

Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds[☆]



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ABSTRACT

Design researchers should inform the commercial production of educational technology by explicating their tacit design practice in workable structures and language. Two activity genres for grounding mathematical concepts are explained: “perception-based design” builds on learners’ early mental capacity to draw logical inferences from perceptual judgment of intensive quantities in source phenomena, such as displays of color densities; “action-based design” builds on learners’ perceptuomotor capacity to develop new kinesthetic routines for strategic embodied interaction, such as moving the hands at different speeds to keep a screen green. In a primary problem, learners apply or develop non-symbolic perceptuomotor schemas to engage the task effectively; In a secondary problem, learners devise means of appropriating newly interpolated mathematical forms as enactive, semiotic, or epistemic means of enhancing, explaining, and evaluating their primary response. In so doing, learners heuristically determine either inferential parity (perception-based design) or functional parity (action-based design) as epistemic grounds for reconciling naïve and scientific perspectives. Ultimately embodied-learning activities may interleave and synthesize the genres’ elements. This taxonomy opens design practice into richer dialog with the learning sciences. An appendix lays out the embodied-design framework in a “how to” form amenable for replication both within the domain of mathematics and beyond.

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1. Objective: systematizing pedagogical design

With the advent of the technological era, we are witnessing an unprecedented proliferation of commercial educational products. Day by day, hundreds of tablet applications that promise to teach children school content are spawned for immediate global consumption, and the rate of this production juggernaut is only increasing. Certainly this is a blessing for all stakeholders in the global pedagogical program. Yet whereas educational apps may be streamlined and engaging, industry is by-and-large uninformed by empirically based theory of learning, and consequently its products are often of suboptimal pedagogical quality, orienting students on the rehearsal of meaningless solution algorithms. Engagement is not enough. What can be done?

In 1896, Fannie Farmer published *The Boston Cooking-School Cook Book*. This compendium of recipes utilized an unprecedented

format: Ms. Farmer specified precise quantities by introducing a measurement system involving standardized spoons and cups. The book was quickly adopted throughout the US and, in so doing, transformed domestic cooking practices to the point that Boston cuisine could be recreated in Berkeley. The analogy should be clear. It is about responding to a schism between design and production caused by the logistical entailments of progress in a New World, namely migration and the dismantling of the nuclear ma-and-pa apprenticeship studio. And it is about taking initiative to reify and disseminate tacit expert knowledge by using new cultural forms that oblige a level of specification that would enable emulation in remote locations. University design labs cannot accommodate all commercial designers, and so it is our ethical obligation to explain what we would consider effective learning products as well as how we go about creating them. We need a design book!

That is, if once it was common for seasoned educators to both envision and prototype instructional materials – names such as Friedrich Fröbel or Maria Montessori come to mind – now these materials are churned out of cyber sweatshops. If education scholars and practitioners find these commercial materials wanting, it is because these educators have certain standards by which they measure these materials. Yet what are these standards? And, moreover, if educational *designers* find these

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materials wanting, what is it that the designers know that industry does not know? We must go beyond hand waving and get our hands dirty. We must be clear and specific.

This article and, more broadly, my entire research program, stems from an ethical conviction that educational designers should articulate their tacit knowledge, so that industry can emulate expert design practices and cook up quality products. Toward these ends, this essay shares the results of one design-based researcher reflecting on his cumulative practice. But it takes a village, and so my hope is to develop useful constructs and perhaps some humble theory that may promote productive dialog with fellow scholars interested in deepening our collective understanding of educational design—its art, craft, and theory [2–4].

For the most, I will be speaking about mathematics education, because that has been my area of endeavor, where I have developed some insights on how to foster meaningful learning. However I will strive to strike a register of description that would be sensible and hopefully useful for designers in other STEM fields, such as science, who wish to evaluate the design framework and join the discussion on reifying design acumen. Elsewhere, we have elaborated on why we believe that diverse STEM content all stems from common cognitive architecture [5].

Design-based researchers, members of a community at the intersection of learning theory and practice, generally find it useful to articulate, disseminate, and debate among themselves philosophical, theoretical, and practical aspects of their *métier* [6–9]. One particular aspect of this dialog that tends to draw the attention of industry, and not only academe, is the building and refinement of empirically evaluated heuristic design frameworks for creating effective learning materials [10–15]. Specifically, the following article is on principled frameworks for designing learning activities geared to foster student re-invention of conceptual cores that the designer identifies for the targeted content domain. Essentially theoretical, this retrospective essay will draw on a body of empirical work to support and exemplify two proposed design frameworks as well as demonstrate their commonalities and hone their distinctions. Both frameworks should be regarded as different manifestations of what I have called “embodied design” [16].

The motivation for sharing the current reflection is that I have noticed structural consistency as well as variation across a set of pedagogical designs I have been investigating over the past two decades. At the time of conception, those designs were intuitively conceptualized. In hindsight, I am now striving to make sense of those designs’ similarity and contradistinction vis-à-vis educational-research literature. In particular, I am spurred by a tension between, on the one hand, what appears to be quite cohesive an approach to mathematics pedagogy underlying those designs and, on the other hand, apparently different ways of implementing this general approach. I am thus looking to develop a useful taxonomy of what I propose to call *design genres*, such as perception-based design and action-based design—two genres of embodied design.

As will soon become evident, this article is not so much about *what* to design but more so on *how* to design. Yet the taxonomy of design genres ultimately pivots on *why* to design in this or that way. That is, I am seeking to characterize educational design on the basis of alleged cognitive dynamics underlying children’s receptivity to STEM knowledge. The emerging taxonomy of designs hinges on an implication of two forms of cognitive receptivity. I will argue that students participating in activities that draw on their innate or early perceptual intuitions (“perception-based design”) are recipient to knowledge that is formulated such that the students can experience *inferential parity* between their intuition and the formal structures. And I will argue that students participating in activities that draw on their capacity to develop new physical coordination patterns (“action-based design”) are recipient to knowledge that

is formulated such that the students can experience *functional parity* between their unequipped and equipped actions. As I explain, it is through the appreciation of parity that students are willing and able to reconcile naïve and scientific perspectives on mundane phenomena and, in so doing, accept techno-scientific forms and process.

This taxonomy, which would avail of critique, elaboration, and expansion, is couched in learning-sciences nomenclature in an attempt to build a coherent account of relations between mathematics-education theory and practice in a way that may inform the work of other researchers and designers. As such, though this budding taxonomy cannot be exhaustive, it may indicate routes toward charting some design waters in the ocean of reform-oriented mathematics education. To the extent that this effort bears appeal to fellow designers and design-based researchers, we may thus all be better equipped to help mathematics students navigate conceptual transitions along meaningful continuums [17]. Optimally, this essay would take strides toward creating *The Berkeley Designing-School Design Book*, so that designs such as ours could be recreated as far afield as Boston....

2. Modus operandi: the designer as a reflective practitioner

Why might I hold so much stock in design? Is design any more than a thoughtless conduit between theory and practice? This section offers a brief *apologia* of design, wherein I argue for the centrality of designers in the core intellectual work of generating theory and shaping practice. In particular, I submit, reflective educational designers are uniquely positioned to generate theory of learning, teaching, and – reflexively – design. In this section I position my own design research as creating opportunities for dialectical synergy between theories of learning.

Winograd and Flores [18] view scholarly discourse on design as part of a larger, interdisciplinary intellectual pursuit that goes beyond how to build this gadget or another to encompass an inquiry into the human potential to navigate transition:

In ontological designing, we are doing more than asking what can be built. We are engaging in a philosophical discourse about the self—about what we can do and what we can be. Tools are fundamental to action, and through our actions we generate the world. The transformation we are concerned with is not a technical one, but a continuing evolution of how we understand our surroundings and ourselves—of how we continue becoming the beings that we are. (p. 179)

Design-based researchers embrace the above urge to perceive the practice of design not only as a compliant operationalization of extant theoretical models of human learning but also as a proactive, critical agent of change that can inform and transform these models. Technology plays a particularly vital role in stimulating reflection on what it means to know, because its architectures, encodings, and encasings often dictate an analytic decoupling of naturalistic form and content, sensation and cognition, semiotic systems and meaning—technology tends to mirror and unpack for us implicit aspects of our reasoning and lay them bare for scrutiny and improvement [19–21]. As McLuhan [22] wrote:

The hybrid or the meeting of two media is a moment of truth and revelation from which new form is born...a moment of freedom and release from the ordinary trance and numbness imposed...on our senses. (p. 63)

In like spirit, I am inspired by the prospects of reconceptualizing mathematics education via identifying within our community’s inventions and empirical data such mechanisms and processes that may challenge our field’s implicit assumptions about how students can and should learn as well as how, accordingly, designers can and should design and teachers can and should teach.

By reflecting specifically on the actual designs themselves that we build, we may also be able to face undertheorized aspects of our creative process and, in so doing, both acknowledge and demystify this process, which is difficult to describe let alone document empirically [23,24]. That is, just because we do not always understand how we invent new instructional devices and lesson plans, we need not ignore, misrepresent, or romanticize this process [16,25]. As Schön [4] cautions,

mystification consists in making knowledge-in-practice appear to be more complex, private, ineffable, and above all more once-and-for-all, more closed to inquiry, than it needs to be.... [D]emystification is not a showing up of the falsity of the practitioner's claims to knowledge but a bid to undertake the often arduous task of opening it up to inquiry. (p. 289)

Finally, by exploring unknown aspects of how designers design, we may illuminate corresponding unknown aspects of how students learn. The rationale here is that design, as an educational enterprise, is enabled by designers and students sharing in biology and cognition [26]. As such, by reflecting on our own designs as projections of our mathematical knowledge – phenomenizations of our tacit schemas [14]—we may better understand, share, and foster core yet covert aspects of this knowledge [27,20].

The designs discussed in this essay were conceptualized intuitively. They resulted from my efforts to build materials and activities that concretize my core tacit images for the targeted mathematical notions [28]. My design process thus begins by introspecting, in an attempt first to elicit, capture, and articulate my own multimodal dynamical scheme underlying the target notion, then to embody the scheme in forms that learners can engage and utilize meaningfully in guided goal-oriented activities. In parallel, I perform cognitive domain analyses with the objective of retro-rationalizing my own intuitive design, and I iteratively evaluate and tweak these analyses vis-à-vis learning theory and consultation with peers as well as pre-pilot empirical results that I gather concurrently. The project then continues to ascend spirally through cycles of implementation, reflection, and modification [29].

The objective of this particular essay is to step back from the creative process so as to survey and sort the products of this process in terms of commonalities and differences in materials, tasks, and facilitation methodology. In an attempt to ground this taxonomy in the learning sciences, the reflection will draw from several theoretical resources, as follows.

I practice design-based research by integrating perspectives from constructivist, sociocultural, and semiotic-cultural approaches. This struggle to hold together under a single auspices perspectives from schools of thought that are often viewed as antinomial [30] has been described as the “dialectical approach” [31]. As such, in analyzing the multimodal behavior of children who participate in implementations of my designs, I attempt to articulate what primitive cognitive mechanisms children bring to bear [32,33], how these mechanisms inform students' sense-making as they co-enact cultural practice with instructors [34], and how instructors steer students to objectify presymbolic notions in disciplinary forms [35,36]. The two design genres surveyed below share in a conceptualization of mathematical content learning as emerging through students' efforts to enhance, communicate, or substantiate aspects of their implicit perceptuomotor schemes—a guided process that is mediated and formulated by the cultural tools that students are encouraged to utilize as the means of accomplishing their ad hoc objectives.

Ultimately, my design framework attempts to respond to persistent calls from leading educational researchers to make mathematical content meaningful to students by helping them construct fundamental mathematical notions [37,17,38]. I view my design framework as spelling out effective methodology for realizing these calls.

3. A tale of (what seem to be) two design genres

This section lays out what I am proposing to view as two related yet distinct design genres for creating mathematics learning activities, the perception-based and action-based design genres. Both genres are different from what a purist constructivist educator or purist sociocultural educator might each recommend: I view these “single malt” positions as incomplete and instead promote a “blended malt” framework. The two genres can be viewed as a sociocultural interpretation of radical constructivist pedagogical philosophy [39], in the sense that they abide with the more tempered accounts of what resources and guidance teachers should provide in fostering student reinvention of mathematical ideas [40]. Namely, per both of my proposed genres students begin from what they can see or do in coping with a problematic situation; yet then this naturalistic capacity enters in dialog with analytical discourse on the same situation, as embodied in the lesson's media, symbolic artifacts, and teacher voice and positioning; via this guided dialog, the students are encouraged to negotiate, coordinate, and reconcile the spontaneous and scientific perspectives [41,42].

Research studies that evaluate design products built according to this framework tend to center on the events of reconciliation. Why are students willing to accept formal disciplinary structures, given that their deepest notions are presymbolic and unarticulated? Upon what epistemic ground do students see fit to appropriate cultural forms that parse the world differently from their intuitive views? We will get to this critical epistemological issue after exemplifying the two design genres.

The objective of this section is not, and perhaps cannot be, to describe in great detail a set of design studies. Rather, I wish to explain and exemplify design genres. Where the reader may wish to learn more about rationales and findings, I provide references to other publications. Finally, any taxonomy per force draws broad brushstrokes—it condenses complex activities into particular essential elements. Yet in practice, elements of these and other genres may often intermingle.

3.1. Perception-based design

In [43,39], I surveyed a set of designs for students to ground mathematical concepts via coordinating tacit and analytical views on situated phenomena. These designs have all been evaluated empirically via semi-structured clinical interviews, and microgenetic analyses of students' conceptual trajectories suggest that these designs bear didactical potential. Aspects of these designs have been integrated into high-visibility units.

Common to this set of designs is that they each target an a/b concept, such as likelihood (favorable events/possible events), slope (rise/run), density (total object area/total area), and proportional equivalence in geometrical similitude ($a:b = c:d$). Further common to these designs is a general lesson plan by which to invite students first to articulate their naïve view with respect to a situation and only then engage in modeling, reflecting, and discourse by which to negotiate the formal view as complementary to, and empowering of their naïve view.

As such, activity sequences in this genre begin by presenting students not with mathematical definitions, notation, and worked examples—in fact, participants often do not know they are “doing math”. Rather, the instructor presents students with a set of materials and asks them to cast a judgment with respect to some physical, figural, or logical property inherent in these materials. Importantly, the materials are crafted such that students' naïve inferences, though qualitative, ill-articulated, or tentative, nevertheless agree with mathematical analysis. That is, I do not attempt to cause cognitive conflict early on in the process by proving the students wrong; rather I attempt to embrace and affirm children's

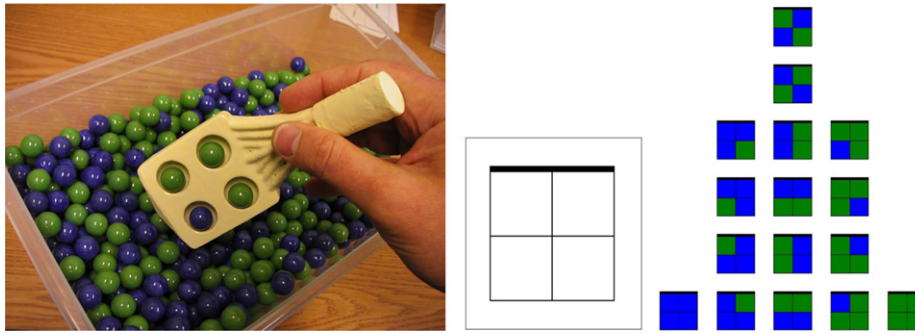


Fig. 1. Selected materials from a design for the binomial. From left: an open urn full of green and blue marbles with a scooper for drawing out four marbles; a card for indicating possible outcomes using green and blue crayons (the thick line indicates “this side up” so as to distinguish rotations); the event space made up of 16 such cards. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

agency in making sense of the world in their natural, uninformed yet often sophisticated ways [44–46]. Thus students are expected initially to apply not analytical views, which they would not as yet share with the instructor, but – explicitly – their naïve views. In particular, students first experience the embedded magnitudes not analytically via a/b structuration but rather holistically, as a gestalt perceptual sensation [47, pp. 46–49]. Only then, in order to introduce the analytical view, the instructor provides students suitable media and guides them through the formal procedure of building a model of the situation. The emerging practical and theoretical question at the heart of research on this design genre has been whether and, if so, how and why students accept the mediated analytical view, proposed by the instructor, as complementary to their own naïve view on the source phenomenon.

Consider the probability subject matter of simple compound-event random generators, such as the rolling of two dice or the flipping of four coins. For this content, middle-school students should learn to perform combinatorial analysis procedures and make sense of resultant event spaces. For example, students are to determine the chance of getting an outcome with 2 heads (H) and 2 tails (T) from flipping four coins. This content has presented great challenges for students as for adults [48], and researchers implicate students’ difficulty in appreciating the importance of attending to variations on combinations [49]. For example, students analyzing a four-coin flip do not discern among the equiprobable yet unique outcomes HTHT and THTH—they typically argue that the order is irrelevant to an analysis of chance (likewise, students view the dice-roll outcomes 3–5 and 5–3 as indexing literally the same outcome). Consequently, any useful intuitions and predictions that the students might have brought to bear on the problem become thwarted by the instructor’s analysis that, as far as the students can tell, dismisses their own view. Students are thus expected to accept probability algorithms that conflict with their “normal thinking” [50,51]. Granted, the algorithms enable the children to solve school assessment items, yet Wilensky [52] has warned of the “epistemological anxiety” ultimately bred by such reluctant acquiescence to ostensibly arbitrary routines (see [53], on deuterio learning; see [38], on the perpetuation of meaningless solution algorithms).

As designers we are thus searching for a concrete situation that embodies the same mathematical problem as does the four-coin experiment yet in a form that is conducive to correct rather than incorrect intuitive prediction of actual experimental outcomes. Toward that design objective, we seek to create an opportunity for learners to express their predictions qualitatively, without any numerical indices. Only subsequent to these predictions will we guide the learners to coordinate meaningfully between their naturalistic view of the situation and the complementary mathematical view. This is the *gestalt-before-elements* principle of perception-based design [39].

In my design solution, the instructor presents the student with a small tub full of marbles – a mixture with equal amounts of green and blue marbles – accompanied by a utensil for drawing out exactly four marbles set in a 2-by-2 square configuration (see Fig. 1). Students are asked to indicate the four-marbles event they believe is most likely to be drawn from the tub. The instructor then provides the students cards as well as a green and a blue crayon and guides them through combinatorial analysis of the stochastic experiment; this process results in the construction and assembly of the experiment’s event space—a collection of sixteen iconic representations of all possible outcomes, organized in five stacks according to k (# of greens).¹

In our studies, Grade 4–6 students, who had not formally studied probability, judged that the most likely four-marbles draw from the tub would have two green and two blue marbles. This is precisely what mathematicians would predict via probability theory, and yet the students did so based not on combinatorial analysis but, I submit, on hard-wired perceptual capacity to infer the representativeness of samples based on comparing color ratios in a sample and its source population [54–56]. The students further judged that an all-green or all-blue draw would be the rarest type of draw, and so on. Importantly, these naïve inferences were couched in terms of the five possible combinations, with no reference to the variations on these combinations. Nevertheless, and critically, students were ultimately able to make sense of the event space as triangulating their naïve expectation, even though the event space does include those variations they had been ignoring. How do students achieve this coherence between tacit and cultural views on a stochastic situation, given that these views apparently carve the phenomenon at different joints [57]—with or without variations?

I have argued that students ground the analysis product as meaningful via a creative inferential process called abduction [58,59]: students bootstrap the design’s targeted mathematical content in the form of a causal *rule* they invent by which to construe a product (the mathematical model) as a *case* that explains as a *result* their unmediated perceptual judgment of the source phenomenon; students initiate this insight heuristically by aligning [60] and interpreting relations among elements of the mathematical model as analogous to relations among elements in their perceptual construction of the source phenomenon. For example, students notice that there are more possible outcomes of the two-green-and-two-blue color combination (6 outcomes) than outcomes of the three-green-and-one-blue combination (4 outcomes;

¹ Strictly speaking, this situation is a hypergeometric, not binomial experiment, because as each marble settles in the scooper, there is one less of its color in the urn. However, the large number of marbles in the urn makes this distinction practically negligible, hence I treat it as binomial for the purposes of this article.

see Fig. 1, on the right); this inference evokes an “ $A_{\text{Outcomes}} > B_{\text{Outcomes}}$ ” notion that coheres with a corresponding comparison of these same two events’ intuited likelihood as inferred from the source phenomenon (Fig. 1, on the left), that is, “ $A_{\text{Likelihood}} > B_{\text{Likelihood}}$ ” (see [61]). This process is greatly supported by the instructor, who guides the child via multimodal discourse toward particular perceptual features, in both the phenomenon and its model, whose highlighting, alignment, and coordination are crucial for achieving the abduction [62,63].

Implementations of perception-based designs such as this have suggested an intriguing finding. Namely, under appropriate design conditions, students are able to make sense of the analysis product, that is, the material assembly that the educator views as a model of the situated phenomenon, before they appreciate the analytic process by which this model was built. For example, first students would succeed in accepting the event space as a meaningful representation of the intuitively anticipated outcome distribution, and only in retrospect would they accept the combinatorial analysis process by which this product was created. This is the *product-before-process* formalization sequence of perception-based design [39].

Let us examine a case study, so as better to contextualize the discussion and to demonstrate the productive “messiness” and emergent character of tutorial interactions. Tamar (pseudonym) is a 6th-grade middle-school female student characterized by her mathematics teachers as “middle achieving”. We will discuss only the first 25 min of Tamar’s hour-long interview, because after that point she engaged in computer-based simulations that go beyond the scope of this paper. This particular episode was selected from the data corpus as paradigmatically demonstrating all participants’ struggle to coordinate tacit and analytic views on situated phenomena, albeit Tamar’s particular resolution of this struggle was unique.

Asked what would happen when we scoop, Tamar singled out the two-green-and-two-blue (2g2b) event as most likely as compared with each of the other four aggregate events (consult Fig. 1, on the right). Asked to support her prediction, Tamar alludes to a perceptual judgment of the source phenomenon:

It’s like a 50–50 chance of getting two-green-two-blue...because it kinda looks like there’s an even amount of them [green and blue marbles in the container], so if you scoop, it’s, like, yeah...²

In her attempt to warrant her correct perceptual judgment, Tamar is constrained by her mathematical knowledge and so she arrived at an incorrect explanation. Here she projects directly from the proportion of green and blue marbles in the urn (50–50) to the proportion of green and blue marbles in a 2g2b scoop (50–50). The chance of getting a 2g2b scoop is actually 0.375 not 0.5.

The researcher encouraged Tamar to explain further or be more rigorous, but she could not offer any more insight on this subject.

Soon after, once Tamar had created the expanded sample space, so that the cards were spread out on the desk in loose order, the researcher asked her whether the cards had any bearing on her earlier prediction. She responded:

I’m not...I think that...I’m not sure...I just...yeah...

Thus whereas Tamar was able to conduct combinatorial analysis *per se*, she did not intuit the practical objective of this activity—neither its process nor its product. In particular, Tamar had yet to discern any relation between the number of variations per event and the relative likelihood of events.

The researcher guided Tamar to organize the sample space cards by the number of green singleton events, and Tamar assembled the sixteen cards into the tower (see Fig. 2). The researcher



Fig. 2. The combinations tower, a compound event space, that Tamar was guided to generate and assemble. The vignette will culminate with Tamar’s reasoning as she compared the likelihoods of two possible outcomes represented by cards on the bottom row of this structure: the all-green outcome on the far right and a three-green-and-one-blue outcome immediately to its left. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

then asked Tamar whether she had any new observations. Tamar surveyed the assembly and offered that she had overestimated the chance of 2g2b:

It actually seems like it could be more...like it’s not exactly 50–50 chance of getting two-and-two [as compared to] getting something totally different, because there are more.... There’re a lot more combinations and stuff.... Now I think there’s actually more chance of getting something different.

We thus see Tamar reasoning about her intuitive judgments of the source phenomenon vis-à-vis the formal analytic structure of the phenomenon. At the same time, we see Tamar becoming aware of the inadequacy of her earlier explanation: she realizes, but cannot yet express, that whereas 2g2b is indeed the most likely event, it appears to encompass the plurality, not the majority, of all possible events in the sample space.

The researcher asked Tamar whether she knew how to express this idea otherwise, perhaps with numbers. Tamar said she does not—“It’s just, looking at it, it *seems* like that”. Thus whereas Tamar’s perceptual reasoning was proportional, her explicit reasoning was not, possibly because she was not sufficiently fluent in rational numbers.

To the extent that the above transcription is of interest to researchers of probability education, I would like to suggest that what is interesting about it is what Tamar did *not* say more so than what she said. Namely, I am referring to Tamar’s facile endorsement of the compound event space concurrent with its stochastic implications. Tamar, who only ten minutes prior was unable to suggest any rigorous means of supporting her prediction for the greatest likelihood of 2g2b, beyond referring to the color ratios in the box, and who still could not offer an explanation once the event space was completed yet scattered on her desk as sixteen discrete items, immediately assumed mathematically appropriate analytical reasoning once the event space was reconfigured so as to make

² Square brackets communicate indexical information with respect to speech referents, which can be gleaned from the agent’s gestures.

salient the number of outcomes per event. Per semiotic-cultural theory and the perception-based design framework, once the event space was more conducive to perceptual reasoning Tamar availed of these material means so as to objectify and modify her qualitative notions. She linked sensations of differential representativeness in the random generator with differential discrete quantities across the five event sets. This heuristical anchoring of qualitative sensation in an enumerable display is striking in its educational significance precisely due to its discursive insignificance.

Still, heuristical anchoring of presymbolic holistic notions in articulated analytic structures does not imply conceptual understanding. In fact, there is much work to do in order to render this implicit reasoning explicit and available for reflection. In particular, such anchoring may engender struggle over contrasting meanings of ambiguous objects—informal and formal meanings [64,29]. For example, does Tamar see a particular 3g1b card as one of sixteen equiprobable elemental events or as one of five heteroprobable aggregate events? Is she conscious of how she is seeing the outcome and why she sees it as such?

The researcher (Dor) lifts up two cards from the completed combinations tower and holds them side by side, well within Tamar's visual field. Looking at Fig. 2, these are the single 4g card on the far right and the 3g1b card immediately to its left³:

Dor: Is one of these patterns more likely to show up than the other?

Tamar: I actually think that this one [3g1b] is more likely to get, because it seems like it's harder to just get four of one color than to have it more mixed.

Tamar views the particular 3g1b card as “more mixed” than the single 4g card. Her assertion would be mathematically correct if she had qualified the “more mixed” as the collective property of *all* the 3g1b cards. Indeed, it is four times as likely to randomly sample any one of the 3g1b cards than the single 4g. Only that Tamar's speech utterance explicitly indexed not the collective of all 3g1b cards but a specific 3g1b card that is in fact equiprobable to the 4g card. As such, Dor and Tamar share a referent – the particular card – but they construct it differently, with Tamar seeing it as 3g1b *per se* and Dor seeing it as the 3g1b card with blue in its bottom-right-hand corner [65]. *Tamar's interaction with the 3g1b card is analogous to seeing HHTH as 3H-1T and inferring that it is more likely than HHHH.* Yet this finding is more striking than the Kahneman and Tversky work, because here the entire event space is explicitly available for inspection. The constructivist tutor's challenge becomes to help Tamar sustain these order-blind presymbolic notions of likelihood while guiding her to re-map these notions onto card *sets* rather than individual cards, in accord with mathematical analysis. In a sense, the tutor has to help the student re-wire a sign:

Dor: Ok, so that's interesting—what you're saying is....

Tamar: It's like a 50–50 [the two patterns are of equal likelihood] but...it's just...to me it seems like that [3g1b] would get more.

Tamar fluctuates between a view of the 3g1b card as a heteroprobable aggregate event and as an equiprobable elemental event. But she is becoming conscious of this tension.

Tamar may not be able to resolve this tension on her own. It is Dor's role, in his capacity as tutor, to facilitate and encourage Tamar's awareness of her competing interpretations, while negotiating language and forms by which she may own, accept, and further articulate *both* interpretations. What Dor chose to do is guide Tamar toward realizing that she is sometimes seeing the particular 3g1b card as an order-less event:

Dor: Now, I want for us to be careful with the definitions here, because you said this [4g] is all green and this [3g1b] is mixed. So...[3 sec. silence]

Tamar: Well, it's actually harder to get just that pattern [the particular 3g1b card], I guess, so it's, like, even [i.e., equivalent chances of getting the particular 3g1b card as compared with the 4g card].

When Tamar says, “It's actually harder to get just that pattern”, she appears to be comparing the specific 3g1b card not to the 4g card but to the entire group of 3g1b cards. Dor asks for clarification:

Dor: Oh, ok. Can you explain to me now what just went on in your mind when you made that observation? 'Cause that's important for me. [He places the two cards back on the desk in their respective locations.]

Tamar: Well, I just, like, saw all of them [the event space] and just.... At first I thought that if you got [the particular 3g1b card]... It could be anyone of those that... [any of the four different 3g1b cards], and then I like just stared at that one [the particular 3g1b card], and I knew that it was, like, just as hard, because you have to get that exact pattern, so....

Tamar is reflecting on her construction of the particular 3g1b card. Initially, she had construed it intuitively as an aggregate event, one of five aggregate events in the entire space, but then she attended to the card analytically as a specific pattern whose likelihood is equiprobable to the other cards in the space. Dor explores how robust this new awareness may be by orienting Tamar to other event columns in the tower and essentially reiterating the previous question. As we will see, Tamar's awareness was not too robust:

Dor: Is there any exact pattern in this field...this space, or collection [the event space of all 16 cards]...that is.... I don't know...easier or harder to get than any other particular pattern?

Tamar: No, I don't think so.

Dor: Interesting. So....

Tamar: Well, I think that this [a particular 2g2b card at the bottom of its column] might be a little bit easier [than 4g], because it's...well, I don't know! It just seems more difficult, to me, to get four of one color than to get them mixed.

We see that Tamar, upon attempting to generalize her new awareness from the 3g1b column to the 2g2b column immediately adjacent to its left, “regressed” once again to ignoring the order of singleton outcomes in individual cards. Tamar is in transition:

Dor: Ok...but do you have the sense of what you're flipping between? On the one hand, you're saying “to get mixed”, and you're kind of referring to the whole thing [the entire 2g2b column], but....

Tamar: Yeah....

Dor: ...then, when you stare at one [the particular 2g2b card at the base of the column]....

Tamar: Yeah...I, I think it's even....

Dor: Do you recognize the little confusion....

Tamar: Yeah.

Dor: ...that there is here between the specific pattern and the group?

Tamar: Yeah.

Dor: Ok, that's a confusion I think we have to sort out, I think, in order to, like, understand this stuff.

The dyad continues to clarify terms. Tamar achieves stability within under two minutes, and the interview moves on to computer simulations of the experiment.

As Tamar's case has demonstrated, perception-based design holds apparent potential not only for mathematics learning but

³ An accompanying video clip of 2' 15" minutes duration can be viewed online at <http://tinyurl.com/dor-tamar>.

also for *research* on learning, because it hones universal tension between students' informal resources for making sense of situations and instructors' formal reconstructions of these situations. I thus believe these empirical findings, theoretical developments, and investigative contexts are not only valid and useful within this design genre but, rather, might elucidate and even inform mathematics education more generally [39,63,66].

A later section will revisit the general question of why students are willing to accept a scientific reformulation of their own naïve worldview. And the Appendix provides a template for creating perception-based design.

3.2. Action-based design

The action-based design genre emerged on the background of a growing body of theoretical and empirical research in the cognitive sciences implicating embodied activity as the source, substance, and process of human reasoning [67]. Within mathematics-education research literature, we witness increasing, converging support for a conceptualization of goal-oriented interaction – whether physically manifest or mentally simulated activity – as the epistemic source and intrinsic phenomenology of problem solving [68–70]. And yet, competent performance in the disciplines, specifically in mathematics, is instantiated within semiotic registers involving signs, forms, and procedures that bear little to no cues as to their spatial-temporal origin and meaning. We are faced with a continuity paradox: How does embodied action give rise to reflection, analysis, disciplinary forms, vocabulary, and inscription? How are these epistemically disparate resources linked through participation in learning activities? How does a teacher guide this process?

In approaching this traditional symbol-grounding problem, I agree with Harnad [71] that knowledge evolves “bottom up”, and I characterize the “bottom” as deliberate embodied activity, yet I complement his position with sociocultural “top down” mediation via guided participation in social practice. In particular, I investigate the conjecture that individuals' mathematical understanding can emerge as they attempt to enhance, represent, or reflect on their own presymbolic situated action by utilizing cultural tools, that is, via enactive and discursive extension of embodied solution procedures [72–76]. In my design-based research work, I attempt to zoom in on this instrumentalization process of children adopting/adapting action-oriented artifacts available to them in the learning environment [6,77,78].

The rationale of action-based design coheres with empirical findings from the dynamic-systems perspective on motor development [79], cultural anthropology research on parentally promoted infant action routines [80], and cognitive anthropological research on vocational instruction of dexterous tool use within manual practices [81]. These disciplines all conceptualize skill development as the guided, repeated solution of similar motor problems via attuning to emerging affordances in the perceptuomotor field of interaction.

Finally, the rationale of action-based design resonates with, and draws inspiration from, Dourish's HCI (Human-Computer Interaction) notion of *embodied interaction*:

[Embodied interaction] is an approach to the design and analysis of interaction that takes embodiment to be central to, even constitutive of, the whole phenomenon.... [E]mbodied phenomena are those which by their very nature occur in real time and real space; embodiment is the property of engagement with the world that allows us to make it meaningful; Embodied Interaction is the creation, manipulation, and sharing of meaning through engaged interaction with artifacts [82, pp. 102–126].

To explore the potential of the action-based design genre for mathematics education, we built a technological device, the

Mathematical Imagery Trainer (MIT). The MIT is an embodied-interaction technological system designed to foster student development of perceptuomotor schemes for grounded formalization of mathematical notions. Our first MIT was engineered specifically for proportion (MIT-P, see [83,84]).

Proportion is a pivotal curricular topic that has been presenting difficulty for many students from late-elementary school and through to college [85]. Research on students' incorrect solutions to rational numbers, more broadly, has implicated “additive reasoning” as underlying their numerical errors [86]. In particular, students attend to additive rather than multiplicative relations within and between number pairs. Looking at $6:10 = 9:x$, for example, students attend to the difference of 4 between 6 and 10 rather than the factor of $10/6$, or they attend to the difference of 3 between 6 and 9 rather than the factor of $9/6$; consequently, they infer that $6:10 = 9:13$ due to the equivalent differences of 4 within both number pairs, or due to the equivalent differences of 3 between corresponding elements (from 6 to 9 and from 10 to 13). In a sense, proportionality presents a novel situation involving the equivalence of number pairs bearing non-equivalent differences among corresponding elements. Somehow, students are to accept a new type of equivalence class in which different differences can be construed as “the same” [87]. The question is how they may develop this new equivalence class.

We maintain that students can and should ground proportional equivalence in additive reasoning, only that doing so requires appropriate cognitive structures, what Pirie and Kieren [28] call dynamical imagistic schemas. And yet, we evaluate, everyday contexts do not occasion opportunities for people to develop these target schemas. That is, mundane activities do not afford the performance and practice of embodied coordinative routines that, with suitable guidance, could be signified quantitatively and symbolically as proportional. We thus wished to design a novel embodied-interaction activity by which children would develop a new pre-numerical schema bearing semiotic potential as a case of proportionality (see [35]). The interaction would *initially* elicit the students' perceptuomotor scheme presumed to underlie their additive reasoning yet *subsequently* treat this scheme so that students would assimilate proportional relations. One might think of this intervention as “Feldenkrais somatic mathematical education” or just “*somathics*”. Once the new scheme is established, we would steer students to signify this new “embodied artifact” as a mathematical artifact (cf. [88,35]), so that the embodied artifact becomes a “conceptual performance” [89]. In short, action becomes concept.

There have been numerous attempts to support the grounding of proportions, and these attempts vary, in part, in accord with the designer's conceptualization of multiplicative constructs. Some designs embark from an iteration rule for combining a and b discrete quantities into a succession of linked cumulative totals, for example: adding \$2 and \$3, respectively, into two separate piggybanks; tabulating the two linked running totals down the columns, such as 2–3, 4–6, 6–9, 8–12, etc.; and then highlighting multiplicative relations inherent to this tabulation as calculation shortcuts for moving between number pairs in the solution of proportion problems, such as scaling by a factor of 4 from 2–3 to 8–12 (e.g., [90,91]). Other designers launched the activities from non-additive situated multiplicative transformation, such as splitting a set of material elements into equally sized subsets (equipartitioning, see [92]).

Yet all those designs scarcely, if ever, considered what I view as the phenomenological core of proportional equivalence, namely the sensory experience of identity between two ratios (“sameness-relational” equivalence, see [93]). How might students experience 1:2 and 2:4 as sensuously identical? Ideally, I reflected, this sensuous identity should be instantiated in forms that are readily conducive to numerical quantification via measurement, so as

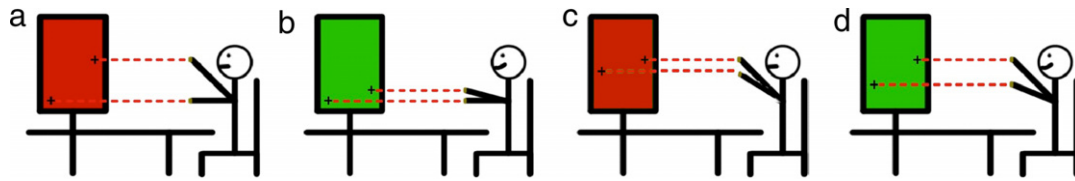


Fig. 3. The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the favorable sensory stimulus (a green background) is activated only when the right hand is twice as high along the monitor as the left hand. This figure encapsulates the study participants' paradigmatic interaction sequence toward discovering the proportional operator scheme: (a) while exploring, the student first positions the hands incorrectly (red feedback); (b) stumbles on a correct position (green); (c) raises hands maintaining a fixed interval between them (red); and (d) corrects position (green). Compare 3(b) and (d) to note the different vertical intervals between the virtual objects. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

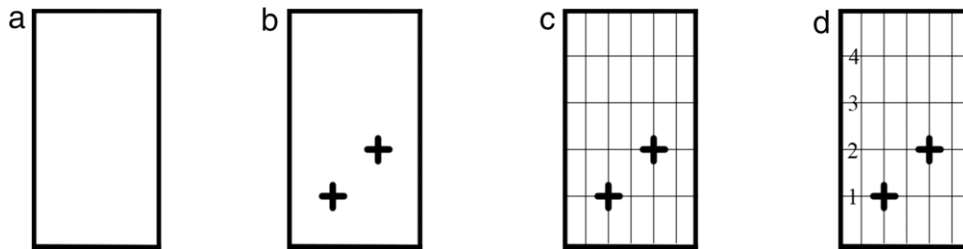


Fig. 4. MIT-P display configuration schematics, beginning with (a) a blank screen, and then featuring a set of symbolical objects incrementally overlaid by the facilitator onto the display: (b) cursors; (c) a grid; and (d) numerals along the y-axis of the grid. These schematics are not drawn to scale, and the actual device enables flexible calibrations of the grid, numerals, and target ratio.

to enable progressive formalization. Perhaps, I wondered, we could use technology to import into a learning environment the familiar “recipe” conceptualization of proportion by which, for example, 1-and-2 units of some substances “taste” the same as, respectively, 2-and-4 units of the same substances. In this design, the a , b , c , and d values of the $a:b = c:d$ proportion would all be extensive quantities from the same measure space, and yet the physical enactment of the a -and- b pair and the c -and- d pair would somehow generate identical sensory effects, borrowing on the idea of $a/b = c/d$ as intensive quantities.

Several inspirational prior designs satisfy some of my own design specifications [94,57,95,96]. However either these designs introduce symbols too early, do not leverage NUI (Natural User Interfaces), or do not offer proportional equivalence as sensuous identity.

I thus sought to create an activity, in which learners could begin to construct proportionality initially by noticing that two physical postures – an a -and- b bimanual posture and a c -and- d bimanual posture – effect the same feedback; learners would then learn to move between the two postures, maintaining the target feedback. This is the *dynamical conservation* principle of action-based design: enacting continuous motion that varies positional/quantitative properties of topical elements yet sustains an overall target feedback. Students would discover and rehearse presymbolic action of proportional transformation as a new perceptuomotor form – a “proportion kata” – that maintains an invariant feedback across the different “ratio asanas”. Only then would we introduce into the problem space mathematical tools, which students would recognize as bearing contextual utility. By appropriating these tools, students were implicitly to represent, reconfigure, and signify their embodied form in mathematical register. As such, the embodied artifact, initially performed as tight perceptuomotor coupling with an interactive technological device, would evolve into a standalone conceptual performance articulated in the discipline’s semiotic system [5,89].

The MIT-P remote-senses the heights of the user’s hands above the datum line (see Fig. 3(a)). When these heights (e.g., 2” and 4”); Fig. 3(b)) relate in accord with the unknown ratio set on the interviewer’s console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display maintaining a fixed distance between them (e.g., keeping the 2” interval, such as raising both

hands farther by 6” each, resulting in 8” and 10”), the screen will turn red (Fig. 3(c)), because the pre-set ratio has been violated. But if she raises her hands appropriate distances (e.g., raising her hands farther by 3” and 6”, respectively, resulting in 5” and 10”), the screen will remain green (Fig. 3(d)). Participants are tasked first to make the screen green and, once they have done so, to maintain a green screen while they move their hands.

The activity advances along a sequence of stages, each launched by the introduction of a new display overlay (see Fig. 4) immediately after the student has satisfied each of successive protocol criteria. For example, consider a student who is working with the cursors against a blank background (Fig. 4(b)). Once he articulates a dynamical-conservation strategy for moving his hands while keeping the screen green, the activity facilitator introduces the grid (see Fig. 4(c)).

We implemented the MIT-P design in the form of a tutorial task-based clinical interview with 22 Grade 4–6 students, who participated either individually or in pairs. Qualitative analyses of video data collected during those sessions suggest that the activities created opportunities for students to struggle productively with core conceptual challenges pertaining to the target content of proportions, at least per our embodied-cognition modeling of this mathematical topic. That is, the students discovered effective non-numerical strategies for utilizing instrumented gesture to enact dynamical conservation and then learned to re-describe these strategies numerically.

Initially, the students explored the space by waving their hands about until they chanced to turn the screen green, whereupon we asked them to find yet another green. All students moved both hands up (or both down), keeping a fixed distance between the hands. Thus, per our hypothesis, students’ default scheme for dynamical conservation is analogous to their typical numerical errors on rational-number problems, such as $6:10 = 9:13$ (albeit we cannot as yet support a claim for a *causal* relation). After further exploration, students articulated a strategy that relates between the hands’ elevation and interval, for example, “The higher you go, the bigger the distance needs to be between them to make it green”. They thus experienced different differences as “the same”.

Next, students engaged the tools we overlaid onto the problem space, adopting/adapting them as enactive, semiotic, and epistemic means of enhancing their performance, discourse, and

inquiry. In particular, students elaborated and generalized their qualitatively expressed, manipulation-based strategies into quantitatively expressed mathematical propositions. For example, they engaged the grid as a frame of reference that appeared better to enable an enactment of the “higher-bigger” strategy, yet in so doing they modulated into a new strategy: in the 1:2 setting they said: “For every 1 unit I go up on the left, I need to go up 2 units on the right” (the “*a-per-b*” strategy, see [97]).

Deeper analyses of students’ conceptual microgenesis revealed that their discoveries of more sophisticated interaction strategies, such as *a-per-b*, were neither premeditated by the students nor directly mediated by the instructors. Rather, these advanced strategies emerged as the students engaged the new mathematical tools to carry out an existing strategy for accomplishing the task (see [98], for sociogenetic modeling of similar phenomena; see [99]). More specifically, in the micro-process of utilizing a new object to perform an existing strategy, the strategy’s implicit perceptuomotor subgoals “hooked” the new object’s embedded affordances, so that the strategy became redistributed and reconfigured. Consequently the strategy “shifted” and, in so doing, both its practical and mathematical power increased (the *hooks-and-shifts* principle of action-based research, see [100]).

In the latter interviews of the study, we introduced a new protocol item: we asked the students to reason about any relations they discern among the different strategies they had devised, which – still using the 1:2 ratio as an example – also included moving the right hand double as fast as the left hand, placing the right hand double as high as the left, increasing the interval between the hands by 1 unit as they both rise, etc. In [101] we demonstrate cases of students coordinating between strategies, and we claim that they achieved this by inventing heuristic logico-mathematical causal mechanisms. One student said, for example, “(the right hand) is always going up by two, and (the left hand) is going up by one, which would mean that (the right hand) is always double (the left hand)”. In the cognitive process of building this causal inference, the students coordinated multiplicative and additive conceptualizations of the dynamical conservation by re-visualizing additive elements multiplicatively. As such, the design achieved the objective of grounding proportionality in students’ additive schemas. Let us examine a case study for this design genre, too.

Naama is a Grade 5 female student indicated by her teachers as “low achieving”. I deliberately selected the earlier part of Naama’s interview for this paper, because her struggle en route to understanding enables me to showcase qualities of embodied learning and, in particular, the notion of *body as vanguard in mathematical learning* [102]. Following, I describe Naama’s early attempts first to move her hands while keeping the screen green and then to articulate a stable rule for doing so. Ultimately, Naama was successful in determining the MIT-P’s multiplicative rules, yet I interpret her behaviors along the way as illuminating the complexity of grounding mathematical concepts in embodied-learning activities. In particular, on the one hand I demonstrate how perceptuomotor fluency may form the basis for schematic reorganization, yet on the other hand I highlight possible difficulties in facilitating this process.

Once Naama had first succeeded in positioning the cursors so as to make the screen green, the interviewers encouraged her to try moving her hands while still keeping the screen green. In her initial attempts to do so, Naama would first move both hands at a fixed distance from each other, which resulted in a red screen, and then she would correct to green either by returning both hands back to the previous green position or by adjusting the distance between her hands. Naama was thus able to inch her way up and down along the display by progressing from each green position to the next one, alternating between fixed-distance simultaneous bimanual actions yielding red followed by mono-manual sequential adjustment actions yielding green. We interpret

the fixed-distance gestures as expressing Naama’s theory-in-action. Because the theory was robust, Naama interpreted the “error” feedback (red screen) as indicating not a problem with her theory but production imprecision that does not bear on the theory-in-action but only on her perceptuomotor acuity and dexterity. That is, Naama tended implicitly to cast her adjustment manipulations as pragmatic “noise” irrelevant to the epistemic “signal” informing the evaluation of her theory.

Soon after, however, Naama appears to have noticed new properties of the situation—she accordingly reverted to a new strategy, which she explained thus:

The higher you go [right hand/cursor], it [left hand/cursor] has to follow, kind of.... If you want it [hands/cursors] to go higher, this one goes higher [right hand jolts twice upward] and then you have to move this one a little bit each time you move [left hand jolts once upward].

Interestingly, whereas Naama does not verbalize the *quantitative* properties inherent to the “two” vs. “one” upward jolts she gestured, these two-vs.-one jolts anticipated the 2:1 ratio setting of the device. Indeed, this right-goes-up-two-units-then-left-goes-up-one-unit sequential strategy proved quite successful in terms of maintaining a green screen, barring the brief red interims. Naama was thus shifting from incorrect to correct theory-in-action—from: (a) simultaneous bimanual action maintaining fixed, un verbalized distance between the hands; to (b) sequential right-then-left quantified and verbalized hand motions. This strategic shift indicates progress along a learning trajectory toward articulating an *a-per-b* proto-ratio conceptual structure (e.g., “two per one”).

Yet then, asked again to explain her strategy, Naama reverted to the naïve theory-in-action:

You have to keep moving the hands and keep them in the same position [relative to each other]... kind of hold them in the same place...you don’t move your hands out of the position.

Executing this naïve strategy, however, again required the performance of adjustment actions. But here we witness dissociation between the mathematical and performance value of strategies, a dissociation that we view as offering challenges for action-based design: Whereas Naama’s theory was mathematically naïve, and we were keen for her to switch to the other strategy that we appreciated as mathematically more advanced, Naama was eager to pursue the naïve strategy, because she was becoming increasingly dexterous at performing perceptuomotor action-feedback cycles, each consisting of a tiny fixed-distance jolt followed by a tiny adjustment. Naama appeared to prefer the naïve strategy—it was more practical. Thus, whereas induced perceptuomotor competence is necessary for action-based design, it is not sufficient: embodied theory-in-action that remains uncultivated may never contribute to mathematical understanding, because learners may revert to safe, workable “detour” solutions that do not navigate unfamiliar territory and are thus less cognitively challenging.

One of the interviewers then asked Naama to lay down the tracking devices upon the desk and demonstrate her strategy using her bare hands. Note that by gesturing *sans* devices, Naama cannot receive online automated feedback on her gestures. One might therefore expect that Naama would move her hands at a fixed distance in accord with her explicit strategy and not perform any secondary adjustment actions, because there would be nothing to adjust by or to. However, and to our great surprise, whereas Naama announced she were moving her hands at a fixed distance, she simultaneously raised her hands miming a changing-distance action that approximated a correct 2:1 growth! We asked her to repeat the demonstration, and she did so twice, still insisting that her hands were moving at a fixed distance. What are we to make of this acute gesture-speech mismatch? It could indicate

that Naama knew more than she could as yet say; that she was prepared for conceptual change (see [103,104]). That is, it appears that the embodied-learning activity entrained Naama to embody a pedagogically targeted action pattern before she could articulate it.

Once the grid and numerals had been introduced onto the display, Naama orally likened the upward vertical gestures to “stacking blocks”, two blocks on the right for every one block on the left.

We shall leave Naama now. For a dénouement of this case as well as other cases, readers are referred to our publications [102,100].

I have now introduced two design genres, perception-based and action-based design. In the reflective process of articulating all the above, I came to ask what these genres might have in common, given that both enable discovery-based learning. As I elaborate below, I believe that both genres create epistemic affordances for grounding conceptual knowledge yet they differ in the particular nature of these epistemic affordances. This proposed centrality of an epistemic factor in the learning process might clarify why I use the appellations “perception-based” and “action-based” to distinguish the genres even though clearly activities in both genres involve perceptuomotor activity! Namely, I am interested in implicating the epistemic root of sense-making – what the new mathematical concepts are grounded in – and differentiating this vital resource from pragmatic aspects of the activities.

4. Jumping to conclusions: comparing two design genres

Both the perception-based and action-based design genres offer learners a subjective sense of continuity from a relatively naïve, immediate form of effectively engaging a situation through to a scientific, analytical, mediated form of doing so. In both genres, the instructor embraces students’ naïve forms of engaging the situation as valid and productive. In both genres, simple perceptuomotor engagement becomes restructured when students appropriate semiotic means of objectification available in the problem space. In both genre procedures, initial interactive embodiment and subsequent numerical signification are staggered rather than concurrent.

Still, by what criterion do learners judge that the naïve and mathematical schemas are commensurate such that the students experience continuity across these modes of engaging the activity? This question is important for the theory of education as for its practice, because the question touches on the old Socratic “learning paradox”—learners’ universal capacity to build conceptual structures larger than the sum of their available parts (e.g., [39,105–107]).

Whether they participate in designs that accord with the perception- or action-based genre, children are led to instrumentalize available mathematical forms by evaluating the forms’ ad hoc contextual appropriateness vis-à-vis their own naïve resources for engaging the situation. In perception-based design, the child compares two *inferences*: (a) the informal inference from looking directly at a phenomenon; and (b) the formal inference from studying its mathematical model. In action-based design, the child compares the *effects* of two strategies: (a) the naïve strategy, in which the body moves in an acquired perceptuomotor kinesesthetic routine that is well coupled with the environment; and (b) the reconfigured strategy that avails of enactive, discursive, and quantification affordances inherent in mathematical tools that are introduced into the interaction space.

Critically, both designs thus appear to afford learners a sense of meaning for the mathematical forms they first engage during the activity. In perception-based design, the sense of meaning emanates from achieving *inferential parity* between the immediate and mediated views on a source phenomenon. In

action-based design, the sense of meaning emanates from *functional parity* across a naïve and an instrumented strategy for effecting the targeted goal state of a technological system. Both inferential-parity in perception-based design and functional-parity in action-based design constitute for learners epistemic grounds for appropriating the mathematical signification of their embodied skill.

Finally, whereas – still prior to formalization – perception-based design avails of the child’s pre-existing capacity, action-based design also constructs new perceptuomotor schemes.

The modest taxonomy of design genres offered in this paper has its obvious limitations by being very much idiosyncratic to the work of one person. As such, the taxonomy might turn out to bear only little if any use to other mathematics-education designers and researchers, because it may be highlighting but a mere corner in what is otherwise a vast, multi-dimensional terrain of designs [108]. Nevertheless, to the extent that the rationale and methodology of this taxonomy agree with fellow interaction designers and design-based researchers, it may be worth their while to qualify or, hopefully, substantiate, complexify, and expand this taxonomy from the wealth of their own experiences. As such, this essay will have achieved its objective: By highlighting the epistemic dimension as critical to learning, the proposed taxonomy treats interaction design at a cognitive grain size that opens the field to richer dialog with the learning sciences. For my part, I would be intrigued by follow-up studies that evaluated the generalizability and scope of this taxonomy by surveying prominent designs and asking: What are the learners’ epistemic grounds for adopting mathematical structures? If it be not inferential or functional parity, what be it?

Note that the two design genres discussed in this article had not been articulated let alone juxtaposed prior to developing the design rationales and creating the products—the very notion of a design genre as well as the typology proposed herein materialized only post hoc through reflection on a personal history of design research. It is therefore quite intriguing that these two genres and their respective epistemic criteria for the adoption of cultural forms map quite well onto a celebrated typology from almost a century ago—Vygotsky’s two types of artifacts that mediate cultural practice: symbol and tool [109].

The probability space is a symbol, an external structure that organizes and thus amplifies the child’s thinking (a process that Vygotsky called “reverse action”, because the symbol operates not out into the world but back onto our thinking). The remote-control interface, by way of comparison, is a tool, a technological artifact that enables the child to cause a perceivable effect in the external world. Yet here, too, the artifact forms the user: in order to operate the tool successfully, the child must reform his own bimanual coordination scheme—the child must move in a new way.

That said, the MIT-P interface is unlike a simple tool, for example a stick. Similar to the probability space, the MIT-P interface bears symbolic elements. In both cases, or design genres, the symbolic elements serve as a frame of reference for enhancing the performance of a task in response to the emerging epistemic demands of the social interaction. Still, unlike the probability space, the MIT-P symbolic elements mediate motor actions, where these are critical for completing the task. Thus the MIT-P, with its symbolic artifacts embedded in a physically manipulatable technological device, is an integrated or hybrid tool-sign, a techno-scientific form that Vygotsky considered a developmentally advanced artifact.

This article began by diagnosing the paucity of quality educational technology as a communication schism between theory and practice, that is, between academic researchers who develop empirically based theoretical models of learning and industry designers who build educational applications for commercial

consumption. I suggested the pivotal role of design-based researchers in bridging theory and practice via reflective demystification of their design process. The objective of this article was indeed to engage in reflective demystification of my own design process. For fellow design-based researchers, this article has hopefully demonstrated the feasibility and utility of such demystification. Such demonstration might encourage my colleagues to join in the efforts of bridging theory gown to commerce town. For designers who are not scholars of education, perhaps the article has revealed some of the nuances of our work and, in particular, how we consider, debate, modify, and generate theory in our attempts to make sense of our empirical evidence. Moreover, I hope commercial designers could avail of this article by emulating my design practice. Toward these ends I have created an [Appendix](#) that spells out the design process.

In closing, I wish to reemphasize that rich learning activities may well include interaction elements availing of both the perception-based and action-based design genres. My objective was not to promote exclusivity of either design genres but rather to examine the root of subjective meaning that educational designers may offer mathematics learners. Ultimately, the ongoing research program is to continue my efforts in developing what I call *embodied design*. Embodied design is a pedagogical framework that seeks to promote grounded learning by creating situations in which students can be guided to negotiate tacit and cultural perspectives on phenomena under inquiry; tacit and cultural ways of perceiving and acting. To realize this vision, I have found, educational designers should keep the body in mind.

Acknowledgments

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Appendix. Embodied-design procedure: phenomenalization, concretization, and their dialogically emergent complementarity

How do we foster meaningful understanding of mathematical concepts? My approach to this general pedagogical problem has been to create situations in which the child can juxtapose and ultimately reconcile naïve and scientific methods of engaging phenomena. These situations should be set up such that the child's own early resources, including naïve perspectives, informal inferences, default sensorimotor coordination patterns, aesthetic preferences, and naturalistic qualitative assertions are in agreement with mathematical theory. At the same time, the child should be led also to recognize and understand the techno-mathematical view on these situations and appreciate this alternative view for its greater precision, control, prediction, and explanatory power.

I attempt to realize this general approach by designing for each targeted concept two perceptual displays – a phenomenon and its model – that together enable an instructor to create for the learner productive dialectical tension. More specifically, I design: (a) a phenomenon affording sensorimotor engagement leading to qualitative inferences that agree with a discipline's theory (this is the process of phenomenalization, see below); and (b) a non-symbolical version of an analytical model for the same phenomenon—a version that affords sensorimotor engagement (this is the process of concretization, see below). The pair of structures – the phenomenon and its model – is crafted and tuned so as to foster learners' appreciation of parity across the naïve and scientific forms of engaging the world. When I interact with the learners, I (c) guide them to discover a way of seeing the model as expressing their intuitive notions about the phenomenon. As

such, I lead students to appreciate parity between unmediated and mediated forms of engaging situations (this is the process of dialog, see below). In this section I offer general steps in a heuristic framework for building instructional activities that induce parity—a framework that I call embodied design.

The objective of this appendix is to offer a template for other researchers to engage in the embodied-design process, so as both to create embodied-design products and set up evaluation studies. I have intentionally avoided any citation of the references, which can be found in the main article. Also, whereas the text will allude to mathematics examples, I attempt to strike a level of description that would potentially make this framework sensible and applicable beyond mathematics.

As I elaborate in this appendix, embodied design process unfolds as follows:

- *a. Phenomenalization—crafting a situation amenable to intuitive engagement*
 - Identify and select a generic schema that underlies reasoning about the target disciplinary notion; and
 - Create a situation in which enacting that schema constitutes a solution strategy to an interaction problem.
- *b. Concretization—crafting a diagrammatic model of the situation*
 - Determine the formal disciplinary model of the problem situation and invent a diagrammatic version of this model;
 - Identify symbolic artifacts by which a learner could signify their solution strategy for the interaction problem in the form of this model; and
 - Devise situated incentives for the learner to appropriate those symbolic artifacts either as semiotic, enactive, or episodic means.
- *c. Dialog:*
 - Elicit the informal actions that solve the problem situation
 - Guide the construction of the formal diagrammatic solution to the situation
 - Engage the learner in reflecting on relations among naïve and disciplinary visualizations of the situation.

The process of preparing embodied-design products is thus two-pronged, involving: (a) phenomenalization, that is, creating a manifestation for an informal notion pertaining to the target concept; and (b) concretization, that is, creating a manifestation for its formal structure (see [Fig. 5](#)). These two lines of work are co-constraining, and their products co-evolve, in the sense that the designer switches attention between these emerging structures, tuning them one toward the other. Guiding the designer are the objectives of both honing learners' prospective experience of cognitive tension between the formal and informal views and optimizing for their prospective reconciliation as complementary.

Phenomenalization begins by carefully noting the multimodal, spatial-dynamical notion that the concept evokes for you, the designer, and gradually devising some activity that would evoke that notion for learners. Your choice of notion may be informed by your earlier research and reading, by which you have become aware of key pedagogical challenges in the conceptual domain—you might choose a notion that would enable you to tackle that challenge head on by making it the focus of tension and reconciliation.

If the notion that the concept evokes for you is a complex sensation about properties of a situation, then you should design an activity by which a learner would experience the same sensation before she analyzes the situation formally. This would lead you

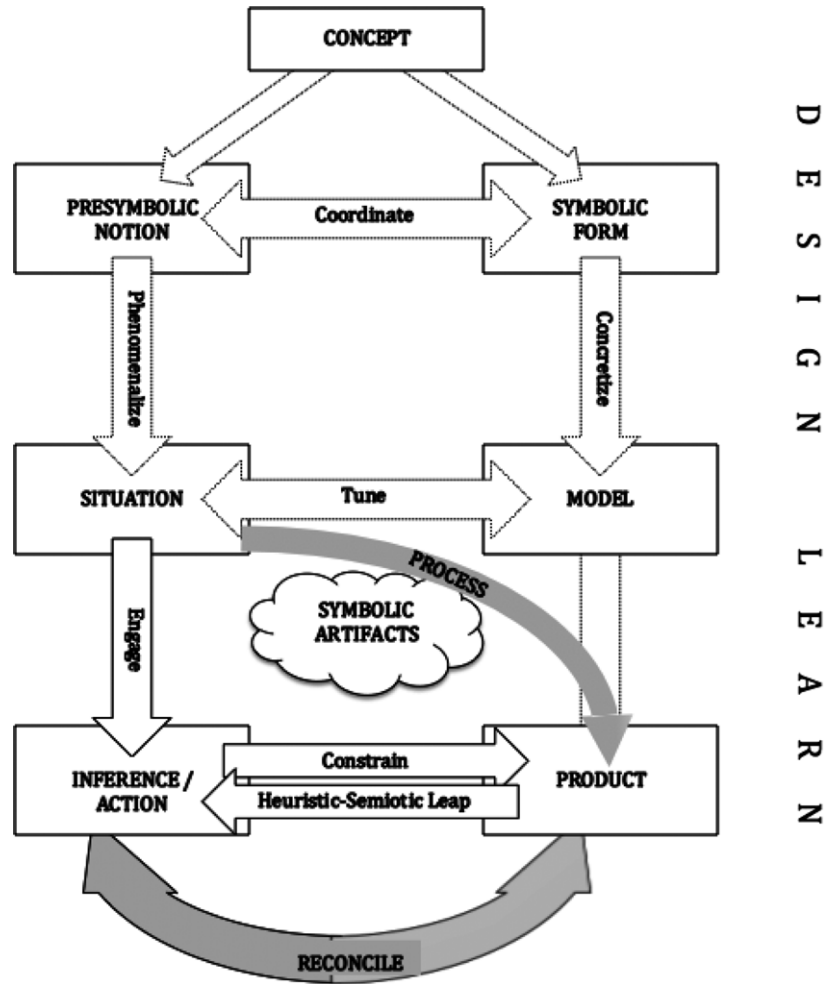


Fig. 5. Embodied-design general architecture.

to working within the perception-based design genre. But if, however, the evoked notion is a complex physical gesture, then you should design an activity by which the novice would learn to perform that gesture as a means of solving an interaction problem. This would lead you to working within the action-based design genre.

Concretization begins by noting the formal structures, such as symbolic and diagrammatic forms, that experts use to express the concept in textbooks as well as to organize the enactment of solution algorithms for problems related to that concept. Maintaining the logical essence of this structure, create a new visual display that portrays the disciplinary analysis for the same situation as in your phenomenalization. This display should be a non-symbolical rendering of the formal representation structure and its elements should be iconically encoded and spatially configured so as to render essential conceptual information perceptually salient.

These two complementary visual displays – a phenomenalization of a concept and an appropriate concretization of the its formal model – are the mainstay material resources of the embodied-design learning environment, and the instructional activity involves working with both resources and reflecting on their distinction and complementarity. At the same time, this general design architecture is realized differently in perception- and action-based design processes, and these alternative design routes will ultimately lead learners to evaluate the distinction and complementarity according to different measures. In action-based design, complementarity will be evaluated according to inferential

parity between naïve and formal structurations of the phenomenon. In perception-based design, complementarity will be evaluated according to functional parity between a physical coordination pattern and its technical re-encoding. These ideas will be elaborated below (see Fig. 5).

In both the perception- and action-based design genres we set up the foundational learning activities such that the critical conceptual work is carried out still prior to conversion into symbol-based semiotic systems. In so doing, we avoid what we believe is a common conflation in discourse about mathematics education between conceptual notions and semiotic systems. Unlike mainstream design, we aim for students to resolve conceptual challenges prior to representing new meanings in symbolic notation. In so doing, we stage learners' formative conceptual evolution within naturalistically meaningful environments, where the instructional dialog is about ways of engaging objects.

In both genres we evaluate the success of an intervention by assessing whether or not the student was able ultimately to appropriate the scientific view via reconciling it with the naïve view and, in so doing, accept the formal analysis process that led to the creation of the scientific product. We also look for evidence that the students themselves were aware of these distinct yet complementary views: the students should be able to articulate the alternative views, describe their confusion, and explain their resolution of this confusion.

Below, we elaborate on the design process in each of these genres—perception-based design and action-based design.

A.1. Perception-based design process

When you engage in phenomenalization for perception-based design, the system you build consists of a generic object (or set of objects) that will constitute for learners the distal stimulus, sometimes called the source perceptual display or, more broadly, the situation or phenomenon. Around this phenomenon you create a task that frames how the learner engages the situated phenomenon, such as orienting the learner toward particular aspects of the phenomenon and implying germane actions for resolving, modifying, or investigating the phenomenon. More specifically, the task is a problem respecting physical or figural or logical properties of this phenomenon, such as a comparison among elements of the phenomenon, where the activity enables the learner to arrive via perceptual judgment at informal inferences expressed in the form of assertions about these target properties. These assertions thus stem from a holistic evaluation of the situation, in accord with the *gestalt-before-elements* principle of perception-based design. Critically, the assertions are *aligned with formal inference from disciplinary analysis*—the assertions are professionally correct, even if they are rough qualitative estimations.

The activity sequence begins by presenting the learners with the phenomenon and eliciting their assessment for a particular property of this phenomenon. The learners use intuitive perceptual judgment to arrive at an informal inference about the target property. Learners are then presented with tools and/or media and are guided to use them so as to enact the professional analysis, even if they do not understand or initially accept the rationale of this analytic process. In so doing, learners are guided to attend to a property of the source display that they had not been attending to because their earlier perceptual judgment had drawn on hard-wired cognitive faculties that are blind to that structure information. Ultimately, the process of enacting disciplinary analysis results in certain products. Examples of such products from using the interpolated artifacts are a tangible visual display assembled from the processed media or a set of numerical values depicting measurements of the phenomenon's physical properties.

The instructional process culminates with reconciling inferences from the informal and formal resources, whereby the learners are guided to discover how to interpret the formal-analysis products as expressing their informal-analysis inference. Learners would thus come to view the mathematical products as bearing inferential parity with their qualitative judgment. In the case of the Seeing Chance project, for example, students achieved inferential parity between informal and formal inference concerning the comparative likelihood of sampling a two-green-and-two-blue outcome as compared to a four-green outcome. The informal inference was warranted by comparative sensations of expected frequency, whereas the formal inference was warranted by comparative counts of possible outcomes in each event class (“ $2g2b >_{\text{freq.}} 4g$ ” because “ $2g2b >_{\#} 4g$ ”). In so doing, the learners accepted the relevance of accounting for variations on each combination. This is the *product-before-process* formalization sequence of perception-based design

Earlier we noted learners' initial reluctance to accept the rationale of the formal analysis process. The learners are all perfectly able to enact the algorithm successfully, only that they do not regard this proposed algorithm as a meaningful solution process for the problem at hand, because the algorithm orients them toward properties of the phenomenon that they regard as irrelevant or superfluous to the task. However once they have completed the process and have ultimately achieved inferential parity, learners tend retroactively to accept the rationale of the analytical process that had led to the product. By *inferential parity* we mean that the student achieves a way of seeing the formal structure as bearing the same information about the target property of the source phenomenon as their informal judgment for that phenomenon.

The crucial event in perception-based design is thus the moment when learners construe the formal product as expressing their naïve inference. We have analyzed that event in great detail and from multiple theoretical perspectives as an abductive heuristic-semiotic leap, whereby learners come to appropriate the new analytic structures and process as viable enactive and discursive means of participating in the cultural practice.

A.2. Action-based design process

When you engage in phenomenalization for action-based design, the technological system you build should create an interaction problem whose solution is a motion analog of the target concept. This coordination pattern is a physical performance, such as a specific bimanual gesture or a form of choreographed steps on a mat, which you have determined as the target concept's underlying spatial-dynamical scheme.

The game mechanic of the interaction task you build should be simple: the players' objective is to receive a designated sensory feedback from the technological system, such as a green screen or the sound of a bell, in response to their physical activity. And so the player moves about physically, abiding any interaction constraints that you might prescribe, until they achieve the sensory feedback. Yet this simple game mechanic soon escalates into a challenging interaction task, because players are required to keep changing their physical input yet maintain the system at the target feedback. As such, the system should be set up so that not just one single physical input yields the target system state. Rather, many distinct physical inputs all yield that output, as in a function.

In order to maintain the system constantly at the target feedback state, the student needs to discover and perform a motion pattern that connects from one successful physical input to the next, then on to the next input, and so on. In a sense, the student needs to enact an embodied mathematical function, that is, a coordination pattern that moves through an n -dimensional input space among physically encoded data clusters that each satisfy the set of game constraints.

The student is perfectly able to learn this new coordination pattern, only that initially this pattern never occurs to him as a means of solving the interaction problem. Rather, the student initially attempts a much simpler pattern that fails to solve the interaction problem. This default interaction pattern is the physical analog of what some scholars have called a student's “misconception” about the content but other scholars have implicated as *bona fide* solution schemas that just happen to be inappropriate for the new content domain but are resources by which to get a first foothold in the new content.

The activity sequence begins by presenting the students with the technological system, explaining interaction affordances and constraints, specifying the task objective, and encouraging the students to explore the physical space. The student will move about in the space until she happens to stumble on a posture that effects the target system state. Often, the student will “freeze” in that posture. The instructor then invites the student to move out of that posture and find another posture that results in the target state. Once the student finds another such posture, the instructor repeats the request and eventually asks the student to move about while maintaining the target state. We have therefore called this class of interaction patterns “*dynamical conservation*”.

When the student departs physically from her first “safe” posture, she likely will still be operating on the basis of the naïve action pattern, and so once again will find herself lost in input space. She will adjust her physical state along the permissible dimensions, until once again she stumbles on another successful position, only once again to be launched into another exploration. The student thus iteratively corrects her physical posture so as

to achieve the target state. After repeated cycles of enacting the inappropriate theorem and then adjusting so as to receive the target feedback, the student notices that her local adjustments are not arbitrary but instead fall into a consistent global pattern. She can therefore anticipate the adjustment for the subsequent iteration and, moreover, *she can assimilate those anticipated adjustments as integral components of her naive scheme itself*. Reciprocally, this scheme accommodates into a new coordination pattern. The student might initially experience the accommodated scheme as a variant on the initial scheme, only later to differentiate it as a new scheme. The instructor encourages the student to reflect on her actions. The student comes to recognize this smooth transition into the new coordination pattern, and she articulates it verbally. At that point we say that the child has learned to perform the conceptual phenomenalization, that is, the physical manifestation of the designer's presymbolic notion for the target concept.

Next, the instructor introduces into the interaction space a set of symbolic artifacts designed to shift the coordination pattern into techno-scientific register, that is, into the concretized manifestation of the disciplinary analysis for the same task objective. Learners discover how to use these symbolic artifacts as frames of reference for their performance of the interaction task. More specifically, learners recognize in these symbolic artifacts interaction utilities for enhancing their performance. These apparent utilities include, but are not limited to, epistemic means of evaluating the quality of their performance, semiotic means of articulating their performance strategy, and/or enactive means of improving their performance. Yet in engaging these frames of reference, the students find themselves shifted abruptly into a drastically new coordination pattern that is geared with techno-scientific equipment. Once again, the student is guided to recognize and articulate the new strategy.

The activity culminates with the instructor initiating a summary discussion. The instructor recounts for the student the various effective strategies she had devised during that session, including the non-equipped and equipped strategies, and asks her to reflect on possible relations among these strategies. The instructor emphasizes that all these different strategies yielded by-and-large the same output, so that it stands to reason that they are not completely independent but might, instead, correspond in ways that are worth noting. That is, the researcher draws the student's attention to the *functional parity* of the various solution strategies: they all share the same effect. The student is expected at this point to discover relations among the various strategies. These relations, which ground the techno-scientific structures in the unequipped physical coordination patterns, often hinge on creative heuristic rules that anticipate formal proofs the student would encounter in future schooling.

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