numerals rising from zero along the grid's *y*-axis. The objective of this section is to present a set of vignettes selected from two interviews so as to demonstrate participants' behavior before and immediately after the interviewer layered a symbolic artifact onto the learning environment. We view these behaviors as cases of students bootstrapping new proto-mathematical forms by first hooking the artifacts and then shifting with them. Namely, we believe students initially appropriate an artifact because they recognize its affordances for enacting, explaining, or evaluating their strategy—what we call a "hook"; further, we believe that through implementing these affordances, students find themselves engaging new forms—what we call a "shift." We will be highlighting in these vignettes aspects of student behavior that we have coded as exemplars both of students' motivation to hook the symbolic artifacts and of the critical interaction dimensions enabling or impeding hooks and shifts (we summarize these in the Conclusions section of this paper).

For purposes of rhetorical clarity, we will not discuss the first artifact introduced to the students, the crosshairs, because discussion of the second and third artifact, the grid and numerals respectively, better enables us to elaborate our proposed construct; furthermore, the crosshairs are analytically problematic due to their unique Human–Computer Interaction issues pertaining to students projecting embodied action onto an instrumented plane (Trninic et al. in press). Therefore, the subsections below are presented in the following order: (4.1) Grid and (4.2) Numerals.

### 4.1 Collaboration and Arbitration with a Cartesian Grid: From Conflicted Qualitative Strategies in a Continuous Plane to a Consensual Quantitative Rule in a Discrete Plane

Following the pair of crosshairs, the Cartesian grid is the second symbolic artifact layered onto the computer display. Introducing the grid onto the display, we maintain, implicitly catalyzed many participants to reconfigure their green-making strategies into pedagogically desirable forms. In this section, we demonstrate how this strategic reconfiguration process can be explicated through the analytic lens of our hook-and-shift construct. In particular, we present and analyze video data from a paired-student interview to argue that the dyad collectively hooked and then shifted by using the grid.

Uri and Eden, two Grade 6 male participants, were selected for a paired interview on the basis of compatible mathematical achievement (both were identified by their teachers as "high achievers"). Their interview was conducted by an apprentice researcher (DT), and the lead researcher (DA) occasionally intervened. Uri and Eden were seated side by side in front of the remote-action sensor system and computer display and each operated one of the two tracker devices (right-tracker device [RT] and left-tracker device [LT], respectively). The students were presented with the task of making the screen green under an unknown 1:2 ratio setting and then, once they had first achieved this objective, moving their hands whilst maintaining a green screen (i.e., the screen would be green as long as the right-crosshair [Rc] were double as high as the left-crosshair [Lc]).

Analysis of a student *pair* co-operating the remote controls to produce green is complex, because an interpersonal coordination task emerges on top of, or mediates the instrumented interaction task (cf. White and Pea in press). Nevertheless, this increase in analytic complexity bears the methodological gains of eliciting authentic dialogue, thus partially circumventing the contrived discursive setting often inherent to interview-based empiricism (cf. Roth 2009) as well as indicating scale-up potential.

#### 4.1.1 Hooking the Grid

Prior to the introduction of the grid, Uri and Eden had been working together for nearly 11 min in the no-crosshairs (blank screen) condition and then another 7 min in the crosshairs condition. So doing, they identified two spatial dimensions—height and distance—as relevant to making the screen green and had articulated two *theorems-in-action* (Vergnaud 1983, 2009) with regard to each of these dimensions: (a) Rc should be higher than Lc; and (b) the vertical distance between Rc and Lc is non-arbitrary. As we elaborate in our discussion of "interaction dimensions" (see Sect. 5, below), both students had thus articulated cognitive *content* with respect to interaction prior to the introduction of the grid and apparently a requisite factor for hooking it. (Below, italicized characterizations of student behavior mark interaction dimensions.)

However, Uri and Eden disagreed as to whether this vertical distance should change or remain constant as the crosshairs move. Whereas both Uri and Eden observed different distances between the Rc and Lc at certain green locations, Uri interpreted this difference as a systemic principle for making green, while Eden attributed it to an HCI issue, as though the physical manipulation were inaccurate (Eden, apparently an avid video-game designer, referred to this error as the "human factor"). Uri articulated a covariant principle relating distance and height, explaining that "it has to get, like, farther away, the higher up we are" and that "the lower you are, the less distance apart it has to be"—a *changing*-distance theorem-in-action. Eden, however, courteously responded with, "Well I'm not sure if it matters if you're lower or higher, but I think it's just, like, you stay the same distance apart"—a *fixed*-distance theorem-in-action.

Thus, Uri and Eden's collaborative hands-on problem solving enabled them each to notice and explicitly articulate a relation between the crosshairs' height and distance (both students bore cognitive *content* with respect to the interaction). Yet whereas Uri concluded from their empirical data that the distance should vary, Eden concluded from the same data that it should not (each student had *validation* for their subjective *content*, yet note that at this point Eden bore a *correct dimension, incorrect value*, i.e. he attended to the distance but judged it to be constant).

The students' disagreement bore practical implications, because the dyad was co-operating the two devices—each student depended on the other to enact a green-making theorem-in-action, yet their respective theorems were mutually exclusive. Consequently, the students' success within this collaboration became contingent on whether or not they could rule between their incompatible changing-distance and fixed-distance theorems-in-action. At the same time, they were apparently under-equipped to arbitrate within the continuous space. Namely, when the grid was subsequently introduced (see below), they recognized its potential for ruling between the theorems—they "hooked" the grid largely for its discursive, argumentation, and arbitration affordances. Specifically, the grid served these boys to quantify the distance between the crosshairs and ultimately determine that this distance should in fact change between green spots, as Uri had believed and Eden soon concurred.

The excerpt below begins immediately after DT had layered the grid onto the screen. In passing, note how both students immediately recognized the grid's mathematical function, i.e., both demonstrated *fluency*. That is, we can assume that both Uri and Eden are sufficiently graph fluent, because they immediately identify the object as "Grid" and orient to it as parsing the working space into enumerable quotas of spatial extension, which was not the case for all study participants.[5]

[5] RT = Right-Tracker device; LT = Left-Tracker device; Rc = Right-hand crosshair; Lc = Left-hand crosshair; //= utterance overlapped by next speaker. We mark spoken utterance with bold characters for readability.

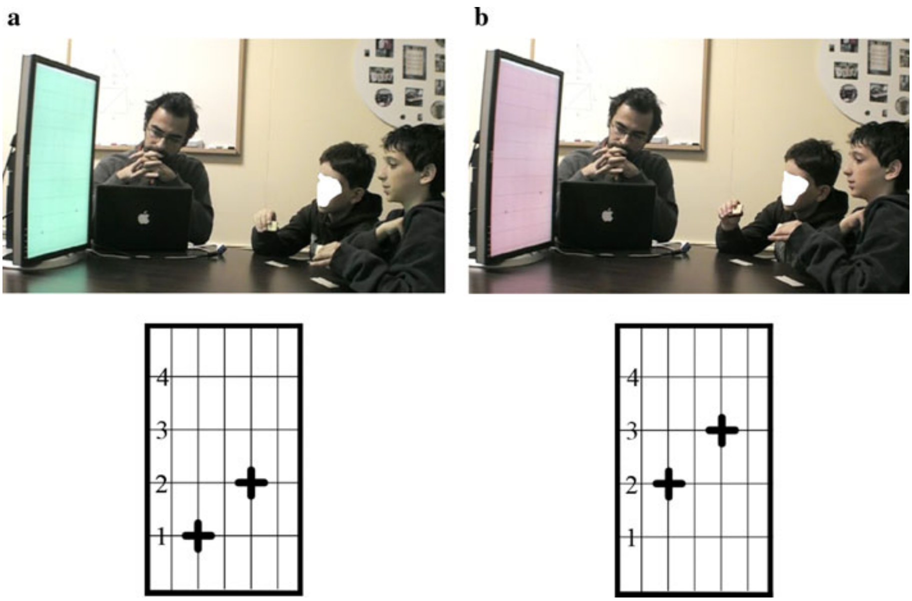


**Fig. 3** **a** After the introduction of the grid, Uri (*middle*) and Eden (*far right*) find green with Rc at 2-line and Lc at 1-line, respectively. Noticing the distance between the Rc and Lc, Eden predicts that the fixed-distance subtends "an entire box." The diagram directly above this caption is a partial schematic recreation of the screen (actually, the *y*-axis ran to 10). **b** Immediately, Uri and Eden reposition Rc and Lc to 3-line and 2-line, respectively. The screen turns red. Upon noticing that the fixed-distance theory does not obtain, Eden says, "Then maybe you should raise it. So maybe the higher you go, the more boxes it is apart." This diagram, too, was recreated for clarity

Eden: <19:36> **Grid**

Uri: **Yeah**. [grabs RT, lifts it, and remote-places Rc on the 1st-from-the-bottom gridline (hence "Rc up to 1-line"). Simultaneously, Eden, too, brings Lc up to 1-line. On the way up, between 0-line and 1-line, the screen flashes green for a moment but then turns red. Eden lowers Lc back down, holds it at .5 units. The screen turns green.] **Oh so you can like show where**… **Let's see, so** [Rc up from 1-line to 2-line]//**if you're on here**…

Eden: **//maybe it has to be two**… [Lc up to 1-line (see Fig. 3a)] **an entire box apart**

Uri: [Rc up to 3-line] **If I go here**…

Eden: [Lc up to 2-line; screen goes red (see Fig. 3b)] **Then maybe you should raise it** [Uri raises Rc to 4-line; screen flashes green]. **So maybe the higher you go, the more boxes it is apart.**[6]

Uri: **Let's just say like I'm here** [Rc down to 2-line], **then he has to be one box under me**…

Eden: [Lc to 1-line; screen goes green] **And then the higher he goes//**

Uri: **//and when I go here** [Rc up to 3-line], **he has to be like in the middle** [Eden moves Lc up to 1.5 units; screen goes green]

Eden: **So the higher//**

[6] In passing, we note different types of pronouns employed to designate action or measurement. Action is attributed to individuals (I and you), whereas measurement is about absolute magnitudes (it). These linguistic marks suggest subtle conceptualization of action as *pragmatic* or *epistemic* (cf. Kirsh 2006).

Uri: **//And here** [Rc up to 4-line, while Eden moves Lc up to 2-line] **he has to be like two boxes under me**

Eden: **So like the higher it goes, the more space there has to be between each [inaudible]**. [both Eden and Uri place their tracker devices on the table]

Thus, it appears that both Eden and Uri immediately appropriated the grid in view of its affordances to arbitrate among their conflicted theorems, however they differed with respect to the nature of their discovery, and this difference can be related to their idiosyncratic beliefs prior to the introduction of the grid. Namely, Uri had articulated a changing covariant relation between height and distance, so for him the grid afforded reiterating and quantifying this qualitative principle. Specifically, the grid enabled Uri to reformulate his continuous qualifier "get farther away" as the discrete quantifiers "one box" and then "two boxes." Eden, who had acknowledged the in-principle possibility of a changing-distance rule yet maintained a fixed-distance rule, soon changed his mind and articulated a changing-distance hypothesis ("the higher you go, the more boxes it is apart"). However, Eden ends with a *qualitative* statement about "space," which suggests that Eden construed the grid as a means not of quantifying the "higher–bigger" conjecture but of *evaluating* whether or not this conjecture even obtained. Thus, Uri and Eden both hooked to the same artifact, yet they utilized the collaborative inquiry activity it enabled for different purposes.

This episode demonstrates a common-sense view that an artifact's subjective utility is contingent on the individual's goals. Yet the episode also suggests that a dyad can engage in physically co-enacting collaborative inquiry even as they develop and hold different theorems-in-action (compare to Sebanz and Knoblich 2009, who suggest otherwise). Finally, the capacity of learners' to engage in deep reflection over *collaborative* manipulation suggests that *embodied reasoning can be distributed intersubjectively, with perception of vicarious action acting as proxy for action, as long as perception is monitoring vicarious action against the enactment of a particular theorem-in-action.*

In the following excerpt, we continue at a point where the dyad initiates further inquiry. As we shall see, the dyad's exploration will shift them from the now-consensual "higher–bigger" strategy toward a proto-ratio *a*-per-*b* strategy. Both strategies can be viewed as expressing covariation—"the more *x*, the more *y*"—that is enacted as coordinated bimanual operations embodied and monitored in the functionally extended spatial medium of the computer interface. However, the former strategy is continuous–qualitative, whereas the latter is discrete–quantitative, so that adopting and articulating the latter strategy is a pedagogically desirable outcome. The students' shift was apparently contingent on their consensus over changing-distance rather than fixed-distance as their theorem-in-action. Namely, some of our study participants, who identified the *correct dimension* (i.e., distance) but inferred an *incorrect value* for this dimension (i.e., constant), did not experience a shift or experienced difficulty in shifting.

### *4.1.2 Shifting with the grid*

Having reached consensus, Uri and Eden elaborated their explanation. At this point, they had instrumentalized the grid to quantify the distance between the hands. This new conceptualization of space is soon to engender the semi-spontaneous emergence of a new mathematical form. In the transcription that follows we will observe that the students shift with the grid from a continuous–qualitative strategy to a discrete–quantitative strategy. In particular, the students are about to change the object of their co-manipulation from: (a) the distance between the hands/crosshairs; to (b) each hand/crosshair's location independent of the other one.[7] Following this excerpt, we offer an interpretation of media and interaction factors inherent in the shift.

Uri: <20:19> **I think that, uhm, when I go up to here** [points to 2-line], **he has to be one** [points to 1-line]. **Then when I go up//**

Eden: **Like for every… for every box he goes up, I have to move, go down**[8]//

Uri: **//You have to go up half////a box**

Eden: **//Yeah//**

The dyad's coordinated production of green tacitly modulated from pre-grid simultaneous motions, in which, ideally, the distance constantly changes and green coloration is maintained throughout, to with-grid sequential motions, in which each hand separately ratchets up to its respective designated destination and green is effected after a brief red interim, once the second ratcheted motion is completed. Imperceptibly, the dyad thus shifted from their "the higher, the bigger" continuous–qualitative strategy to an *a*-per-*b* discrete–quantitative strategy.

We wish to highlight several properties of discourse that contributed to the shift beyond each child's strategic perceptuomotor interactions: (a) the sequentializing (linear) constraint of the speech modality, which introduces order into originally simultaneous actions; (b) the indexing or deictic affordance of the new symbolic artifact, which enables unambiguous reference to particular physical locations germane to successful enactment of strategy *sub*goals; and (c) turn-taking norms of conversation about distributed actions, which suggest splitting the description into respective complements.

Eden and Uri thus co-discovered that in order to maintain green, they should progress at coordinated intervals of 1 (Uri) and ½ (Eden) vertical units, by either both going up the screen or both going down. It is through this serendipitous discovery that their earlier observation, "the higher you go, the more boxes it is apart," a covariation between height and distance, transformed (shifted) into a covariation that foregrounds the independent actions of the left and right entities, "For every box he goes up—you have to go up half," a new strategy that is closer to normative forms for ratio (i.e., *a*-per-*b*). We wish to underscore that whereas the general *x*-per-*y* covariation form was maintained, its semantic–mathematical content was replaced (see Table 1, below).

In addition to explicating our hook-and-shift construct, our analysis of the case has demonstrated that collaborative mathematical learning processes are impacted by nuances of personal/interpersonal framing to the extent of dissociation between a dyad's mechanical and epistemic actions. Namely, whereas the two dyad members collaborated on using a single symbolic artifact (the grid), their joint experiment simultaneously enacted an exploration of two *different* hypotheses ("different distance" vs. "same distance"). Uri was quite comfortable from the very onset with the higher–bigger principle, so he did not

[7] Note that at this point in the interview, these independent locations of the two hands or crosshairs are not yet absolute or explicit—they do not designate the hand/crosshair's unitized distance from the desktop or screen bottom. Rather, the students will conceptualize the locations as self-referential, with each hand motion recursively departing from the previous location toward the new location. Study participants typically referred to locations in absolute terms only once the numerals appeared on the screen.

[8] Close examination of Eden's actions reveals that rather than directly going up by half a unit each time that Uri went up by a whole unit, Eden was going up by a whole unit but then lowering by a half, as though correcting each time.

**Table 1** Consistent "covariation" linguistic structure across strategy micro-shifts

| Mathematical properties | "The more *x*, …" | "the more *y*" |
|---|---|---|
| Continuous-qualitative | "The higher you go, …" | "the bigger the distance" |
| Discrete-qualitative: | "The higher you go, …" | "the more boxes it is apart" |
| Discrete-quantitative | "For every box he goes up, …" | "you have to go up half" |

need arbitration but refinement, whereas Eden, who challenged Uri with a same-difference theorem, needed resolution. Once a common ground had been established, the dyad was able to continue mathematizing the mystery artifact–phenomenon and jointly articulate a new mathematical form, which we recognized as pedagogically desirable.

### 4.2 Numerals: From Location Indexes to Computable Quantities

Following the pair of crosshairs and the Cartesian grid, the *y*-axis numerals are the third symbolic artifact layered by the interviewer onto the computer display, in accord with the interview protocol. This section presents a case of a student who, we argue, hooks and then shifts with the numerals. Our case-analysis participant, Siena, is a 6th-grade student identified by her teachers as low achieving. She, too, was interviewed by an apprentice researcher (DT), with the lead researcher (DA) occasionally intervening.

About 20 min into the interview, the grid was introduced. Siena immediately responded that the grid would "make it easier to say where it is," gesturing the second "it" to mean the hand locations effecting green. As such, Siena hooked to the grid as enabling her better to *explain* her strategy *content*. However, she did not go on to use the grid so as to "say" or otherwise demonstrate green locations. In order to probe her statement, DT held the RT so that the Rc fell precisely on a gridline and asked Siena to predict where the Lc should be so as to effect a green screen. Initially, Siena did not lift the LT but instead communicated her predictions for the Lc's location by pointing with a finger to one of the horizontal lines on the screen. Siena was seated too far away from the screen so as to literally place her finger on the particular line she was referring to, and so the interviewers, who were positioned to her sides, could not know unequivocally which line she was referring to. A need for repair action thus emerged in the conversation. The following excerpt begins shortly before the introduction of numerals onto the screen.

But first, a brief clarification should help the reader make sense of Siena's otherwise abstruse statement, below. The computer display monitor was rotated 90° from its normal "landscape" orientation to a "portrait" orientation, so as to accommodate the vertical interaction space required by our design. Consequently, the silver "DELL" logo, which is usually located directly below the screen at the center of the framing panel, was located halfway up the left-side frame panel and oriented downward. Siena will be using features of this logo to refer to a particular location on the screen.

DT: <22:15> **So how about for here?** [places Rc on 8-line]

Siena: **I think**… [lowers RH toward the LT that is lying on the desk, lifts it slightly, but then lets go; RH rises, index drawn out] **Uhhm, I think**… **it would be** [RH index points toward the left-side panel of the screen, then glides horizontally to the right along the 5-line to an empty space in the Lc column, then back to the left-side panel (see Fig. 4a)] **right at the E on the DELL**

DA: **Here?** [places LH index finger upon the "E," such that the finger is pointing horizontally across the screen to the right, along the 5-line that Siena had indexed (see Fig. 4b, c, below, noting DA's index finger)]

Siena: **Yeah**

DA: **Let's see.** [moves finger away from screen]

Siena: [lifts Lc toward the height of 5-line. On the way up, the screen flashes green at 4-line; she hesitates very briefly then continues upward to 5-line. The screen turns red] **Oops! Never mind. Guess it was down here** [lowers Lc to 4-line, where she had briefly effected green on the way up]. **On the line below**

DA: **On the line below, uh-huh**…**m'mm**

DT: **Okay, so, maybe instead of having to, you know, point to every line, it would be easier if we had**… **names for them?** [chuckles]

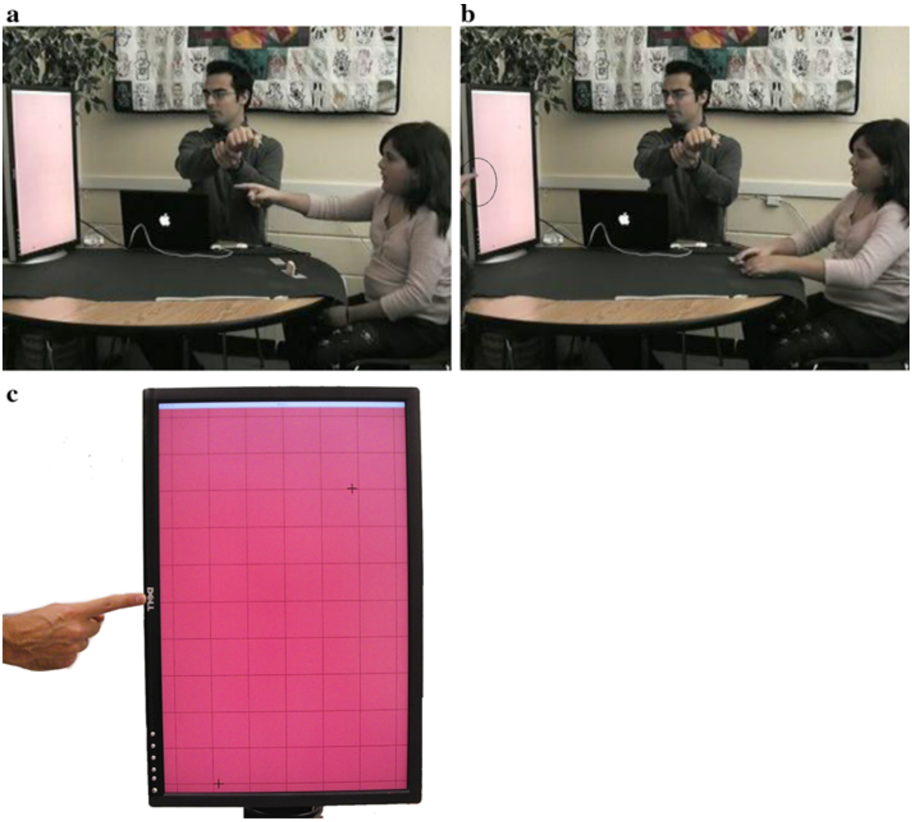

**Fig. 4** **a** DT holds Rc at 8, and Siena is asked to predict the position of Lc. She utters, "Uhhm, I think… it would be right at the E on the DELL." Her right index finger points to "E" then sweeps back-and-forth horizontally along 5-line. **b** Immediately following Siena's gesture, DA places LH index finger upon the "E" (see *middle left*), such that the finger is pointing horizontally across the screen to the right, along 5-line. **c** An example of using fortuitously available features of the environment to resolve the ambiguity of a speech referent. One interviewer (DT) had placed Rc on 8-line. Siena estimated that Lc should be "right at the E on the DELL." The other interviewer (DA) points accordingly along the 5-line. (Scene recreated for this paper.)

DT's utterance, which might be analyzed grammatically as an interrogative probe, as though DT is seeking information, in fact serves pragmatically as more than a mere question. Namely, DT is implicitly communicating to Siena that it is in the interest of the collective activity that Siena now assume agency in elaborating on the symbolic artifact, and in particular by interpolating appellations for the gridlines, such that the lines can be referred to unambiguously. As such, DT, who is about to introduce the numerals onto the screen, is *framing* the numerals' designed function in advance, so that when they appear, they will be immediately construed as serving a particular goal, a discursive goal of regulating the conversation by repairing the ambiguity of referents. DA will now follow up on DT's pragmatic framing, as though it has been established that indeed it is in the general interest of the interlocutors to label the referents. Note also that the new symbolic artifacts, the numerals, will appear immediately after their intended function has been established. As such, Siena has been *primed* to frame the numerals as serving a particular function. Priming is necessary and critical, because even if students have established the prospective affordances of a symbolic artifact in anticipation of its appearance and in accord with the design, still they might forget this framing by the time it appears.

DA: **What would be good ways of naming those lines?**
Siena: **A, B, C, D**… [Following this utterance, she performs three descending chopping gestures with her right hand, her palm facing downward, marking that the top gridline, at shoulder height, should be named "A," the line below it—"B," etc.]
DA: **We could do that. We chose//**
Siena: **//…or numbers**

In passing, we note that Siena is fluent in certain basic representational strategies that avail of common symbol strings, such as "A, B, C,…" or "1, 2, 3, …" Though this particular fluency may appear trivial for a 6th-grade student, the dimension of *fluency* becomes more pertinent when students are expected to use symbolic artifacts that they have not mastered sufficiently, such as when some 4th-grade students behold a Cartesian grid.

DA: **Or numbers. We chose numbers.** [DT operates the console, and the *y*-axis numerals appear on the screen]

In analyzing the above excerpt, we find it helpful to orient ourselves with the following question: Why does Siena suggest to label the horizontal gridlines as A, B, C, D,…descending—rather than 1, 2, 3, 4, … ascending?

To adequately answer this question, we must trace Siena's activity leading to the "A, B, C, D" utterance. Specifically, recall that when DT placed the Rc on a particular gridline and challenged Siena to find its green Lc counterpart, Siena responded by pointing toward the screen so as to indicate her suggestion for its location. Yet pointing became cumbersome, because her distance from the screen prevented unambiguous deictic indexing. Thus a local discursive goal emerged for Siena to better index the location—the specific gridline she was gazing toward. The "E" in "DELL"—a contextually salient perceptual landmark located in the appropriate vertical position, if off to the left of the intended crosshair position—occurred to Siena as a direct practical means of inviting the interviewers to co-attend with her (see Fig. 4c).

It thus appears reasonable to assume that Siena did not, at that point, assign any mathematical/quantitative meaning to the grid. Moreover, her initial evaluation that the grid would help her "say where it is" notwithstanding, Siena struggled to utilize the grid as a means of indexing the crosshairs' locations. Indeed, it appears she valued the discursive utility of the grid at best as equal to the alphabetical characters in the "DELL" logo—that is, purely as spatial placeholders. Moreover, she viewed the "DELL" as a superior index, compared to the grid, possibly because the stylized, tilted "E" is a unique landmark on the monitor, whereas the many gridlines are indistinguishable save for their serial location. That is, enumerating the lines did not apparently occur to Siena as a viable means of distinguishing among these many lines. Whereas a graph-fluent person would very likely count the gridlines from the bottom and up toward the line in question, to Siena the gridlines per se did not afford any means of remote disambiguation that would support the interaction flow.

Siena, then, scanned the environment for a means of indexing a particular horizontal line on the screen. Seen from this perspective, her suggestion to label the lines from top to bottom is a fine solution to her localized discursive goal of unambiguously indexing the lines. Siena thus anticipates the numerals' contextual indexing affordance and can therefore *hook* the numerals, once they actually appear on the screen.

We continue at the point where the numerals have appeared on the screen. Again a brief clarification is due. The *y*-axis numeral "8" marks the gridline where DT had been holding the Rc just earlier, when Siena had guessed that the Lc should be at the "E" so as to effect green. The "E" is adjacent to, and roughly at the height of the "5."

DT: **See those?** [sweeps hand downward toward the numerals on the screen]
Siena: **Mmhm**
DT: **So, let's kind of do the same thing//**
DA: **//Even before you do that—so if, if Dragan** [DT] **puts it** [Rc] **up at eight** [8-line], **where do you think… the other one** [Lc] **should be?**
Siena: **Five**
DA: **Let's try**. [DT lifts Rc to 8-line and holds it at that location]
Siena: [raises Lc to 5-line] **Oops. Ah! I always do it wrong**. [lowers Lc to about 4-line] **I'm always wrong… I always won't find it … Wait!! Wait, wait… Go to ten.** [Still holding LT in her LH, with the Lc suspended at 4-line, she points her RH index finger up toward the "10" gridline. DT lifts Rc to 10-line and holds it at that location; Siena lifts the Lc to 5-line] **Oh!** [glances at DA] **It's always half!**

With her utterance "It's always half" Siena first enunciated the multiplicative constant that relates all Lc and Rc locations co-effecting green under the 1:2 setting.

Whereas we view the annotated transcription, above, as evidencing another case of "hook and shift," we cannot veritably claim that at this point in the interview Siena's insight was grounded through solid chains of signification to an embodied sense of relative heights (cf. Sáenz-Ludlow 2003). Indeed, Siena's inference that the Lc should always be half as high as Rc was an abductive appropriation of a known fact (i.e. the multiplicative relation between 4 and 8), which occurred to her as a feature of the situation, followed by an inductive trial and ultimate confirmation. Thus, Siena's reasoning consisted essentially of evaluating the contextual utility of applying an emergent affordance toward the solution of an unresolved problem.

Some sociocultural theorists (e.g., Newman et al. 1989) might thus view this episode as validating the hypothesis that naïve and expert views are cognitively incompatible—that Siena's learning experience is marked by a clear break from before to after she engaged the new semiotic potential of the cultural artifact. Whereas we agree with the judgment that the process of learning ought to be viewed as strongly framed by its sociocultural context, we would disagree with an assessment that Siena's personal experience is disconnected. Namely, whereas an expert may identify logical discontinuity in Siena's reasoning, we must be vigilant against positing this discontinuity *inside her personal experience*. Simply stated, no third-person account of a dance captures the dancer's first-person experience.

Furthermore, for the student to perceive the alleged discontinuity, she herself would have to be an expert! Thus the danger we run here, to paraphrase Marx, is mistaking our modeling of the world for a world of models. We posit instead that a personal connection exists and is phenomenological rather than logical: it emerges reflexively in the sequencing of Siena's interactions with the artifact as expressed in her utterance sequence. That is, we suggest that Siena's sense of connectedness is grounded in spatial–temporal continuity of intentional, embodied activity. Her *shift* to a new mode of thinking cannot be explained as a logical continuation of indexing but rather as a connection that emerged in the activity of doing and telling (cf. Roth and Thom 2009).

Lastly, we underscore that, in cases such as this, it is deceptively easy to commit historical revisionism and claim that the student interacts with the numerals *because* she sees them as enhancing her reasoning in the manner they would enhance the reasoning of an expert. Precisely the opposite happens here: Siena recognizes the expert utility of the numerals *because* she interacts with them in the first place (cf. Wertsch 1979). Specifically, what we might call "progressive mathematization" (Freudenthal 1986) emerges through the student's active interaction with the numerals: mathematization can be caused by rather than be the cause of interaction. We thus agree with Shank (1987) and Prawat (1999) that the so-called learning paradox is such only inasmuch as we conceptualize learning as a purely logical-deductive process. Accordingly, we recognize the *opportunistic* nature of problem solving and acknowledge abductive inference as par-for-the-course distributed and explorative logical activity. Nevertheless, whereas we evaluate the conceptual utility of hook-and-shift as a paradigmatic reinvention process, still the burden is upon us as educators to evaluate how *conceptually* connected are the various strategies and meanings that sprout from students' somewhat serendipitous shifts.

## 5 Conclusion: From Ontological Innovation to Design Heuristic

In this paper, we introduced the theoretical construct "hooks and shifts." The construct was proposed as a means of explicating unanticipated student discoveries of mathematical forms of reasoning that we observed in the implementations of our instructional activities. The rationale of this paper has been to present the theoretical construct as an ontological innovation emerging from our design-based research studies and argue for its plausibility through existence-proof case analyses featuring study participants whose interactions purportedly exemplified the construct. To our evaluation, the paper succeeded in warranting the new construct as marking an under-conceptualized yet important phenomenon of interest to mathematics education research and practice. Namely, whereas our qualitative data analyses have greatly availed of a range of Vygotsky-inspired theoretical models of learning, and in particular those in dialogue with Piagetian genetic epistemology, we propose that these models require elaboration. Specifically, we encourage theoreticians to account for reflexive micro-processes apparently typical of distributed problem solving, to appreciate the potential cognitive–pedagogical utility of these incidental emergent consequences, and to speculate on how these processes might be designed. That is, students' apparent capacity to reinvent cultural knowledge through engaging emerging affordances of technoscientific artifacts, we maintain, bears important implications for both the theory and practice of mathematics instruction.

"Hooks and shifts" is a dialectical cognitive–sociocultural construct that models mediated discovery. When objects are introduced into a problem space, students are motivated to "hook" them, because the students recognize elements of these objects as better affording the enactment, explanation, and/or evaluation of their interaction goals—the objects appear to the students as satisfying their ad hoc "intellectual needs" (cf. Harel in press). More specifically, students hook symbolic artifacts so as to:

(a) *Enable* orient in the instrumented interaction space, such as when the crosshairs are first placed on the display monitor and users gain traction on the screen by figuring out "where they are"[9];

(b) *Enact* perform, rehearse, and master the skillful control of a system by progressively increasing the precision, efficiency, collaboration affordances, and predictive capacity of their strategy;

(c) *Enhance* avail specifically of the greater precision the mathematical instrument affords, such as moving from qualitative to quantitative statements by measuring features of the interaction;

(d) *Explain* support clear discursive representation of personal solution strategies in accord with the pragmatic framing of the interaction as proto-mathematical (i.e., as demanding unequivocal, generalizable, quantitative indexing of procedures); or

(e) *Evaluate* arbitrate among conflicted theorems-in-actions held among students and instructors participating in collaborative inquiry

Having hooked, students may "shift" to new mathematical forms, because in the course of enacting the intended actions with the new artifacts, previously unnoticed affordances of these objects become contextually attractive and salient as means of accomplishing interaction goals or subgoals. Attending to these affordances and incorporating them into the original operatory schema may surreptitiously modulate or reconfigure the students' action plan, possibly into a different strategy, which the instructor might endorse as more mathematically powerful. Although the new strategy is structurally distinct from the initial strategy, the learner does not experience any rupture but only the rapture of discovery.

However, not all our study participants demonstrated such behaviors, and these cases were, by juxtaposition, instrumental to our development of the construct and, by implication, the theory, design, and practice of the hook-and-shift discovery-based mathematics instruction. In particular, in Gutiérrez et al. (2011) we compare in depth across eighteen cases of individual participants who did and who did not hook or shift with the artifacts (in this paper we only looked at two of these interviews and reported only on selected vignettes within them). Emerging from this comparison is the following set of interdependent interaction dimensions as predictive of students' prospects of hooking and shifting with the symbolic artifacts layered onto their problem-solving space.[10] In order for hooks to occur:

[9] See Trninic et al. (in press). Also note that the five motivation dimensions are not orthogonal but overlap in ways that are difficult to tease apart. This list should not be taken as offering an exhaustive blueprint for the development of a precision coding system as much as to lay out in general the scope of motivations we have witnessed and characterized in our data analyses (see also diSessa 1995; Sarama and Clements 2009, for the affordances of computer-based "concrete" interactive objects).

[10] We reflect that whereas study participants who generated and connected more strategies and meanings were typically the students who had been characterized by their teachers as higher achieving, the other participants were impeded as much by our experimental methodology as by their own knowledge. Namely, at times we marched on through the interview just to ensure that we "cover" all protocol items for subsequent analysis, regardless of whether or not participants were optimally prepared to work with the new symbolic artifacts. Only during analysis did we fully appreciate the detrimental cumulative effect of our facilitation practice on the quality of the experimental implementation as a *learning* experience for some participants. Whereas the dual role of research interviewers in both agitating and measuring behavior is universal and familiar, the hook-and-shift construct underscores a reliability problematic of data gathered in multi-item interviews requiring cumulative feedforward reasoning.

1. *Content* students must have conjectured an effective interaction strategy;
2. *Validation* students must have confirmed the plausibility of the strategy;
3. *Priming* the strategy should be cognitively available when the artifact is introduced;
4. *Fluency* the students should be minimally familiar with the symbolic artifact introduced by the instructor; and
5. *Framing* the instructor should pragmatically position the artifact as potentially instrumental in solving the interaction problem.

Shifting, too, is contingent on students' preparedness. In particular, the following three additional interaction aspects partially predict the likeliness for a shift to occur:

6. *Dimensions/Values Compatibility* students may have hooked an artifact whose affordance dimensions cohere with their strategy, only that their strategy had assumed particular values, orientations, or relations along this dimension that turn out to differ from the values the designer had embedded into the system's interactivity; this is an impediment to shifting, and these students need still to experience and resolve cognitive conflict along this dimension; and
7. *Perceptual Compatibility* students may have hooked an artifact yet their perceptual construction of a key phenomenal property may differ from the disciplinary view implicit to the design; this mismatch, which is an impediment to shifting, surfaces only with the introduction of the artifact and may be aligned through discourse.
8. *Facilitation* to experience a shift, students who have hooked a symbolic artifact need sufficient time to further explore it; indeed, students' productive engagement with an artifact is greatly impacted by the quality of the interviewer's facilitation.

Thus, many interaction contingencies appear to stand in the way of hooking and shifting toward strategic reconfiguration and conceptual understanding, and designers and instructors should be aware of these contingencies and anticipate them in their practice. That said, whereas designers play a pivotal role in determining the conceptual affordances and experiential trajectories of learning environments, still they can by no means guarantee learning outcomes and must rely on the key role of instructors. Yet in some paradoxical sense, hooks and, moreover, shifts, are not directly facilitated but instead require a certain *laissez faire* pedagogical practice that cannot necessarily guarantee these targeted reactions from students. Indeed, as John Olive remarks soberly in concluding a report on an educational-technology research study, "Even the most carefully designed tools can be used in ways that were unintended by the designers of the tools" (Olive 2000, p. 260; see also White 2008). We concur and wish to supplement that steering students' embodied interactions toward spontaneous appropriation of disciplinary forms requires of researchers and instructors to listen very closely to the students, because though their ways of using the tools may be unintended or unanticipated by the designer or teacher, these idiosyncratic ways embody the students' heterogeneous understanding and, hence, their subjective learning potentials.

In particular, hooks and shifts are ultimately possible and productive when the design enables students to set off from reasonable, if naïve, solution strategies and then enact and explain them in progressively sophisticated mathematical forms. As such, evidence of learners' considerable cognitive agency in reinventing cultural forms when working with the Mathematical Imagery Trainer further supports the utility of designs wherein learners enact qualitatively sound solutions even before they articulate their solutions in mathematical registers (Abrahamson 2009b, c). Notwithstanding, in order to substantiate our claims for the conceptual utility of hook-and-shift activities, we still have to determine whether and how the new instrumented schemas students shift toward are grounded in their prior and emerging schemas, fluencies, and practices. Assessing for student coordination of emerging meanings into conceptually coherent structures and fluencies is accordingly a future goal of this project.

At the same time, our serendipitous *post facto* discovery of the hook-and-shift reinvention process within our data only underscores designers' need for a coherent, research-based set of principles by which to instantiate reinvention pedagogical philosophy in the form of enticing objects-to-think with. Elsewhere, we offer some principles toward a heuristic design framework specifically for embodied-interaction mathematics learning activities (Abrahamson and Trninic in press). These principles will develop further as we build, refine, scale up, and research improved designs with the Mathematical Imagery Trainer. In a sense, we are hooking and shifting with our empirical data, just like students learning within our designs.[11]

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[11] A recent large empirical dissertation study implemented our Mathematical Imagery Trainer as part of a 2-week curriculum in a controlled experiment in which fourteen classrooms participated (Petrick and Martin 2011). Students involved in actual embodied manipulation sessions later outperformed on relevant conceptual tasks other students who watched screenings of the embodied sessions but did not have any agency in controlling the virtual objects nor saw a person actually manipulate them. The researchers interpret their data as supporting our own team's conjecture pertaining to the learning gains inherent in physical solution procedures that enact kinesthetic image schemas underlying target curricular concepts (Abrahamson and Howison 2008; Abrahamson and Trninic in press).

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