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ORIGINAL ARTICLE

Bringing forth mathematical concepts: signifying sensorimotor enactment in fields of promoted action

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Abstract Inspired by Enactivist philosophy yet in dialog with it, we ask what theory of embodied cognition might best serve in articulating implications of Enactivism for mathematics education. We offer a blend of Dynamical Systems Theory and Sociocultural Theory as an analytic lens on micro-processes of action-to-concept evolution. We also illustrate the methodological utility of design-research as an approach to such theory development. Building on constructs from ecological psychology, cultural anthropology, studies of motor-skill acquisition, and somatic awareness practices, we develop the notion of an "instrumented field of promoted action". Children operating in this field first develop environmentally coupled motor-action coordinations. Next, we introduce into the field new artifacts. The children adopt the artifacts as frames of action and reference, yet in so doing they shift into disciplinary semiotic systems. We exemplify our thesis with two selected excerpts from our videography of Grade 4–6 volunteers participating in task-based clinical interviews centered on the Mathematical Imagery Trainer for Proportion. In particular, we present and analyze cases of either smooth or abrupt transformation in learners' operatory schemes. We situate our design framework vis-à-vis seminal contributions to mathematics education research.

All doing is knowing, and all knowing is doing.
(Maturana and Varela 1992, p. 26).

1 General introduction and objectives

Varela, Thompson, and Rosch (1991) state the following:

> In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided. (p. 173)

We concur with this view on the irreducibility of cognition, perception, and action in human behavior. We view motor action as tacitly omnipresent in human reasoning, including reasoning during the enactment of cultural practices pertaining to concepts traditionally perceived as "abstract", such as mathematical concepts. However, we take pause to revisit the implications of Enactivism for a theory of learning (see also Ernest 2006). Varela et al. (1991, p. 178) endorse George Lakoff and Mark Johnson's cognitive semantics theory of conceptual metaphor. Yet, we respectfully submit that for Enactivist philosophy to continue contributing to the development of theoretical models of mathematics education, a theory alternative to conceptual metaphor might be adopted that better accounts for the dynamical, interactionist, emergent, sociocultural, distributed, and developmental aspects of teaching and learning—all properties of human practice that Enactivism in fact acknowledges.

Thus our thesis is thus inspired by, yet in dialog with, Enactivism. We offer a view of conceptual learning that highlights the role of motor problem solving in the process of cultural mediation. We argue for our view's theoretical plausibility by framing it within a broad reading of the learning sciences, and we explain the view's pragmatic utility by demonstrating its instantiation in a pedagogical intervention. The objective of this paper is to: (1) outline an

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*a*cognitivist, anti-representationist account of how perceptuomotor doing becomes disciplinary knowing; and (2) argue for the methodological utility of the design-oriented investigatory approach our research team has undertaken in the empirical pursuit of these ideas.

Our account of how doing evolves into knowing treats conceptual learning as grounded in motor problem solving (Abrahamson et al. 2011). The account attempts to integrate and elaborate on conjectures previously raised by cognitive scientists engaged in research on learning. Namely, we embark from the conjecture that mental actions, including those characterized as mathematical cognition, are "embodied" in the sense of being grounded in simulations of sensorimotor processes through the use of the same neural resources as are active in bodily perception and action (Barsalou 2010). At the same time, we view action as constrained yet enabled by contextually relevant features of the world, including the cultural practices that induce, frame, and form the agent's goal-oriented participatory activity.

In fact, as educational researchers with dual commitment to both pedagogy and scholarship, we deliberately attempt to illuminate tensions and opportunities inherent in the dialectics of naïve individual agency and formal cultural structures as they play out in instructional regimes. And yet this dual commitment leads us to critique extant instructional practices. Consequently, our research program involves designing new environments that embed learners in interaction systems wherein naïve sensorimotor coupling is nurtured into formal mathematical activity. In this sense, we see our work as a continuation of design frameworks advocated by pioneers of embodied mathematics education. For example, Zoltan Diénès maintained, and we concur, that "children will learn by acting on a situation" (in Sriraman and Lesh, 2007, p. 61). Yet whereas Diénès, Dewey, Montessori, Gattegno, Skemp, Freudenthal, and other pedagogical visionaries looked to mathematical *products* of acting on a situation, our research focus is more on the actions themselves, that is, the *process* of this doing-to-knowing evolution. Specifically, our focus on physical and simulated action as the cognitive underpinnings of mathematical concepts shifts and recalibrates our gaze away from what the students produce to how they go about producing it.

In this article, we describe learners' experiences of attempting to accomplish a given task in a novel computer-based environment and argue for the pedagogical importance of attending to emergent transformations in motor action patterns that learners experience, acknowledge, and articulate in the course of their interactions. In our view, conceptual knowledge, or at least the manifest competence usually labeled as such, emerges for the agent from guided sensorimotor interaction with cultural artifacts. This emergence is not "one way" from concrete to abstract, as in traditional cognitivist views, because the phenomenology of reasoning itself is *still* embodied, quasi-physical, concrete—that is, best described as a form of doing (Gallese and Lakoff 2005; Kieren et al. 1999; Melser 2004; Noss and Hoyles 1996; Wilensky 1991). It is in this sense that we find our work broadly aligned with Enactivist vision, exemplified by Varela's (1999) aphorism: "The concrete is not a step toward something else: it is both where we are and how we get to where we will be" (p. 7). To contextualize and support our thesis empirically, we model the epigenesis of learners' schemes, or goal-oriented sensorimotor actions, as they participate in technology-enabled activities designed to support the development of a rudimentary mathematical notion, proportionality.

Our thesis, and more broadly our call to leverage yet adapt the Enactivist view, hinges on some analogous process we have discerned across diverse instructional practices. Namely, we will be comparing our own pedagogical activities for the concept of proportion with practices that cultures have developed to foster novices' motor skills, including informal indigenous praxis (Reed and Bril 1996) and formal vocational methodology (Becvar Weddle and Hollan 2010). We view this surprising similarity as an opportunity to broaden Enactivist theoretical scopes.

For scholars of embodiment, particularly design-based researchers of mathematics education, the phenomenon of motor learning bears practical allure, because motor learning is an easier research problem than mathematics learning. Within *explicitly* embodied domains of practice, expert and novice actions are physical and pragmatic, affording researchers more transparency onto those subjects' reasoning processes as compared to the case of *implicitly* embodied mathematical activity. Our research program's overarching intellectual gambit is thus to sidestep from mathematics to motor skill, learn over there what we can about practices and processes of teaching and learning, and then sidestep back to mathematics, where we search, research, and design for useful parallels. In other words, we hope to avail of humanity's ancient pedagogical heritage as we struggle to fashion contemporary pedagogical practices.

The following literature review (Sect. 2), methods outline (Sect. 3), and empirical findings (Sect. 4) explore intellectual terrain that is mostly uncharted by learning scientists, particularly mathematics education researchers—a dynamical-systems model of conceptual acculturation. Yet this *terra incognita*, we argue, is well aligned with Enactivism and, what more, offers viable means of honing its central tenets. As such, we hope to exemplify *how* "perception consists in perceptually guided action" and *how* "cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided".

2 Learning emerges from the situated solution of motor problems: scientific antecedents

2.1 Scaffolding motor development: Edward Reed and Blandine Bril

Edward Reed, an ecological psychologist, and Blandine Bril, a social anthropologist, describe indigenous practices that apparently foster infant development of culturally valued physical capabilities. Mothers in remote Sub-Saharan villages were observed to enact shared routines of handling their infants so that they learn to move in new ways. Society thus intervenes in shaping infant development by creating circumstances—*fields of promoted action*—that encourage the building and exercising of particular motor capacities required for effective participation in cultural activities (Reed and Bril 1996). In fields of promoted action, the field is constituted and administered by experienced cultural agents, and the action performed by novices. This global phenomenon, we submit, bears more than allegorical resemblance to mathematics education. Rather, it epitomizes pedagogical practice, or at least the type we envision, in which learners are thrust into motor problem situations and begin learning to move in new ways before they signify these motor plans mathematically. Mathematics education researchers, we argue, should benefit from closer scrutiny into fields of promoted motor action—their sociogenesis, resources, processes, contributing factors, and context sensitivities (see also Antle et al. 2013). In particular, researchers should closely examine the inception of personal solutions via the coupling of goal-oriented actions and emerging affordances of objects (Schwartz and Martin 2006).

2.2 Dynamical Systems Theory: Esther Thelen

The work of developmental psychologist Esther Thelen and collaborators (Thelen and Smith 1994) lends insight into scheme epigenesis at the interface of organism and environment. Building on Gerald Edelman's *Theory of Neural Darwinism* (Edelman 1987), Thelen theorized cognitive and motor capacity as continuously co-emerging via resonance loops among overlapping neural groups that govern situated and goal-oriented multimodal perceptions and actions. The dynamic-systems view of neural development—as an alternative to the then-prevalent neuromaturational view—was popularized through Thelen's groundbreaking demonstration that infants can walk many months before their normative schedule, but only if their body mass is supported, such as in a water tub. Walking, Thelen posited, is not an innate capacity per se but emerges interactively *as the solution to a motor problem*: where local goals interact dynamically with contingent circumstances, physical actions emerge that, proven effective, are rehearsed and thus potentially reapplied upon other terrains.[1]

Thelen's theory was dramatically validated by its proposal to explain infant behavior on laboratory tasks that hitherto had been considered "purely *cognitive*" (e.g., the "A-not-B" task) by appealing to the perseveration of perceptuomotor routines. As such, goal-oriented *motor actions* that agents perform in the service of gathering information (epistemic actions, see Kirsh and Maglio 1994) endure into explicitly and implicitly embodied performance, perhaps as much as the information these actions recover.

2.3 Motion learning as functional integration: Moshe Feldenkrais

The physicist, martial-arts leader, and autodidact motor-action scientist Moshe Feldenkrais created a somatic awareness practice celebrated as a means both of self-development and alleviation of physical ailment. With regards to the latter rehabilitation technique, empirical research has repeatedly demonstrated its effectiveness, even indicating its superiority over conventional physiotherapy (e.g., Lundblad et al. 1999). The rationale of the Feldenkrais somatic education methodology, its practitioners maintain, bears relevance to any pedagogical approach seeking to foster new action patterns as the basis for conceptual development.[2]

An essential principle in the Feldenkrais practice is to create guided opportunities for students to untangle their action complexes into simpler motor components, modify these components, and then foster their selective reintegration into more salubrious complexes. Importantly, students must assume a degree of agency in achieving novel motion complexes. As Ginsburg (2010) clarifies: "Learning itself is not conscious. The integration process itself is not conscious. Nevertheless, the process depends on conscious processes in feeling and detecting changes. The consequence is felt as difference" (p. 185). This notion—that unconscious, subtle interactions drive adaptations to behavior, and that consciousness plays a *post facto* appraisal role in making sense of these changes—is crucial

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[1] See Clancey (2008) for a survey of complementary intellectual antecedents to the situated/embodied/enactive paradigm, such as the cybernetics research of Gregory Bateson and the robotics work of Andy Clark.

[2] We acknowledge that Feldenkrais scholarship is unconventional as an academic perspective. Notwithstanding, we value its conjectures regarding the roles of embodiment and awareness with respect to learning. These conjectures are original and grounded in a practice that is empirically shown to be effective. Moreover, the conjectures parallel many of our own findings, some of which we arrived at prior to our exposure to Feldenkrais practice.

to our thesis of conceptual knowledge emerging from guided interaction through a felt sense of difference.

2.4 Learning as motor problem solving: Nikolai Bernstein

Independently of the work of Piaget, Feldenkrais, or, for that matter, Vygotsky, and half a century before Thelen's discoveries, the Soviet neurophysiologist Nikolai Bernstein (1896–1966) stated that motor skill development is "repetition without repetition" (Bernstein 1996). Bernstein's insights on cognitive processes underlying motor-skill development bear implications for pedagogical investigations. In particular, Bernstein's work bears on the study of learning environments designed to foster conceptual development via engaging learners initially in the solution of perceptuomotor coordination tasks, what Abrahamson (2013) calls the *action-based genre* of embodied design. Bernstein writes as following:

> The actual importance of repetitions is quite different [than what has formerly been believed]. *Repetitions* of a movement or action are necessary in order to *solve a motor problem* many times (better and better) and *to find the best ways* of solving it. Repetitive solutions of a problem are also necessary because, in natural conditions, external conditions never repeat themselves and the course of the movement is never ideally reproduced. Consequently, it is necessary *to gain experience relevant to all various modifications* of a task, primarily, to all the impressions that underlie the sensory corrections of a movement. (Bernstein 1996, p. 176, original italics here and below)

Thus the mainstay of skill learning is not in performing a would-be idealized motor routine but precisely in the repertory of agile *fixes* to emerging contingencies, which Bernstein called *automatisms*.

Bernstein also warns educators, "The fact that the 'secrets' of swimming or cycling *are not in some special body movements but in special sensations and corrections* explains why these secrets are impossible to teach by demonstration" (p. 187). Thus, a master craftsperson may demonstrate an idealized enactment of a skill for the neophyte to emulate, but the neophyte learns only through attempting to imitate this enactment. This principle has been repeatedly reported in ethnographic studies of craft training, such as carpentry (Ingold 2011) or pottery (Churchill 2014). Indeed, you can show me how you apply saw to wood and you can guide my actions as I myself wield the tool, but I will have to learn the appropriate *felt sense* of this action on my own terms, and on myriad different types of wood, by responding to the felt difference. It is once again a field of promoted action, yet a field including an artifact that the novice learns to apply to an object—it is an *instrumented* field of promoted action. Bernstein's automatisms are thus the cognitive and sensuous residue from having engaged in goal-oriented activity within an instrumented field of promoted action.

2.5 Fostering instrumented fields of promoted action: schematic preview of a design

We agree with scholars who argue that instrumented interaction gives rise to conceptual reasoning (Melser 2004; Roth and Thom 2009; Vérillon and Rabardel 1995; Vygotsky 1978). Yet how this happens and the roles designers and instructors play in this process is not entirely clear. Moreover, mathematical activity involves also operations in a symbolical system, so that a thesis on embodied mathematical learning should also address somatic–semiotic coordination.

In our design, detailed below, learners develop a new type of bimanual coordinated action. That new action turns out to be moving the hands while changing the spatial interval between them correlative to the hands' elevation above a datum line such as a floor or desk. Learners thus experience a somatic phase shift, from a default preservation of a *fixed* interval to the new coordination parameters of a *changing* interval. This shift is semi-inadvertent, as it begins with nuanced *local* adjustments absent of a *global* motor-action plan. Yet these local adjustments are consistent—the interval grows with elevation and vice versa—and this consistency, as well the learner's increasing dexterity in executing these automatisms, eventually compels the learner to notice a global interaction pattern. Then actions become concepts through the learners' semi-spontaneous appropriation of available semiotic elements they perceive as bearing ad hoc enactive, discursive, and/or epistemic utility for accomplishing their objectives. Moreover, engaging these semiotic means not only shifts the learner's discourse into mathematical register but in turn restructures the action within the task constraints.

Our study participants' interactions occur within a carefully orchestrated context—a designed and monitored instrumented field of promoted action potentially formative of conceptual development. Left alone to solve the coordination problem, learners are quite unlikely to derive its mathematical implications—they need framing and steering from the instructor so as to move from an "artifact-sign" to a "mathematical-sign" (Bartolini Bussi and Mariotti 2008). By way of analogy, all bicycle riders regain balance by steering toward the direction of fall, and yet few will ever articulate this pervasive scheme as an analytical generalization. It is the framing of the interaction as part of a larger cultural activity that may prompt an agent to reflect on an otherwise tacit inference from a learning process and preserve this explicit inference for future reference and elaboration (Trninic and Abrahamson 2013).

3 Designing and analyzing instrumented fields of promoted mathematical action

In this section, we outline our research methodology as well as its relevance for our investigations of embodied knowing and learning. In sequence, we introduce *design-based research*, a general approach to studying human learning, *embodied design*, a pedagogical design framework for conceptual learning, and *microgenetic analysis*, an intensive qualitative-analysis technique for making sense of complex human performance occurring over a relatively brief period of time.

3.1 Design-based research

The overarching approach of our study is that of design-based research, in which theory and design co-develop iteratively (Cobb et al. 2003). Design-based research (DBR) is not a methodology per se but a rich disciplinary context within which we carry out investigations of learning and teaching. We do so because we have a conjecture as to how learning could be better, and yet current learning environments are unsuitable for addressing the conjecture; therefore, we design and evaluate a novel learning environment. Furthermore, because we cannot consider ahead of time all the consequences of our actions, the process of design and its implementation provide opportunities to generate novel conjectures about learning that are then incorporated into the theoretical framework driving the design. In turn, this framework drives the next iteration of design, and so on. This type of reflective practice (Schön 1983), while vital in all aspects of scientific inquiry, is foregrounded in DBR. In our experience, the high frequency of observations emerging from DBR, many of which are unexpected, make it ideal when dealing with a novel research space, such as issues of embodiment, because it stimulates the generation of conjectures necessary for further experimental evaluation and theory building.

3.2 Embodied design and the case of the Mathematical Imagery Trainer

Our design project began with the following assertion: To the extent that mathematical knowledge is grounded in motor action schemes, constructivist instruction should attend to motor action knowledge—its nature, construction, and interaction with enactive, semiotic, and epistemic means in the learning environment. One model for such instruction, *embodied design* (Abrahamson 2009, 2012, 2013; Abrahamson and Lindgren 2014), is to design technologically enabled fields of promoted action that elicit existing motor schemes yet challenge the learner to adapt and articulate new schemes and ultimately signify them within the discipline's semiotic system. Embodied design is thus a framework that seeks to promote grounded learning by creating situations in which learners can be guided to negotiate tacit and disciplinary ways of perceiving and acting. In turn, the hands-on nature of embodied design learning activities typically renders users' implicit mental actions physically explicit and thus accessible for non-invasive investigation. The Mathematical Imagery Trainer for Proportion (MIT-P, see below) is an example of the action-based design genre for fostering embodied mathematical learning.

Our experimental design was driven by a general conjecture that some mathematical concepts are difficult to learn due to a resource constraint of mundane life. Namely, everyday being does not occasion opportunities to embody and rehearse the particular dynamical schemes that would form requisite cognitive substrate for meaningfully appropriating the target concepts' disciplinary analysis of situated phenomena. Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—"additive" solutions (e.g., "$2/3 = (2 + 2)/(3 + 2) = 4/5$")—may indicate a lack of multimodal kinesthetic–visual action images with which to model and solve situations bearing proportional relations (e.g., Pirie and Kieren 1994).

In response to the design problem articulated above, we engineered an embodied-interaction computer-supported inquiry activity for learners to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the *Mathematical Imagery Trainer for Proportion* device (MIT-P; see Fig. 1).

The MIT-P measures the heights of the users' hands above the desk. When these heights (e.g., 10 and 20 cm) relate in accord with the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display at an appropriate rate, the screen will remain green; otherwise, such as if she maintains a fixed distance between her hands while moving them up, the screen will turn red. Study participants were tasked first to make the screen green and then, once they had done so, to maintain a green screen even as they moved their hands. For more technical details, see Howison et al. (2011).[3]

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[3] For a brief video demonstration of the MIT-P, see https://www.youtube.com/watch?v=n9xVC76PlWc.

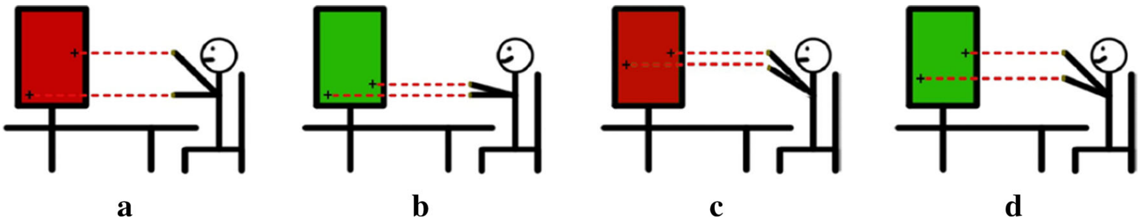

Fig. 1 The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the target sensory stimulus (*green background*) is activated only when the right hand is twice as high along the monitor as the *left*. This figure encapsulates participants' paradigmatic interaction sequence: **a** the student first positions the hands incorrectly (*red feedback*); **b** stumbles on a correct position (*green*); **c** raises hands maintaining a fixed interval between them (*red*); and **d** "corrects" position (*green*). Note the difference between **b** and **d**

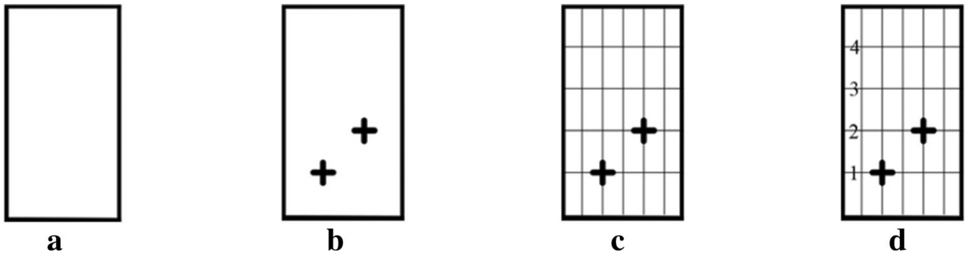

Fig. 2 MIT-P display configuration schematics, beginning with **a** a blank screen, and then featuring a set of symbolical objects incrementally overlaid by the facilitator onto the display: **b** cursors; **c** a grid; and **d** numerals along the *y*-axis of the grid. These schematics are not drawn to scale, and the actual device enables flexible calibrations of the grid, numerals, and target ratio

At first, the condition for green was set as a 1:2 ratio, and no feedback other than the background color was given (see Fig. 2a). Then, cursors were introduced that "mirrored" the location of participants' hands (see Fig. 2b). Next, a grid was overlaid on the display monitor to help learners plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions (see Fig. 2c). In time, the numerical labels "1, 2, 3,…" were overlaid on the grid's vertical axis on the left of the screen to help learners construct further meanings by more readily recruiting arithmetic knowledge and skills and more efficiently distributing the problem-solving task (see Fig. 2d).

In the interest of clearly laying out our investigate approach, it is worth noting that the structured task described above evolved through iterations of MIT-P implementations. We spread the implementation of the interviews thinly (no more than two per day), such that from day to day we would be able to introduce changes to the materials, activities, and protocol in light of the emergence and refinement of theoretical constructs. These rapid-prototyping changes were based on fieldnotes, preliminary analyses of multimodal utterance, minutes from our team's daily debriefings, and collaborative, editable online postings. Thus, both the interview protocol and the interactive affordances of the instructional materials evolved as we progressed through the pool of participants. We gradually incorporated into the protocol any activities and prompts that arose during interviews and that, during our debriefing, we evaluated as eliciting "researchable moments" from the participants. These were moments in which unexpected behavior from a participant suggested new theoretical constructs that we wished to test in subsequent interviews.

In addition to preliminary analysis undertaken during the project's implementation phase, we engaged in more intensive retrospective analysis, using a technique we now explain.

3.3 Microgenetic analysis

Our primary approach to making sense of the video data collected during the implementation of the MIT-P is microgenetic analysis. Generally speaking, microgenetic analysis is an intensive investigation of a relatively brief period (often much less than a minute) of rapidly changing competence, with the aim of making sense of processes underlying this change. Why undertake this type of intensive investigation, rather than, say, pre- and post-test analysis? Before we share our own reasons, we wish to share Siegler's (2006, p. 468):

> If learning followed a straight line, [microgenetic analysis] would be unnecessary. Yet, cognitive change involves regression as well as progression, odd transitional states that are present only briefly but that are crucial for the changes to occur… and many other surprising features. Simply put, the only way to

> find out how children learn is to study them closely while they are learning.

Our own reasons for this time-intensive analysis involve not only our commitment to investigating the embodied aspects of learning, but also our vigilance against forming plausible-sounding yet faulty rational narratives of the learning process. This statement bears some explaining. First, in our experience, intensive interrogation of video data is necessary to detect embodied nuance that may go otherwise unnoticed. What may have been ignored as a series of "inconsequential" gestures is instead evaluated as potentially a vital aspect, and thus indication, of the gesturer's understanding. Transcriptions, even highly detailed ones that include bodily movement, cannot capture in vivo activity with sufficient fidelity for explorative analysis. Instead, we value an investigative approach where a human observes another's activity, typically over and over again.

Second, there is a constant lure to interpret learning process as teleological, as though the child necessarily reasons rationally toward a logical conclusion. This is a form of *historical revisionism*, that is, reading onto the beginning of a micro-process something that emerged only at its end; ascribing to a child an understanding that was not present at a particular moment. This type of revisionism is liable to ignore aspects of activity that do not fit the researcher's "tidy" narrative, such as bodily movement at odds with verbal utterances. To echo Maturana (1987), "Everything is said by an observer" (p. 65). We find microgenetic analysis a strong guard against historical revisionism and confounding our expert observations with those of the students we study (see also Maheux and Proulx, 2015 [this issue], on the methodological challenges of analyzing data within an Enactivist framework).

We conclude this section with a few practical comments about the application, reliability, and validity of microgenetic analysis. Making inferences on the basis of videotaped performance data is a "highly subjective and interpretive enterprise" (Schoenfeld et al. 1991, p. 70). To this end, analysis is collaborative, a form of competitive argumentation where each member of the research team puts forth an interpretation of recorded events. An interpretation is considered viable if all members of the research team are convinced by it. This includes members witnessing it for the first time as a video recording. Interpretations that fail to gather sufficient support are not discarded but instead recorded as such. In fact, we encourage and welcome challenging interpretations, because they force us to embody the observed learner's actions. We mean this literally: occasionally we may physically act out what we observe, thus acquiring unexpected perspectives. Through personal mimetic reconstruction of recorded multimodal activity we may arrive at completely unexpected insight, for example, that perhaps our subject lowered her arms not as an exploration action but simply because she was physically fatigued from holding them up in the air for too long! While intensive in time and effort, we find microgenetic analysis invaluable in our investigations of embodiment. The high frequency of observations and emergent conjectures make it an excellent companion to design-based research.

4 Findings: from motor problem solving to conceptual learning

In accord with our thesis, our discussion of empirical findings will focus on the processes by which our study participants changed their motor action patterns through engaging in problem-solving activities within our instrumented fields of promoted action. We will be looking at both smooth and abrupt transitions.[4] With respect to smooth transitions, we will consider the case of learning to control new phenomena that demand variations on existing schemes (Case 4.1). With respect to abrupt transitions, we will consider the role of artifacts—auxiliary stimuli—as dramatically perturbing schemes previously established as effective for controlling phenomena, thus prompting individuals to establish new schemes still broadly within the task demands (Case 4.2).[5]

4.1 Smooth transition: solving a motor action problem by adjusting a scheme

Our first case analysis, Siena, is a 6th-grade student identified by her teachers as low achieving. This case followed the standard protocol: with the interaction condition set at a 1:2 ratio, Siena was tasked to "make the screen green" and encouraged to explore the interactive space. Quickly, she generated a green screen and affixed her hands at that posture. DA (first author) encouraged her to look for green elsewhere, and she continued exploring. As with all learners in our study, Siena initially kept the distance between her hands fixed as she attempted to find another green, resulting in a red screen. Two minutes later, DT (second author) asked Siena to explain her discoveries to another (hypothetical) learner. She noted three green "places" on the screen, corresponding to high, middle, and low bimanual postures. Siena added, "You'd have to have

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[4] Interestingly, dynamical-systems research into coordination of bimanual action (Kelso and Engstrøm 2006, p. 208) has demonstrated a dichotomy between "smooth" and "abrupt" transitions in the development of motor skill, analogous to our findings.

[5] For further empirical results from this line of work, see Abrahamson et al. (2014).

the right hand higher up slightly… You'd have to be slow and careful".

Next, DT asked Siena to reproduce the "high" green. Once she had done so, DA attempted to steer Siena toward discovering the effective interaction rule by asking her to reproduce the low green, then the high, then the low again. She did so with relative precision. *For an observer it is evident that the distance between her hands was changing correlative to their elevation over the desk, and yet Siena did* not *account for this change*. DA then asked Siena whether she could move from low green to high green keeping the screen "green all the time". Siena tried to accomplish this by moving very fast, resulting in a red screen. She continued, now moving slower, until a minute later she was able to keep the screen green continuously while raising and lowering her hands, making corrections as needed. At this point, the interviewers agreed (communicated via glances) that Siena had apparently determined a working theorem for "making green". And yet, the following exchange ensued.

DT: You seem to have figured something out. Can you share with us what it is?
Siena: [shrugs, smiling] Not really
DA: So what have you learned so far?
Siena: [pauses, searching for words] It's really hard to keep steady, to go high up [mimes smooth bimanual motion upward] but it's possible
DA: Is there any rule? How would you explain this to someone else?
Siena: Um, just try to keep focused on the screen… keep steady and try to keep like that at equal, and got to be sort of equal and like move your hands at the same time. Like, if it's like this [demonstrates by holding hands apart at a fixed distance], then you have to move them both up at the same amount apart [moves hands up at a fixed distance]

Recall that just prior we had observed Siena skillfully "keeping the screen green" and so assumed that she had formulated a new action theorem. Yet it may be that Siena became proficient at the motor action in the absence of a conscious action theorem: she articulated the fixed-interval strategy *despite* manually demonstrating the appropriate changing-interval action. It appears that Siena performed a rapid succession of minute *local* corrections to green even through the consistent pattern of these automatisms had not yet emerged as a *global* awareness.

DA then guided Siena to evaluate her fixed-interval rule empirically. Complying, Siena found a low green and then raised both hands, keeping a fixed interval, which resulted in a red screen. DT then took over the right-hand device, and they worked together, moving up slowly.

Siena: Always the right hand should move up a little higher… this one [right] should move slightly faster than the other one. The right hand should be slightly faster than the left, but they should still keep at sort of the same pace
DA: What do you mean by pace?
Siena: Like, if this one's [right] going like this [moves upward quickly] then this one [left] should be going slightly slower than the other one

Several minutes later, DA asked Siena to explain further what she meant by "pace". She offered that "[pace is] sort of a continuous speed. They should be at a different speed, but they're both at their own continuous speed". During this utterance, she gestured forward, in a series of away-from-body saccades. When asked, again, to summarize for the sake of the hypothetical fellow student what she had learned, she offered: "Make the right hand go a little bit faster, but let them both be at their own continuous pace". We consider this a smooth transition from the fixed-interval theorem to a different-pace theorem. We labeled such transitions "smooth" because the change from one motor pattern to the other was gradual, a series of micro-adjustments. Some of these micro-adjustments were so subtle that Siena herself seemed unaware of them!

Thus we see Siena's progress from: (a) performing fixed-interval action; to (b) performing changing-interval action yet articulating a fixed-interval theorem; (c) reverting to fixed-interval performance; and (d) both physically and verbally expressing a right-is-faster-than-left strategy. It is worth noticing that Siena had solved the motor action problem before she could describe it. This may be an indication that, at the time, she was not yet consciously aware of the 'difference' (see Sect. 2.3). Moreover, by mis-describing her effective motor action skill and then operating on this mis-description, Siena temporarily regressed to ineffective motor action. By asking Siena to articulate verbally her effective motor action, the interviewers imposed on Siena what Bamberger and diSessa (2003) have termed "ontological imperialism", conventional semiotic systems that warp the perceptions of the uninitiated by parsing phenomena into static formal units that jar with action-oriented perceptuomotor interaction schemes. It is as though, asked to explain where his arms are as he walks, a person asserted that the right arm swings forward when the right leg does *and then actually walked that way!* When Siena finally achieved a global description of her effective physical enactment, this articulated theorem enabled her to re-visualize and enhance her tacit local coupling with the technological device.

4.2 Abrupt transition: mathematical artifacts reconfigure motor solutions

When the grid is overlaid on the screen (see Fig. 2c), learners tend to respond in a behavior pattern we have termed “snap-to-grid”—their hitherto continuous motions along the screen become parsed into discrete moves from one grid line to the next. Their utterances, too, transition from qualitative to quantitative language.

Our second case analysis is Amalia, a 5th-grade student identified by her teacher as average achieving. Amalia was quick to locate her first green and even quicker at solving the motor problem of moving the hands upward while keeping the screen green. Interestingly, introducing the grid artifact onto the screen actually *impeded* her green-making performance, even as it granted predictive and communicative power. Unlike Siena, who occasionally struggled with the requisite manual dexterity, Amalia had no such issues. And this, we believe, makes Amalia’s case particularly interesting, for she demonstrated a high degree of adroitness in effecting green yet forsook this adroitness in favor of a grid-based alternating right-hand–left-hand ratcheting-up strategy, which we call *a*-per-*b*: or, in the case of the 1:2 ratio, “For every 1 that I go up on the left, I go up 2 on the right”.

Amalia, similar to all other participants, began by finding green and then attempting a fixed-interval action, resulting in a red screen. Immediately, Amalia tuned into the interactive phenomenon—she developed the skill of performing rapid local adjustments so as to maintain a green screen while moving both hands. The following conversation then ensued.

DT: You’re doing really well here. Do you have some sort of rule you’re following?
Amalia: [continuing to move her hands up and down, creating a more-or-less continuously green screen with occasional flashes of red] Um, I’m trying to keep them at a different distance
DT: How so?
Amalia: Well, I’m keeping this one [right] higher

Shortly, DT asked Amalia to say more on this “different distance” strategy.

Amalia: Well, I’m keeping [left-hand held level at a fixed height, she moves right-hand up and down]… I’m seeing if… which one… [moves both left and right up and down] how high I should keep them apart.…
Amalia: You have to keep this one [right] higher, and then you just have to try to see where it wants to be

Clearly, Amalia was operating on the basis of local adjustments. Next, we introduced the grid.

DT: So, what does this look like?
Amalia: A grid

Unprompted, Amalia lifts the controllers from the desk, raises the left-hand cursor to the first gridline and the right-hand cursor to the second gridline, resulting in a green screen. After a brief pause, she moves both objects upward, retaining only some of the smoothness that had characterized her earlier performance.

DT: Can the grid help us in some way?
Amalia: Yeah, because you can measure how high… how much farther one of them should be

Immediately, Amalia snapped to grid. She utilized the measuring affordance of the grid, yet in so doing became grid-bound: her hitherto continuous and simultaneous actions became saccadic and sequential.

Ten minutes later, we removed the grid and asked Amalia if she could “keep it green now”. To our surprise, she did not revert to the pre-grid continuous actions but instead perseverated with the gridded, ratcheting motor actions even though the grid lines themselves were no longer visible. Afterwards, she explained her strategy as follows.

Amalia: I’m remembering where the… things were. I’m trying to do it in my head, remembering where they were

This we found interesting because Amalia had previously done remarkably well by “trying to see where it wants to be” in a sequence of micro-adjustments. Yet the abrupt and radical shift in her behavior instigated by the grid artifact was so substantial as to effectively replace her earlier scheme. In terms of instrumental genesis theory of activity situations (Vérillon and Rabardel 1995), by instrumentalizing the grid as a means of accomplishing the task objective, Amalia instrumented herself with a new utilization schema. Effecting green *with* the grid bore residual effect on her effecting green *without* it (Salomon et al. 1991).

In summary of findings here presented, our first case study learner, Siena, smoothly adjusted her motor action to accommodate the embedded interaction rules of a new phenomenon long before she was able to explain her scheme as a global action plan, so much so that attempting this explicit articulation boomeranged by supplanting her implicit action procedure. Our second case study learner, Amalia, abruptly changed her action scheme once she had recognized the potential enactive utility of an artifact (grid). As it turned out, the resulting motor action was so robust that Amalia sustained and applied it even in the absence of the external resources that had initially prompted the new action pattern.

We are struck by the situatedness of our young participants' proto-mathematical utterances, as they express regularities they discern in the interactive phenomena they learn to control. These are not conceptual metaphors projected from the concrete source domain to the abstract target domain. To the learners' phenomenology, the would-be abstract domain does not pre-exist but is brought forth via dialogic reflection on the perceptual guidance of action.

5 Conclusions

If we are to take seriously the thesis that mathematical reasoning is embodied and emerging from goal-oriented situated interactions within a particular ecology, then research on motor-skill development may offer useful perspectives for research on mathematics learning. In particular, when instructors foster new schemas by creating structured interaction opportunities centered on manipulating pedagogical artifacts, motor-learning research can help us understand the relation between task, context, and action as well as the effect of semiotic systems on bringing forth mathematical concepts by perturbing and signifying learners' budding operatory schemes.

The embodiment approach to mathematics pedagogy bears implications for the future of Enactivist scholarship. When scholars of Enactivism approach the problem of conceptual learning through the lens of perception–action irreducibility, they acknowledge that individuals' cognitive development is circumscribed by opportunities to engage in, and reflect on, motor problem solving. As educational designers, we take the Enactivist thesis as a cue to foster structured opportunities for students to engage in the solution of motor problem solving oriented toward conceptual development. Yet as educational researchers we consistently critique existing theoretical models, including our own, that allegedly explain how motor problem solving begets conceptual understanding. In this paper, we have sought outside mainstream learning sciences literature for accounts of ontogenetic development alternative to the ever-popular theory of conceptual metaphor.

That said, we agree that the ontological status of static material displays such as mathematical diagrams should be interpreted in light of the spatial–temporal dynamics of the cognizer's sensorimotor phenomenology (Sinclair et al. 2013). And yet one need not warrant these ontological observations with psycholinguistic-developmental models of cognition, especially given cognitive developmental findings of innate cerebral inclination to coordinate spatial, temporal, and quantitative perceptions (de Hevia et al. 2014). There is much methodological appeal in models of conceptual metaphors and blends (Fauconnier and Turner 2002; Hutchins 2014), however, these models of cognition maintain vestiges of representationalist epistemology. Meanwhile, alternative accounts of human behavior are being put forth that do away entirely with the assumption of mental content, including philosophers of cognitive science offering radical theory of embodiment (Chemero 2009; Hutto 2013) and cognitive developmental empiricists offering an action-oriented emergent constructivism (Allen and Bickhard 2013). In fact, we view our research program as contextualizing and examining the implications of this paradigm shift for the theory and practice of mathematics education. In turn, the unique empirical contexts of our design-based research may offer the field more nuanced and more comprehensive accounts that draw on the wide spectrum of the learning sciences.

In particular, this paper draws attention to the changes learners incorporate into their perceptuomotor schemes when they engage in problem-solving tasks that demand manual coordination. Inspired by the theoretical construct of a "field of promoted action" (Reed and Bril 1996), we interpreted these adjustments as resulting from a culture's intervention in individuals' development. These coordinative adjustment automatisms may lead to conceptual development, and we have demonstrated the implementation of a pedagogical ideology and design framework for nurturing new schemes into disciplinary knowledge. As designers of mathematics learning environments, our practice has been to create instrumented fields of promoted action that elicit, challenge, and destabilize learners' habitual coordination, stabilize these into new coordinated action structures, and steer learners to signify these structures in mathematics register.

Our case studies, we believe, support a view that the analysis of motor problem solving can shed light on learners' emergent conceptual understandings. Akin to scholars of motor learning, we view learners' idiosyncratic actions as more than background noise that may at best provide a helping hand in learning—"the uniqueness of knowledge, its historicity and context, is not a 'noise' concealing an abstract configuration in its true essence" (Varela 1999, p. 7). Indeed, our research program supports a view of learning as emerging in the complex, dynamical synergy of, on the one hand, individual goal-oriented actions and, on the other hand, acculturating agents who construct and administer fields of promoted action.

We find of particular value the role of *conscious interpretation* of transitions to novel motor solutions, both smooth and abrupt. That is, for the learners in our study, discovery came literally as that—an emergent discovery—as they discovered *in their own actions* something they could not have predicted beforehand. These eureka moments resulted from bringing conscious awareness to the felt difference of their action or potential for action. As such, our findings on the emergence of mathematical reasoning through embodied interaction seem to support previous findings in therapeutic and academic studies of motor learning (Ginsburg 2010).

Within education, motor learning and conceptual learning have long been seen as separate research programs, complete with their own theories and methodologies. Yet recent advances suggest that these disciplines share some—perhaps much—common ground. One need only consider that motor learning is conceived of and studied as *solutions to motor problems*. After all, actions are not mere movement but involve goals and meaningful coping with circumstances. Motor learning, in our view, involves a degree of sense-making in the world, which is close to the heart of conceptual learning.

It might appear to some readers that we have ventured toward relatively esoteric intellectual fields in this article on mathematics teaching and learning. We therefore wish to conclude with von Glasersfeld (1983), who writes on the dynamical foundations of mathematical ontogenesis and emphasizes the pedagogical importance of consciousness.

> [T]he primary goal of mathematics instruction has to be the students' conscious understanding of what he or she is doing and why it is being done. This understanding is not unlike the self-awareness the athlete must acquire in order consciously to make an improvement in his physical routine. …[W]hat the mathematics teacher is striving to instill into the student is ultimately the awareness of a dynamic program and its execution—and that awareness is in principle similar to what the athlete is able to glean… from his or her performance. (pp. 51–52)

In the same paper, Glasersfeld offers that, while research in mathematics education has under-delivered,

> this disappointment—I want to emphasize this—is not restricted to mathematics education but has come to involve teaching and the didactic methods in virtually all disciplines. To my knowledge, there is only one exception that forms a remarkable contrast: the teaching of physical and, especially, athletic skills. There is no cause for disappointment in that area. (p. 42)

von Glasersfeld is implying that we ought to look to those domains that have been the exception. We agree. In our view, the disciplines of motor and conceptual learning stand to draw increasingly closer. For all involved, learning is moving in new ways.

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