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From Gesture to Design: Building Cognitively Ergonomic Learning Tools

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Abstract

Examining data from a design-based research study on mathematical cognition, I analyze gestures performed by mathematically oriented undergraduate/graduate students engaged in a probability-related activity. I reveal three aspects of embodied reasoning in situated problem solving. Participants spontaneously: (a) overlaid upon physical objects manipulated images of these objects; (b) appropriated mathematical representations that had been specifically designed to accommodate their images; and (c) indicated when available media constrained their expressivity. I explain and demonstrate how new learning tools, implemented in subsequent studies, responded to students' elicited needs by introducing interactive mathematical representations that materialize students' spontaneous imagistic constructions. These special tools offload aspects of the expressive acts so as to free cognitive resources, thus enabling students to hone and elaborate their mathematical inquiry in accord with the design's objectives and rationale. I conclude that mathematical reasoning is the enacted embodied negotiation and coordination of multiple multimodal resources, including material and symbolic artifacts. In the context of a situated problem-solving task, students manipulate elements of their pre-reflective apprehension, rendering the situation better conducive to the discursive practices of a professional community, that is, as modeled, encodable, and expressive in normative epistemic forms such as diagrammatic and symbolic expressions. Gesture, and especially gesture that has not yet been articulated verbally, is embodied reasoning groping for epistemic form.

From Gesture to Design: Building Cognitively Ergonomic Learning Tools[1]

1. Introduction

1.1 Background and Objectives

Why should designers of mathematics learning tools care about gesture? In this paper, I argue that situated mathematical problem solving transpires as negotiation and coordination among available resources that include material objects and epistemic forms. Gestures reveal this embodied mathematical reasoning in progress, indicating for designers any expressive limitations inherent to the learning tools under development. I support the argument empirically through qualitative analysis of data garnered in design-based research on college students' probabilistic cognition, and I demonstrate design responses to these analyzed data.

I conduct research in mathematics education using *design-based* methodology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, 1992): In order to study student mathematical cognition, I closely examine students' behavior as they engage innovative tools I introduce into the learning environment. My research is geared to advance several objectives: to understand the nature and process of student mathematical reasoning, to develop effective learning tools, and, reflexively, to improve my design framework. Ultimately, I am attempting to develop a principled methodology for implementing constructivist pedagogical philosophy (Freudenthal, 1986; Papert, 1980; von Glasersfeld, 1990) in the form of content-targeted objects, activities, and facilitation guidelines, so as to contribute to the theory and practice of mathematics education (Abrahamson & Wilensky, 2007).

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Design-based research progresses in iterated cycles of design (building tools), implementation (working with students and teachers), data analysis (interpreting students' understanding and difficulty), and back to further design, and so on. This paper focuses on how data analysis informs design, i.e., on how analysis of videotaped interventions with participant students guides the selection, creation, and/or improvement of the learning tools under development. In particular, I discuss how students' gesturing and manipulation of objects indicate the multimodal resources students use in their mathematical problem solving and where the learning tools potentially fall short in enabling students to avail of their resources. For illustrative examples, I present and analyze three data excerpts from a design-based research study of students' probabilistic cognition, where the participants were undergraduates and graduates majoring in mathematically oriented programs. Each data excerpt is interpreted as demonstrating an aspect of students' situated problem solving that is revealed through scrutiny of gesture. In particular, I discuss whether available media enabled the student to spontaneously appropriate mathematical forms and, if not, how I chose to respond to this shortcoming through further design. I conclude with conjectures as to the nature of embodied mathematical reasoning.

1.2 Theoretical Framework

In this section I lay out the intellectual underpinnings of the study. I begin with an explanation of the relevance of gesture studies to design-based research methodology, so as to clarify my embodied-cognition approach to data analysis. Next, I discuss general pedagogical perspectives and specific content considerations informing my design for probability, the subject content matter which constitutes the context of these data.

1.2.1 Gesture indicates embodied mathematical reasoning. I depart from a premise, grounded in phenomenological philosophy and cognitive-science research, that embodied mechanisms—including kinesthesia, visuo-spatial images, audiated sound, proprioceptive motorics, etc.—are constitutive of reasoning, i.e. they are not mere epiphenomena of some would-be proposition-based symbol-processing reasoning (e.g., Barsalou, 1999; Merleau-Ponty, 1992; Rizzolatti & Arbib, 1998; Varela, Thompson, & Rosch, 1991). One window onto such embodied mechanisms is gesture. Gesture—aspects of hand/arm motion that do not physically manipulate utensils or the environment—is associated with dedicated aspects of reasoning that may not be communicated through verbal utterance (e.g., Kendon, 2004; McNeill, 1992; Schegloff, 1984). I focus on gesture in the context of mathematical learning and practice.

Gestures people perform as they reason about mathematical problems are informative of the multimodal resources they are bringing to bear (e.g., Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999). In particular, gesture acts as a unique window onto reasoning associated with artifacts, such as tools, e.g., an abacus, or inscribed representations, e.g., an equation (Alibali, Flevares, & Goldin-Meadow, 1997), whether or not these artifacts are physically present. That is, gesturing, like interaction with physically present mathematical artifacts, transpires within the spatial medium, and so gestures associated with mathematical artifacts may evince spatial features of these interactions, revealing aspects of the artifacts that the person is attending to as significant for a problem at hand. It follows that some aspects of mathematical learning and reasoning are grounded in interactions with physical objects or manipulations of imaged percepts thereof (Presmeg, 2006). Thus, the routinization of mathematical artifacts, whether voluminous or inscribed, renders them as much cognitive constructions as they are physical (Hatano, Miyake, & Binks, 1977; Vérillon & Rabardel, 1995; Vygotsky, 1978/1930).

In the case of artifacts that are new to a problem solver—a typical scenario in design-based research studies—gesture may uniquely reveal conceptual construction in action, such as when a student interacting with the artifacts builds connections among personal resources (Case & Okamoto, 1996). Thus, in studying learners' gesture-based interactions with mathematical artifacts under development, I investigate the nature of multimodal resources these artifacts afford. Do the artifacts anchor conceptual blending (Fauconnier & Turner, 2002; Hutchins, 2005)? What developmental trajectories do these artifacts enable toward the appropriation of normative mathematical forms? What limitations might the artifacts be imposing on students' expressivity? What design-theory principles do the data suggest in terms of developing effective learning tools?

1.2.2 Pedagogical and cognitive premises of the design for probability. Education researchers focused on the teaching and learning of probability agree over the importance of helping students connect theoretical and empirical activities pertinent to the domain. Students should come to perceive mathematical relations between combinatorial analysis—determining the anticipated results of a probability experiment by methodologically examining a given randomness generator such as a pair of dice—and results from actual experimentation with these generators—grapho–numerical representations of distributed outcomes (Jones, Langrall, & Mooney, 2007). Elsewhere, I have detailed the particular pedagogical and cognitive considerations informing the design of the learning tools discussed in this paper (e.g., Abrahamson, 2006b; Abrahamson, Janusz, & Wilensky, 2006).

Underlying my design is the following conjecture relating embodied reasoning to constructivist pedagogy. Embodied mathematical reasoning recruits cognitive–perceptual mechanisms that facilitate the synoptic apprehension of quantitative relations within a situation. Thus certain mathematical constructs, such as proportion, are privileged by cognitive–perceptual enabling mechanisms (Gelman & Williams, 1998; Gigerenzer, 1998). Moreover, some researchers believe that conceptual understanding of mathematics is necessarily embodied (Lakoff & Núñez, 2000). But the environment needs to be suitable for these embodied-reasoning mechanisms to operate. Thus, one role of mathematical artifacts, such as representations and procedures, is to manipulate the world so that it becomes conducive for embodied sense making.

Mathematical artifacts in use today, such as spatial–numerical mathematical representations, evolved over millennia. In this process, humans' embodied strategies played roles in creating, selecting, and perpetuating these objects. These same embodied strategies could enable today's students to re-invent the historical artifacts—an assumption of ontogeny/phylogeny recapitulation underlying both the 'mediation' rationale of socio–constructivist theory and (Vygotsky, 1978/1930) and the implementation of constructivist pedagogical philosophy (von Glasersfeld, 1990; Wilensky, 1997).

This design-oriented natural-selection perspective on mathematical artifacts is related to the notion of *cognitive ergonomics* (Artigue, 2002). The term 'ergonomic' refers usually to substantive artifacts or bio-mechanical processes or even just postures that are suitable for humans' physical well-being, short-term and long-term functioning, and comfort. The 'cognitive' modifier refers to the cognitive system rather than the physical, but the association with physical artifacts and the focus on process-with-tools as the unit of analysis are still germane to the study of situated problem-solving tasks using learning tools (see Norman, 1991, on 'cognitive artifacts'). An argument emerging from the current study is that attention to students' gesturing is important in engaging a comprehensive human-with-instruments approach to the design of cognitively ergonomic learning tools.

2. Methodology

2.1 Objectives

The data examined in this study come from a series of interviews conducted at the Embodied Design Research Laboratory at UC Berkeley as part of a design-based research project examining students' probabilistic cognition (e.g., Abrahamson & Cendak, 2006). The interviews were designed to elicit students' spontaneous multimodal problem-solving behaviors, as expressed in a variety of channels, such as utterance and gesture, and with a range of expressive artifacts, such as paper and pencil or computer-based interactive simulations. Examining video data, we employ *microgenetic analysis* techniques (Schoenfeld, Smith, & Arcavi, 1991) to evaluate whether the historically young tools of our own design enabled the students to fully embody their thoughts or whether the tools require further evolution cycles toward becoming cognitively ergonomic. That a design researcher—an educator—responds to a student's gestures by way of increasing the learning opportunities inherent in the environment might be seen as a variant on the findings of Goldin-Meadow and Singer (2003), that teachers are implicitly responsive to gestured nuances of student expression (albeit, unlike the teachers, we *explicitly* attend to gesture in the *hindsight* of repeated video screenings).

2.2 Participants, Materials, and General Procedure of the Instructional Task

We worked with 25 undergraduate and graduate students, all enrolled in mathematics or mathematics-oriented programs. The students were self-selected and compensated with $20. The interviews were conducted individually and lasted about 70 minutes each. Students worked with learning tools from *ProbLab* (Abrahamson & Wilensky, 2002), an under-development unit initially created under the umbrella of the *Connected Probability* project (Wilensky, 1997).

[Insert Figure 1 about here]

The interview begins with students analyzing an experimental procedure in which four marbles are drawn out randomly from a box containing hundreds of marbles, of which there are equal amounts of green and blue marbles. To draw out these samples, we use a special utensil, the *marble scooper*, consisting of four concavities arranged in a 2-by-2 array (a *4-Block*; see Figure 1a). Thus a scoop with 1 green marble and 3 blue marbles could have four different appearances (permutations), depending on the specific location of the green marble relative to the scooper handle. According to probability theory, the most likely outcome has exactly 2 green marble and 2 blue marbles in any order. Specifically, the distribution of expected outcomes by number of green marbles is 1:4:6:4:1, corresponding to 0 green, 1 green, 2 green, 3, green, and 4 green, respectively. (The ratio of sample size, 4 marbles, to population, hundreds of marbles, renders the 'without-returns' issue negligible.) We ask students, "What will we get when we scoop?", which is an ambiguous question since it does not specify whether we are referring to a single scoop or to the long run.

Next, students are guided to use a set of stock-paper cards, each depicting an empty 4-block (a blank 2-by-2 grid), and green and blue crayons so as to create all the different 4-block patterns one could possibly draw out of the box with the scooper. This general procedure is called in probability theory *combinatorial analysis*, albeit the students may or may not be aware of the mathematical implications of the activity in which they are engaging. Once students have found these 16 unique green/blue configurations (2^4), the students are guided to arrange these cards in columns by number of green cells in the block. The result is the *combinations tower* (see Figure 1b), the sample space of the 4-block stochastic device that is distributed such that it anticipates the shape of things to come for a .5 *p* value (see Figure 1c, Abrahamson, 2006b). This hybrid nature of the combinations tower—that it shares figurative properties with two constructs that students need to coordinate toward deep conceptual understanding—a sample space and an experimental outcome distribution (compare Figures 1b & 1c)—is a hallmark of *bridging tools* (Abrahamson, 2004b; Abrahamson & Wilensky, 2007).

Students who have worked with the combinations tower are then asked to describe outcome distributions they expect to receive in a computer-based probability experiment that simulates the operation of the 4-block stochastic device and accumulates the experimental outcomes according to the number of green in each scoop. We ask, “What will the histogram look like?”; “Now, what if we change the probability of getting green from .5 to a greater number?” As we will see in the data excerpts, students spontaneously elect to use the combinations tower, on the desk in front of them, so as to describe the outcome distribution they anticipate in a computer-based simulation. Thus the theoretical and empirical are juxtaposed.

3. Data and Analysis

The following data excerpts were selected to demonstrate students’ embodied mathematical reasoning across interview tasks and media contexts. More importantly, these excerpts, which are each typical of many students’ behavior, each show a unique aspect of embodied mathematical problem solving that is revealed through scrutinizing students’ gesture. Common to these excerpts is that featured students each attempt and possibly succeed to ground within available media what appear to be embodied images and, in so doing, they appropriate a mathematical or proto-mathematical form. The first excerpt, ‘Reflexive Artifacts,’ shows a student who overlays upon a mathematical object her imaged re-configuration of that same object to support mathematical elaboration. In the second excerpt, ‘Appropriation of a Mathematical Artifact,’ a student’s spontaneous gesture adumbrates structural elements of a physically absent artifact without alluding to it explicitly; this artifact is then introduced physically, enabling the student to elaborate and extend his reasoning. In the third excerpt, 'Obduracy of the World,' a student struggles with available media to express an image. Analysis of the material affordances of these available media framed a set of design specifications for an innovative computer-based simulation that would enable the desired expressivity. I will explain this design process and compare student work without and then with this new interactive artifact.

3.1 Reflexive Artifacts: Gesture Indicates Embodied Substrate of Mathematical Reasoning

3.1.1 Description of the gesture. While referring verbally to equal proportions of green and blue in the marble box, students typically gestured to one side and then to the other, as though the hundreds of marbles were separated by color to the left and right of the box. In so doing, several students performed a left–right gesture away from the box without any clear referent (the gesture was not deictic), some gestured either toward the box itself or to a box they constructed by gesture, and some of the participants indicated—even touched—the middle point of the physical box immediately prior to gesturing to the "blue half" and the "green half." We will focus on RG, a recently graduated statistics major, who was typical of students who gestured "half–half" toward the marble box (see Abrahamson, in press, for a detailed analysis of a corresponding case of a Grade 6 student, in an earlier study).

Three minutes into the interview, the interviewer asks RG what she would expect to "get" when she scoops. RG had scooped twice and both scoops had yielded a '2 green, 2 blue' sample. The marble scooper is on the desk where RG has just placed it, with two blue marbles on the left and two green marbles on the right. RG responds that she expects to receive a '2 green, 2 blue' sample, "because that's what I've gotten this far, and it looks like it's about half-and-half in there." In saying "half and half," RG flapped her hands up and down above the marble box, with the left hand over the left side of the box and the right hand over the right side. The interviewer responds that in fact the ratio is indeed half-and-half. This response changes the situation for RG from one of statistical investigation—attempting to determine the green-to-blue distribution in the marbles "population"—to a probability experiment, where one can apply the Law of Large Numbers and/or compute expected values (see Abrahamson, 2006a, on nuanced relations between statistics and probability). RG then says (see movie ESM-Abrahamson_RG.mov):

> Now that I know that it's 50 percent blues [gestures to the left half of box surface] and 50 percent greens [gestures to the right half of box surface], I would guess that [gaze shifts to the marbles scooper] I would get two blues [fingers touch two blue marbles on left side of marble scooper] and two greens… [fingers touch two green marbles on right side of marble scooper] is the most likely combination of marbles to come out when I do the scooping [right hand gyrates rapidly in mimed scooping motions].

Note the spatial analogy RG has built between the marble box and the scooper—she maps the left and right sides of the marbles respectively onto the left and right sides of the marble scooper. More significantly, note that it is not the case that there are 50 percent blue marbles on the left of the box and 50 percent on the right. RG's gesture-based utterance appears to be counterfactual. Why would RG gesture a statement incompatible with the distal stimuli? The answer may be in the interaction between RG's expressivity and the nature of the media.

3.1.2 The 'half–half' gesture explained. The epigenesis of a specific gesture is in actual physical manipulation (Vygotsky, 1978/1930). Thus, RG's 'half–half' gesture may be grounded in prior physical actions of sorting and partitioning. Ideally, RG would sort the marbles physically to compare the color groups. Yet it is precisely because the marbles are not readily given to physical grouping that RG selects an alternative medium, gesture, for articulating this grouping. That is, gesture kicks in when available media constrain the flow of expressivity.

Yet why did RG wish to sort the marbles at all? In Abrahamson (2004a) I suggested that people engaging in situated problem-solving use *embodied spatial articulation*, a type of dynamic visuo–spatial reasoning, to negotiate between personal and mediated mathematical schemas. That is, people "massage" percepts—whether physically or imagistically—so that they can apprehend situations through available schemas evoked as conducive to expressing pertinent mathematical properties of the situation. RG interprets the task as requiring an *a*:*b* part-to-part template. She engages in embodied spatial articulation—she is assimilating the marbles into a mimetic embodiment of this candidate *epistemic form* (Collins & Ferguson, 1993) as a means to extract the green-to-blue ratio. For example, RG would enact a different embodied utilization schema, 'counting,' if the objective were to determine the total number of marbles in the box.

Roth and Welzel (2001, in abstract) maintain that gestures can "provide the material that 'glues' layers of phenomenally-accessible and abstract concepts." In the case of mathematical reasoning, I submit, these purported "abstract concepts" are, granted, not phenomenally accessible, but they are as phenomenologically present. That is, what is at stake is not the materiality per se of apprehended objects but the active schema—the "how" of seeing objects. Thus, the *a*:*b* ratio template is phenomenologically of equal status as the apprehended topology of the scooper, and either may constitute an embodied form superimposed on the marbles. That is, both physical and epistemic objects constitute available resources for mathematical reasoning.

Finally, the marble box plays a unique role in RG's reasoning. This particular vessel, due to prosaic reasons, is structurally simple, featuring a rectangular surface, which faces the student. The 'half-half' gesture loops from an object and back to it—that is, from the attended categories (color property) of an object (mixed marbles) and back to the object (mixed marbles), now serendipitously recruited to ground, elaborate, and communicate embedded aspects of its own quantitative property (half–half). The box of marbles is thus both an originary resource and a vehicle embedding cognitively ergonomic expression of its own properties—it is a reflexive artifact. Specifically, a *reflexive artifact* embodies as an affordance an epistemic form for indexing its own properties, thus constituting a cognitive bridge from the phenomenal to the mathematical.

A future study is necessary to determine whether students would use similar gestures for other-than-half-half proportions. Also, if an actual number line, running from 0 to 100, were attached to the rim of the marble box, students' spontaneous part–part gesture could index a part-to-whole numerical value, thus bridging from the preverbal to the numerical.

3.2 Appropriation of Mathematical Artifacts: The Emergence of a Number Line

MT is a senior economics major who has taken many courses in probability and statistics. Below, I compare two data excerpts from his interview, before and after he has constructed the combinations tower (see ESM-Abrahamson_MT-withoutCT.mov and ESM-Abrahamson_MT-withCT.mov). This case study demonstrates how spontaneous imagery evoked by one artifact introduced into the learning environment interacts with another artifact introduced later.

3.2.1 Working with the marbles box. The first data excerpt begins 8 minutes into the interview (see Figure 2a). By that point, it has been established that there are equal numbers of green and blue marbles in the box. MT had scooped only once and received a sample with 3 blue balls and 1 green ball. Now, he is building an argument for his expectation that in an experiment with numerous trials we will receive equal numbers of green and blue balls:

> So when the times that you have 3 blue balls [left hand gestures to the left] will be weighted against when the times you have 3 green balls [left hand gestures to the right, lightly touching desk], and that will make it a combinations that… weighted to be 2 green

> balls and two blue balls [hands parallel on desk]…. And... these two combinations in expected value, it's the same….as the scenario that you got 2 green balls and 2 blue balls.... And, for the same reasons… the 4 blues balls will also be weighted against the 4 green balls, and in expected value they should be in the same ratios. So that would.... eventually give you the same expected values, when you calculate it that way.

MT's manner of speech is hesitant and his gestures often precede his verbalizations, suggesting that his argument emerges as he develops it. Also, in his argument MT uses the mathematical term 'expected value,' yet his explanation does not abide with the formal procedure for determining an expected value but rather it is a qualitative argument for why the mean sample should have 2 green balls and 2 blue balls. (To calculate the expected value, MT should add up the four independent .5 probabilities of getting green, one for each concavity, to receive the sum of 2 green balls as the expected value of a single scoop.)

Whereas each of MT's gesture constructions is intact with his verbalized ideas of balance and compensation that are central to his reasoning—3 blue against 3 green, 2 blue for 2 green, and 4 blue for 4 green—these gesture constructions also relate *to each other* spatially in terms of location, including order and distance. Specifically, MT aligns and equispaces on the desk the outcome categories, abiding with key structural properties of a number line or, perhaps, a type of interval scale running to the left and right of his body center. This is not a mathematically normative '0-1-2-3-4' scale, which could index the single dimension "number of green," but rather a palindrome scale, '4-3-2-3-4'—"4 blue; 3 blue; 2 blue and 2 green; 3 green; 4 green." This form indexes two juxtaposed dimensions, "number of blue" and "number of green," and is thus an appropriate form spontaneously selected for the embodied argument of balance.

The gestured edifice emerges dynamically as an attraction of available media (body, desk) and conceptual topology (semantic categories in a proto-bar-chart). Note how MT's initial gesture ("3 blue balls") is in the air, the second ("3 green balls") touches the desk, and thereon he places the categories upon the desk. As in the case of the marble box, so this typical desk, with its flat surface and straight edge, is fortuitously recruited as an auspicious medium for accommodating the progressively elaborated complex construction that includes categories of events and relations among them. Note also how the first gestures are conducted single-handed and sequentially, yet the latter are ambidextrous and simultaneous, as though once the notion of symmetry has been evoked through the left–right placement of the initial categories, this embodied notion is promoted as skeletal to the mathematical argument. Thus, the desk and the body are recruited and coordinated as media of expression distributed over space and time, each with its unique affordances. Finally, note how the palindrome scale is co-centered with the marble box. This spatial co-positioning of artifact and gestured construction may appear epiphenomenal to the normative bio-mechanics of tool use and, thus, barely pertinent to that which we may wish to call mathematical reasoning. However, a cognitive-ergonomics approach to artifact-based mathematical learning and design should tend to this spatial relationship.

MT, similar to RG in the earlier data excerpt, coordinates topological elements of his mathematical argument with structural properties of the artifacts in question. Namely, just as RG apprehended the marble box as a bipartite object resonating with the 2-and-2 structure of the 4-block array, so MT coordinates his emergent outcome categories with the 2-and-2 concavities of the 4-block. In particular, over a span of 8 seconds MT laboriously negotiates the coordination of two embodied structures—the gestured structure of two instantiated outcome categories and the material structure of two physical sides of the 4-block. Specifically, whereas a commitment to the emergent ordering of *conceptual* categories implies that ‘2 blue, 2 green’ is a single embodied element, MT experiences interference from structural properties lifted from the *material* object, the 4-block, so that MT initially considers two separate categories, ‘2 blue’ and ‘2 green.’ The dilemma is resolved, perhaps serendipitously as a result of a bio-mechanical constraint, when MT draws his palms against each other to embody the middle-outcome category, whilst using his thumbs—which perforce become adjacent—to indicate the expected value upon the material scooper. That MT has “only” two hands serves as an enabling constraint, because this limitation on his expressivity impelled a compact amalgam. That is, the material and conceptual resources are reconciled and bound in a new co-located and co-expressive blend.

Note that MT’s hands-as-columns are of equal height, as in a histogram representing a flat distribution. Indeed, MT’s focus has been on the symmetry of compensation and not on relative frequencies among all categories. In the following complementary excerpt, when MT works with the combinations tower, we will see how the flat distribution takes on *y*-axis verticality. This verticality will be expressive both of the cardinality of the five category sets qua sample space and, alternatively, implicative of expected frequencies in projected experiments.

3.2.2 Working with the combinations tower. Ten minutes later, MT has constructed the combinations tower and is interpreting it (see Figure 2b). He has already said that the relative height of the middle column—the six 2-green permutations—connotes an expected plurality of outcomes of that category in experiments with the marble scooper. Now, MT reiterates his earlier argument, this time attending to the verticality of the distribution:

> And also I… As I say before [left hand on the 4-blue card, right hand on the 4-green card], the 2 columns, in having.... 3 greens and 3 blues [left and right hands on these columns, respectively], will be weighted against each other in expected value terms.

MT is conscious of this utterance being a reiteration, and he uses the same symmetry/balance metaphor of categories being weighted against each other in a hypothetical experiment. However, MT’s gestures indicate that he is attuned to the artifact before him, the set of cards he has constructed and assembled: he fits his forearms onto the 3-blue and 3-green columns, thus responding to the verticality of the distributed sample space, which supplements a contextually meaningful dimension to the flat and ordered set of categories he had earlier initiated.

The combinations-tower column thus acts as “glove” into which the student’s forearms fit smoothly—it is a template that accommodates the student’s previous image ergonomically. Despite the added verticality, MT tacitly recognizes the combinations tower as a medium to ground his earlier reasoning, possibly because the reasoning had hinged on the implications of symmetry, a salient figurative feature of the combinations tower. The combinations tower is thus a substantive artifact that materializes the student’s ideas—it anchors and carries key imagistic aspects of MT’s argument, so that the he can offload this information. That is, MT needn’t commit cognitive resources (working memory) or embodied representational resources (forearms) just to make the category columns intersubjectively present. Yet the combinations tower enables MT to elaborate his thoughts (see Figure 2c). Specifically, the particular malleability of this discrete set allows MT to condense the columns horizontally so as to demonstrate an expected diminution in the variance of an experimental outcome distribution as the number of samples increases. MT ignores the anticipated vertical growth of the graph and focuses on the overall shape of the combinations-tower-as-histogram. Thus, the combinations tower, essentially a sample space, enables the student to coordinate aspects of theoretical and empirical probability, in line with the design’s overall rationale.

3.2.3 Interim comparison of data excerpts. The first data excerpt featured RG who, through gesture, partitioned the hundreds of marbles into groups of green and blue, anchoring this bipartite image back upon the box of marbles. The second excerpt featured MT who anchored an embodied experimental distribution upon the combinations tower, a distributed sample space. Thus, whereas RG reflexively appropriated the stochastic device itself (the marbles box) as a representational means of communicating about its stochastic properties, MT used another available medium (the combinations tower) to express stochastic properties of the device. We now discuss the case of MK who, similar to MT, modified the layout of the combinations tower so as to express aspects of the stochastic experiment, only that now the experiment is embodied not in the original marble box but in its computer-based simulation.

For the interviews that are the focus of the current paper, we had prepared computer models for running simulated probability experiments under a range of p values. Thus, students could reflect on relations between the sample space, which does not vary unless p is either 0 or 1, and the experimental outcome distribution, which is sensitive to the p value. What we did not have—indeed, what we did not realize that students might need—was a dedicated tool that would enable students to examine the impact of the p value on the *expected* binomial distribution, even before any actual experiment is enacted; that is, a simulation for theoretical probability. In this section we discuss data that led to the development of Histo-Blocks, a theoretical-probability interactive simulation that supports students' elicited reasoning and amplifies it electronically.

3.3 Obduracy of the World: Creating Learning Tools to Increase Student Expressivity

MK, a senior statistics major, has been working with a computer-based simulation of the 4-block marble-scooper experiment, with the p value set at .5 (reflecting the half–half green-to-blue ratio in the marble box; see ESM-Abrahamson_MK-media.mov). An on-screen histogram, which tracks the accumulation of outcomes, has progressively converged on a 1-4-6-4-1 distribution. The interviewer asks what the distribution might be for higher *p* values. MK replies:

> It would be... If it was more likely to be green, it would be skewed (see Figure 3a). This [left-side histogram columns] would get sma[ller]... This would get bigger [right-side]....

Recognizing that she cannot manipulate the on-screen histogram directly, MK remote-manipulates it. She frames the histogram—her gaze peering through her hands toward the screen yet not focusing on the transparent plain subtended by her hands—and tilts the histogram to the right. She then turns away from the screen, and her right hand hovers momentarily over a pen (see Figure 3b), as though she means to draw the expected histogram upon the available sheet of paper. But she abandons that medium and turns to the combinations tower physically, saying: "…but it would shift, like..." (see Figure 3c). Moving her hands in opposite vertical directions—left hand down, right hand up—she has made the 1-column shorter and the 3-column taller. Now she is working with the single card on the right of the tower. She wishes to show that this 4-green column, too, would become taller. She lays her right hand on this single card and raises the card to the desired height (see Figure 3d). Yet, once raised, the card is no longer aligned with the bottom of the tower, and so MK returns the card down to its original location (Figure 3e), stating that the card medium ultimately limits her in expressing her embodied reasoning, "You can't really do it on these cool things [cards], but it would be more like that."

These observations of student behavior directly informed subsequent design. Specifically, the Histo-Block model (see next section) was created as a response to conclusions from the analysis of such data—the model enables students to express expected outcome distributions as transformations on the sample space, transformations that were not possible with the then extant media. I now explain the Histo-Blocks model in detail and then describe a data excerpt from its pilot implementation.

3.4 Histo-Blocks: A Data-Responsive Model Built to Facilitate Student Expressivity

Histo-Blocks (see Figure 4) was designed as an attempt to structure into the design the affordance of "electronic gestures" that resonate with the types of attempted manipulation observed in the interview data. Thus, the model is designed to enhance the expressivity, vividness, and precision of the observed manipulations (an online interactive applet of the Histo-Blocks model is available at http://edrl.berkeley.edu/Histo-Blocks.html).

Histo-Blocks is a computer-based interactive visualization designed to foster deep understanding of the binomial function $P(X = k) = n$-choose-k * $p^k(1\text{-}p)^{n\text{-}k}$. For example, if you flip a coin 4 times ($n = 4$), the chance of getting exactly 3 Heads ($k = 3$), when the chance of getting each Head is .6 ($p = .6$; an "unfair" coin), is a product of 4-choose-3 (= 4) and of the compound probability of the four coins, .6 * .6 * .6 * .4, i.e., 4 * .6 * .6 * .6 * .4 = ~.346. Analogously, the binomial function can express the probability of sampling a particular 4-block combination, for instance any 4-block with exactly three green squares, when the chance of getting a green square is .6 (i.e., when 60% of the marbles are green). In order to help students understand the formula, the two factors of the binomial have been distributed over unique yet interlinked visual elements in the model, as follows.

[Insert Figure 4 about here]

Figure 4 shows the interface of the Histo-Blocks model. For all p values, the sample space does not change—it consists of 16 unique configurations of the 4-block stochastic device (compare the combinations tower, below, with the outcomes displayed, and above, in the special histogram). The number of blocks within each column represents how many unique outcomes there are in that event class, and the between-column relative sizes of these blocks represents the comparative likelihood of getting those specific outcomes (the absolute likelihood of each block is the ratio of its space to the entire space of the histogram). Note in Figure 4 that the chance of getting a *particular* 4-block with exactly 3 green squares is greater than getting one with exactly 2 green squares (the rectangular slabs are each taller in the 3-green column as compared to the 2-green column), yet the chance of getting *any* of one type is equal to getting any of the other type (the columns are equally tall).

When the ‘Go’ button is pressed, manipulating the “*p*” slider (see under the combinations tower) directly redistributes the total area of the expected outcome distribution in the histogram (above), just as MK attempted to do with the combinations-tower cards. Yet, unlike the cards in the combinations tower, the blocks in each column in the Histo-Blocks histogram are given to simultaneous and precise stretching and shrinking, uniform within columns, different between columns. Note how the 3-green column in the histogram is taller relative to the combinations tower and the 4-green column is stretched up without losing its base as it did for MK. It is thus that the histogram maintains the constancy of the 4-block sample space—16 unique objects—yet enables manipulation that results in high-precision visual dynamics and interlinked numerical representations expressing the binomial function (see the monitors on the left).

In the later interviews of our study, we had opportunities to engage several students in pilot activities involving the Histo-Blocks simulation.

3.5 Working With the Histo-Blocks simulation

LB, a graduate student in bio-physics and statistical biology, is working with the Histo-Blocks simulation, comparing it to the combinations tower. She gazes at the computer screen, where *p* is set at .6. LB notes that the 2-green and 3-green columns in the “Expected Outcome

Distribution" histogram are equally tall. LB is manipulates the cursor, mouse down, moving it back and forth between the 2-green and 3-green columns in the on-screen combinations tower. She observes the effect of this motion on the monitor readings. In particular, LB is compares the monitor readings for the 2-green and 3-green columns (see Table 1).

[Insert Table 1 about here]

Toggling between these two sets of monitor readings, LB notes the equivalent products of the binomial function, .346. LB begins to realize that whereas the 6 "becomes" 4 (compare, in Table 1, the two readings for "Number of items in this column"), the .4 "becomes" .6 (compare the two readings for "Probability of each item in this column"). She interprets these two changes as compensatory with regards to the product and, thus, explaining of the equivalent heights of the respective histogram columns. Moreover, LB construes the compensation as though the numeral '6' "moves" from the top monitor to the middle monitor (from 6 to '.6'), while the numeral '4' (in '.4') "moves" up to become '4.' Thus, as LB toggles between the two column readings, the compensation is animated as a switching of places.

In sum, in the course of her inquiry using the Histo-Blocks simulation, LB had opportunities to ground in a dynamic visual metaphor the meaning of the symbolic elements of the binomial function. In particular, note how the interlinked interface features—the combinations tower, the histogram, the monitors, and the slider—enabled LB numerically expressed insight into relations between the sample space of the stochastic device, the *p* value, and the anticipated experimental distribution.

4. Discussion and Concluding Remarks

A foundational socio–cultural conceptualization of pedagogical activity (Vygotsky, 1978/1930) is that teachers, or adults in general, intuitively attune their didactic practice in real time in response to how a student *sees* (Wittgenstein, 1953) mathematical artifacts. Such rapid, ongoing discourse-based attunements (Schegloff, 1996) enable the interlocutors, and primarily the teacher, to maintain the intersubjectivity necessary for co-constructing a complex web of inter-referring aspects of artifacts, to be "on the same page" (Stevens & Hall, 1998).

In the luxury of video-based investigative hindsight, education researchers practice methodically an elaboration of this same intuitive pedagogical attuning, only that this post facto attuning projects not to the particular participant in question but to future participants in a prospective study. Design-based researchers, in particular, examine how artifacts in the learning environment either facilitated or hindered students' reasoning and expressivity and how these artifacts might be modified in light of these observations. It is such attention to the affordances of learning tools that this paper has attempted to foreground. The argument is not that design-based researchers *should* be doing so, because this appears to be common practice (Collins, 1992). Rather, the objective of this paper has been to expose and frame this tacit practice of designers from an embodied-cognition perspective, as a means of offering routes to systematizing the practice as a recommended course of action. To do so, I have presented several data excerpts so as to reveal and scrutinize moments when problem solvers attempted to appropriate available media to materialize, communicate, and elaborate their reasoning. Specifically, I have investigated the nature of the available media vis-à-vis the task and the design rationale and whether the media afforded student reasoning evaluated as conducive to conceptual development along desired trajectories. My examples were selected to demonstrate the utility of methodological attention to gesture as a means of eliciting students' embodied reasoning. Once such reasoning in interpreted, the designer is in a better position to develop deeper understanding of learning processes as well as improved tools supporting these processes.

In terms of insight into student cognition, this study bolsters the prevalent notion that students' mathematical problem solving is not either with or without objects, perceptual or conceptual, situated or symbolic, concrete or abstract. In fact, these pairs of constructs assume an ontology that does not capture the phenomenology of mathematical reasoning. Rather, mathematical reasoning is the enacted coordination of multiple multimodal resources, including material and symbolic artifacts distributed over space and time. Thus, the embodied mind engaged in situated problem solving operates at the nexus of bottom-up apprehension of the situation and top-down problem-solving objectives and vis-à-vis available media of expression. In this embodied auspices, the problem-solver spontaneously recruits, conjures, and mimetically embodies cognitive artifacts bearing representational affordances—a symbolic form, a diagram, a figure of speech—as kinesthetic–imagistic substrate supporting the extraction and coordination of relevant phenomenal aspects of situated objects. In so doing, the problem solver conducts a dynamic, embodied, exploration for reciprocity of substance and form, bringing to bear a host of kinesthetic–imagistic tools for evaluating and comparing, such as re-arranging, weighing, and measuring imagistic objects.

In the context of a situated problem-solving task, gesture is set into play when available media impede expressive constitution and distribution of imaged elements. Gesture manipulates elements of the pre-reflective apprehension, rendering the situation better conducive to the discursive practices of a professional community, that is, as encodable and expressive in normative epistemic forms such as symbolic expressions. Thus, embodied action negotiates the situated–semiotic divide. That said, cultural epistemic forms are assumed to be non-arbitrary and in fact reflective of our species' shared neural and ecological history, so that these normative forms are likely conducive to accommodating spontaneous expression of embodied reasoning.

Gesture plays a central role in situated problem solving: gesture bridges from prereflective absorption to reflective attention, from direct intuitive grasps to processes of conscious reasoning and communication. Gesture, a physical action with spatial–dynamic properties, concretizes personal kinesthetic negotiation for inspection, verbalization, and intersubjectivity—gesture grounds embodied quantitative relations for further elaboration.

Gesture, like physical manipulation, is performed by the hands upon objects of mathematical reasoning, be they imaged or concrete. Are gesture and manipulation, thus, phenomenologically akin? Note that MK initially superimposes both her hands upon her view of the computer image and skews it—using iconic gesture, then later uses both hands, again, upon the combinations tower, which she skews. The like ambidexterity of "hands off" gestured manipulation (computer screen) and "hands on" physical manipulation (combinations tower) that serve the same communicative objective—sharing an image—raises the possibility that gesture and manipulation are phenomenologically more akin than a media-only analysis might suggest (see Kosslyn, 2005, for a possibly analogous relation between vision and mental imagery).

Situated mathematical reasoning (modeling) is an epistemic game of fitting a phenomenon to an epistemic form. That is, gesture supports the assimilation of material substance into utilization schemes. Gesture concretizes the epistemic form, at times superimposing it upon the phenomenon. The epistemic form needs concretization, because it is disadvantaged as a percept vis-à-vis the imposing presence of the substantive phenomenon.

Finally, there may be interesting connections between my own observations of students' embodied reasoning and the robust finding that gesture often indicates students' readiness for conceptual change, when the students cannot yet articulate their new thoughts verbally (Church & Goldin-Meadow, 1986). Namely, it could be that the gesturing student is manipulating an image but has not quite located a resonant and pertinent symbolic form that can accommodate and express it. Gesture, and especially gesture that has not yet been articulated verbally, is thus embodied reasoning groping for epistemic form. Gesturing is an epistemic game at play.

5. Implications for Design and Future Work

An embodied-cognition perspective on situated mathematical problem solving implies that learning tools are more than disposable signifiers of content—the structural properties of these artifacts are to become the very stuff of reasoning—the embodied texture of cogitation. Thus, artifacts should be designed so as to facilitate the negotiation of multiple multimodal resources (see Abrahamson & Wilensky, 2007, on 'bridging tools'; see Case & Okamoto, 1996, on 'bridging contexts' fostering central conceptual structures; see Hutchins, 2005, on 'material anchors for conceptual blends').

However, a designer’s quest to streamline the learning environment such that it afford students’ tacit appropriation of representational forms may compete with a pedagogical principle that students should develop meta-cognitive problem-solving facility (Schoenfeld, 1985). That is, the obduracy of the world may in fact foster cognizance and articulation of a thought process and not just its content, so that students become more generative in their problem-solving capacities. Namely, usability per se, such as through avoidance of ambiguity, need not necessarily be the golden standard in educational design as it is in industrial and instructional design (cf. Norman, 2002). Rather, educational design is to some extent a craft of tradeoffs between usability and pedagogy. For example, by virtue of facilitating students’ electronic manipulation of the combinations-tower columns, I am liable to deny students opportunities to reflect on the media and content. Perhaps it was important that MK manipulated the cards physically—perhaps she learned by ensuring that the middle column becomes shorter so as to compensate for the growth of the columns to its right?

As a designer who believes in the efficacy of learning-through-insight, I submit that it is upon the designer to prioritize learning objectives and then foreground and problematize the core concepts, while backgrounding procedural aspects of secondary importance. I posit that students should appropriate mathematical inscriptions and procedures as problem-solving tools, and that learning is more effective when students can articulate what tools they need. Thus, I would want for future students to struggle with the cards and only then, once they have articulated the limitations of the medium, would I suggest using the computer-based model. Future studies will provide opportunities to examine the utility of this design principle.

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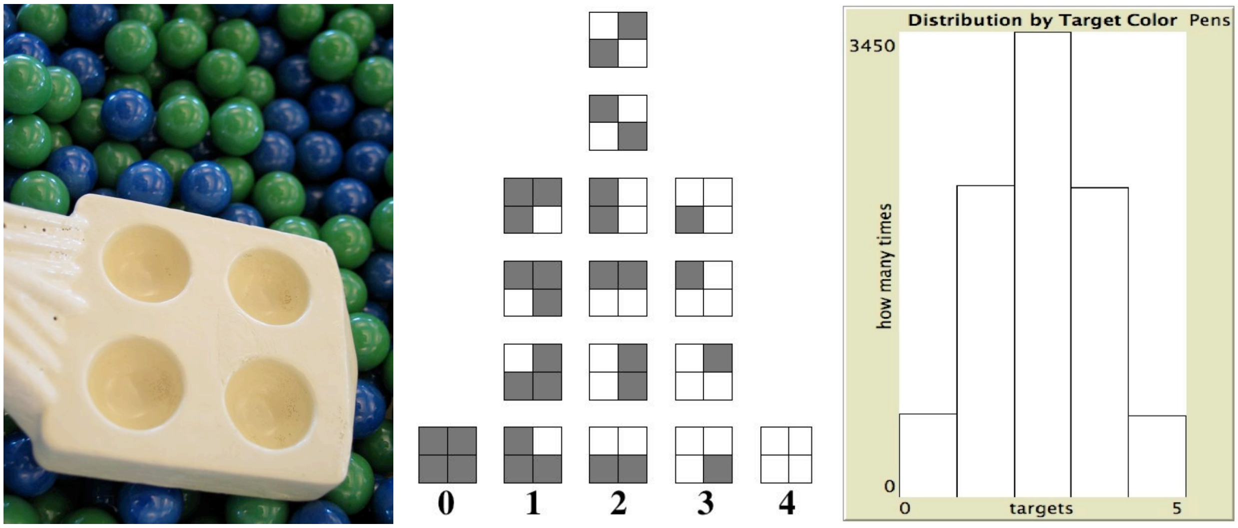
Supplemental material:

Figures 1 – 4

Table 1

Movies

http://edrl.berkeley.edu/ESM-Abrahamson_RG.mov
http://edrl.berkeley.edu/ESM-Abrahamson_MT-withoutCT.mov
http://edrl.berkeley.edu/ESM-Abrahamson_MT-withCT.mov
http://edrl.berkeley.edu/ESM-Abrahamson_MK-media.mov

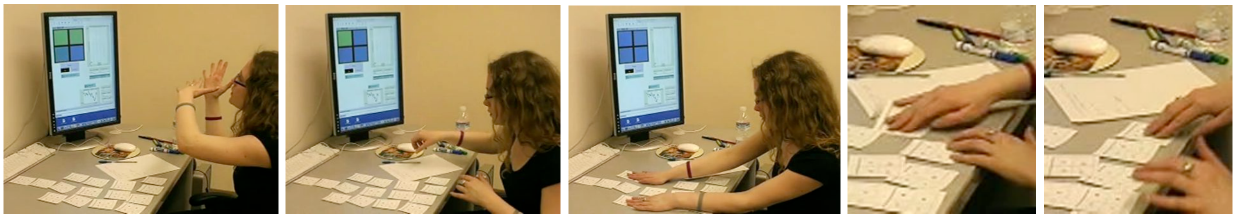


a. *b.* *c.*

Figure 1. ProbLab materials used in the study—theoretical and empirical embodiments of the 4-Block mathematical object: (a) The marble scooper; (b) the combinations tower; and (c) an actual experimental outcome distribution produced by a computer-based simulation of the marble-scooper probability experiment.

a. *b.* *c.*

Figure 2. MT, a senior economics major, explaining why he expects a ‘2 green and 2 blue’ sample as the central tendency of a projected empirical experiment with the marble scooper: (a) using hands as event categories; (b) using columns of the combinations tower; (c) elaborating on expected variance by squeezing cards together.



a. *b* *c.* *d.* *e.*

Figure 3. Negotiating media constraints on image expressivity, MK: (a) manipulates the on-screen histogram “hands off”; (b) considers pen and paper, but declines; (c) manipulates the on-screen histogram “hands on”; (d) shifts a card up to show the expected shape; but (e) returns the card because the shift violated constraints of the representational form.

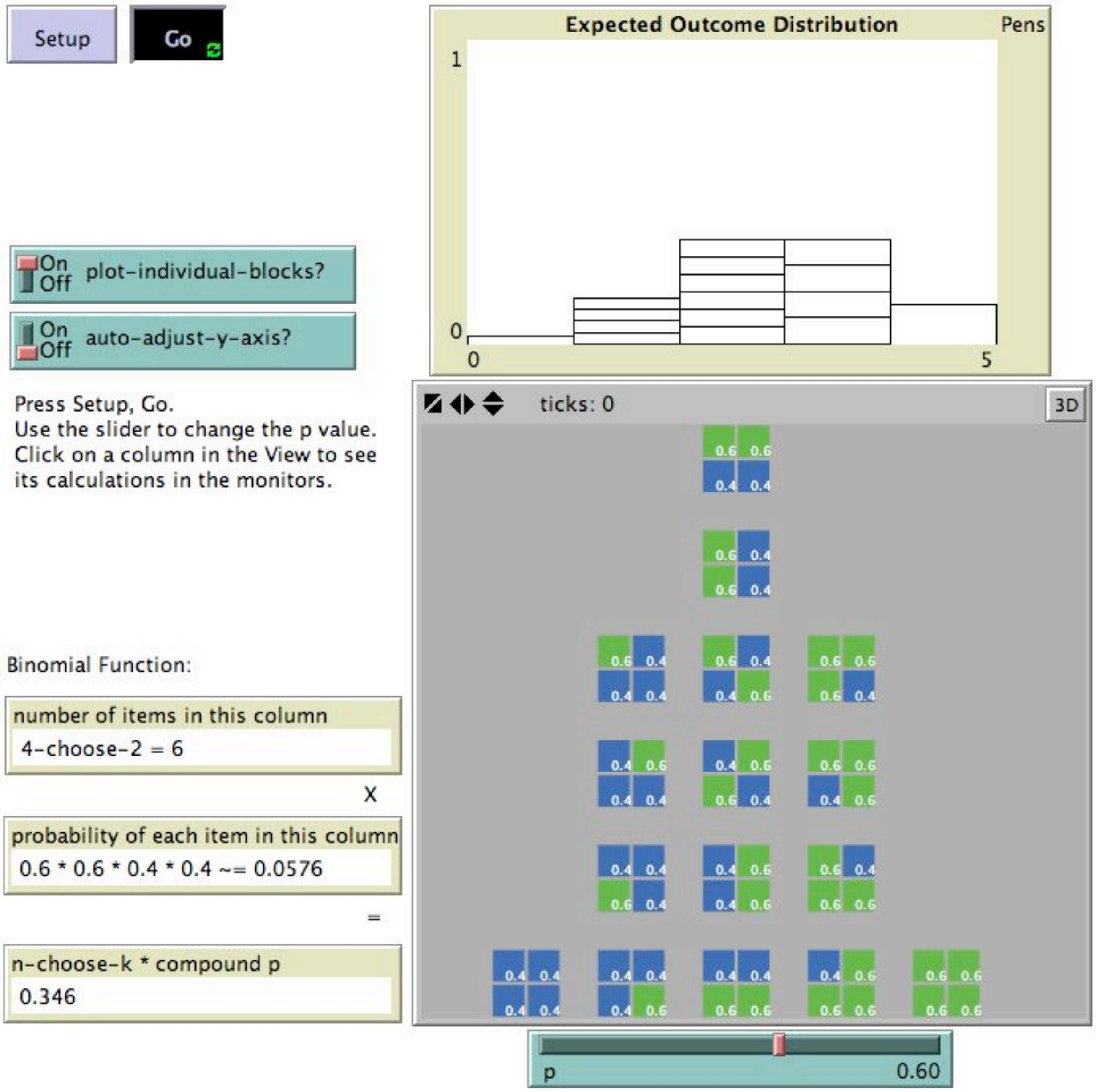


Figure 4. Interface of the ProbLab model Histo-Blocks, built in NetLogo. In this screen-shot, the probabilities, histogram, and monitors all express properties of the middle column of the combinations tower for a *p* value of .6. The corresponding middle column in the histogram, directly above the combinations tower, is equal in height to the column immediately adjacent to the right, because the properties of that column are 4 * .6 * .6 * .6 * .4 = ~.346.

Table 1.

Comparison of Monitor Readings for the Binomial-Function Factors Corresponding With the 2-Green and 3-Green Combinations-Tower Columns in the Histo-Blocks Simulation Under a p *Value of .6*

	Combinations-Tower Column	
	2-Green	3-Green
Monitor		
Number of items in this column	4-choose-2 = **6**	4-choose-3 = **4**
Probability of each item in this column	.6 * .6 * **.4** *.4 = 0.0576	.6 * .6 * **.6** *.4 = 0.0864
n-choose-*k* * compound *p*	0.346	0.346