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MATHEMATICAL REPRESENTATIONS AS CONCEPTUAL COMPOSITES: IMPLICATIONS FOR DESIGN

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Positing that mathematical representations are covert conceptual composites*, i.e., they implicitly enfold coordination of two or more ideas, I propose a design framework for fostering deep conceptual understanding of standard mathematical representations. Working with* bridging tools, *students engage in situated problem-solving activities to recruit and insightfully* recompose *familiar representations into the standard representation. I demonstrate this framework through designs created for studies in three mathematical domains.*

Introduction

Objectives

This is a theoretical paper on design for learning mathematics. I propose a design framework for fostering deep conceptual understanding of standard mathematical representations. Mathematical representations, I posit, are *conceptual composites*, i.e., they enfold a coordination of two or more ideas. For example (see Figure 1, across), part-to-whole diagrams representing the idea of a fraction integrate the multiplicative relation between a part and a whole, e.g., 2-to-3, and the logical relation of inclusion, i.e., the part is integral to the whole (see the Design section for further elaboration of this figure). The composite nature of mathematical representations is often covert—one can use these representations without appreciating which ideas they enfold and how these ideas are coordinated. Consequently, learners who, at best, develop procedural fluency with these representations, may not develop a sense of understanding, because they do not have opportunities to build on the embedded ideas, even if these embedded ideas are familiar and robust. For example, students' difficulty with rational numbers (e.g., Post, Cramer, Behr, Lesh, & Harel, 1993) could be related to an absence of opportunities to understand how fractions are related to familiar multiplicative constructs, such as multiplication and division. Lacking a sense of understanding, in turn, is detrimental in terms of students' mathematical cognition and affect: it hinders the potential generativity, creativity, and satisfaction of engaging in mathematical reasoning (Wilensky, 1997) and compromises students' performance in solving problems.

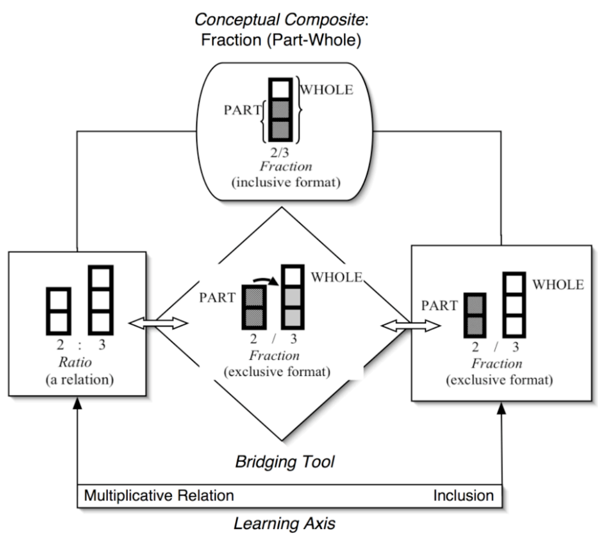

Figure 1. A part-to-whole fraction representation as a conceptual composite.

In the proposed framework, a designer creates a cluster of mathematical representations that *decompose* and "satellite" the target representation, highlighting its covert conceptual components. Students learn through engaging in situated problem-solving activities that recruit representations from the cluster to *recompose* them into the standard representation (so learning, in a sense, is in the reverse order of design). The teacher leads classroom discussion of situated

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problems to illuminate how the satellite representations are embedded in the target representation. Working with *bridging tools* (Abrahamson, 2004)—"ambiguous" representations interpretable as either of the complementary composite components—students *recompose* the components insightfully, as a reconciliation of the tension caused by the ambiguity, into the composite captured in the standard representation. For some concepts, this methodology implies a critique of conventional instructional sequences. For example, if understanding a multiplicative relationship between two quantities is a helpful prerequisite for composing an understanding of fractions, then perhaps ratio and proportion should be taught prior to fractions.

To further demonstrate mathematical representations as conceptual composites, I will overview three designs and then compare them and offer my conclusions. First, I explain the background of this paper and its theoretical framework.

Background

The paper emanates from reflection on a decade of studies my collaborators and I have conducted on students' mathematical learning. These *design-based research* studies (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) were part of three separate projects and included designs for the content domains of fractions, ratio and proportion, and probability, respectively, with K-16 participants. Implementations of these designs have had positive effects on students (Abrahamson, 2000, 2002, 2003; Abrahamson & Cendak, 2006; Abrahamson, Janusz, & Wilensky, 2006; Abrahamson & Wilensky, 2004, 2005, in press; Fuson & Abrahamson, 2005).

The proposed framework is 'emergent' in the sense that it had not been outlined as a principled methodology prior to the creation and study of the designs described herein. Yet, an assumption underlying this paper is that educators are not always fully conscious of their choices in responding to students' needs, and that effective educational practices, which may be intuitive, should be spelled out (e.g., Lampert & Ball, 1998). Thus, the framework described in this paper constitutes a post facto summary of structural similarities I am discerning across these designs. To the extent that these similarities are not coincidental but in fact illuminate important aspects of students' engagement with mathematical objects, the framework could develop into a theoretical model for design—a manual that spells out intuitive aspects of the craft of design and formulates these aspects. Such a manual would constitute a tool for designers to progress from domain analysis and diagnosis of learning problems toward design, implementation, and data analysis.

Note that this paper focuses on mathematical *objects*. Important issues of facilitation, such as principles and suggestions for using these objects within a classroom forum, will not be discussed here. Further more, the paper glosses over auxiliary objects and activities that are not pertinent for the discussion of conceptual composites (but see references to other manuscripts that describe the designs in full and report on implementations of these designs).

Theoretical Framework: Learning as (Re)-Invention

The foundations of the proposed design framework are in the philosophies of constructivism and phenomenology (Heidegger, 1962; Piaget & Inhelder, 1952). Learning is understood as a creative process of discovery grounded in individuals' embodied interactions in the world.

Some mathematicians and scholars of creativity (e.g., Poincaré, 2003; Steiner, 2001) describe creative insight as the act of joining together ideas that had not previously been combined. These innovative composites are captured in mathematical representations that subsequently become tools of the trade and part of the historical heritage that is mediated to students. Traditional mathematics education can arguably be described as students learning to use historical tools for solving suitable classes of problems. Yet, to the extent that deep understanding is contingent on understanding how mathematical tools work (Wilensky, 1993), students need to experience the separate ideas of these composites and only then recombine them, thus reinventing the concepts (Abrahamson, 2004a, 2004b, 2006).

If deep understanding is contingent on personal reinvention, educators are to provide as much support as is necessary for such reinvention to occur, with the objective that students engage in mathematical inquiry and experience conceptual continuity (on 'guided reinvention,' see Freudenthal, 1973; Gravemeijer, 1994). The proposed framework for supporting students' personal reinvention of mathematics builds on the Realistic Mathematics Education (Freudenthal, 1986) and the 'radical constructivism' (von Glasersfeld, 1992) frameworks. In particular, the proposed framework attempts to provide specificity—a design template—for implementing these philosophies of didactics in terms of objects and activities for mathematics learning environments. Towards providing such specificity, the proposed framework interprets normative mathematical representations as conceptual composites. The framework also provides guidelines for creating *bridging tools*, new representations and activities that highlight each of the embedded aspects of a composite and help students construct (recompose) the composite (Abrahamson, 2004b). This principle has been applied for the teaching and learning of fractions (Abrahamson, 2000), ratio and proportion (Abrahamson, 2004b; Fuson & Abrahamson, 2005), and probability (Abrahamson & Wilensky, 2002, 2004). I will now overview these designs, explaining how the framework applies in each.

Designs

This section presents three designs, focusing on the mathematical representations students used in the learning activities. For each design, I state the standard mathematical representation interpreted as a conceptual composite and the bridging tools that we created so as: (a) to help students build on their previous and emergent understandings; and (b) to support students in seeing and coordinating these understandings in the form of the target conceptual composite captured in the standard mathematical representation.

Fractions

Fractions, and rational numbers in general, are famously infamous, constituting the first major learning crisis for many elementary-school students yet also constituting a gateway to advanced mathematical ideas, e.g., algebra and probability (Post, Cramer, Behr, Lesh, & Harel, 1993). Figure 2 (across) shows an *a/b* fraction in canonical part-to-whole (inclusive) form (on top), decomposed into: (a) a ratio pair depicting a multiplicative relation (on the left); and (b) an exclusive fraction, in which the part is exterior to the whole (on the right). The multiplicative

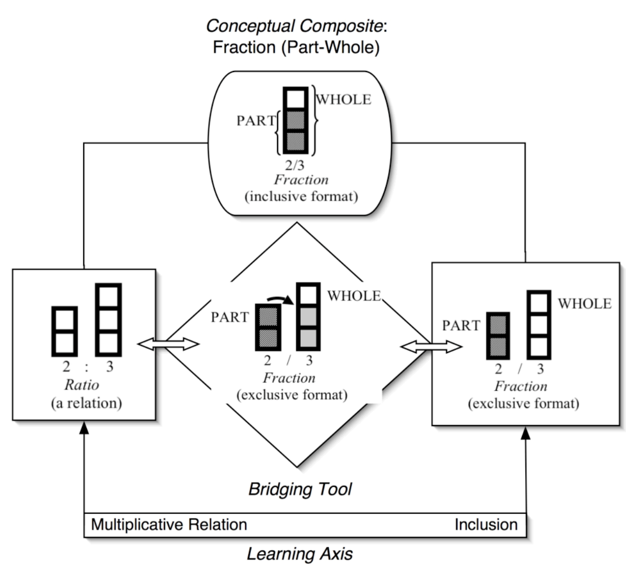


Figure 2. Decomposition of the fraction *a/b* part-to-whole diagrammatic composite.

relation within the ratio pair is communicated through visually comparing proportionately-equivalent pictures of this pair of rectangles (the 'eye trick' optical illusion, Abrahamson, 2002) and measuring their true lengths with an extensible ruler (the *equistretch*), e.g., 2&3 vs. 4&6. These measurements are inserted into a table the students create, thus forming a ratio table depicting proportional progression. Inclusion is communicated by placing the smaller piece on top of the larger one, then removing the smaller one completely, marking only its extent on the larger one. Terms shift then from ‘2-to-3’ (marked ‘2:3’) to ‘2-of-the-3 of this 1’ (marked ‘2/3’). A fraction is, thus, that part out of a whole that relates to the whole as 2:3 (or '2/3 : 1'; see Abrahamson, 2000, 2002).

Ratio and Proportion

The topic of ratio and proportion is traditionally taught briefly, after fractions. Yet, some researchers of mathematics education, e.g., Vergnaud (1983), regard proportion as a potential bedrock of the ‘multiplicative conceptual field,’ implying that this topic should receive more attention in mathematics instruction.

We decomposed proportionality into *repeated-addition* and *factor*factor* action models of additive–multiplicative structures. The repeated-addition model of proportion was represented in a ratio table as two linked repeated-addition sequences, e.g., +3 and +7 going “hand in hand” down two columns as a proportional progression (see Figure 3, across, on the left). The factor*factor model of proportion was represented in a *proportion quartet*, a 2-by-2 structure (see Figure 3, across, on the right; see also Confrey, 1995; Vergnaud, 1983). To enhance the multiplicative affordance of the quartet, the multiplication-table factors are included around the cells of the proportion quartet. The multiplication table served as the bridging tool—the ratio table is two columns of the multiplication table, and the proportion quartet is four cells configuring a rectangle in the multiplication table.

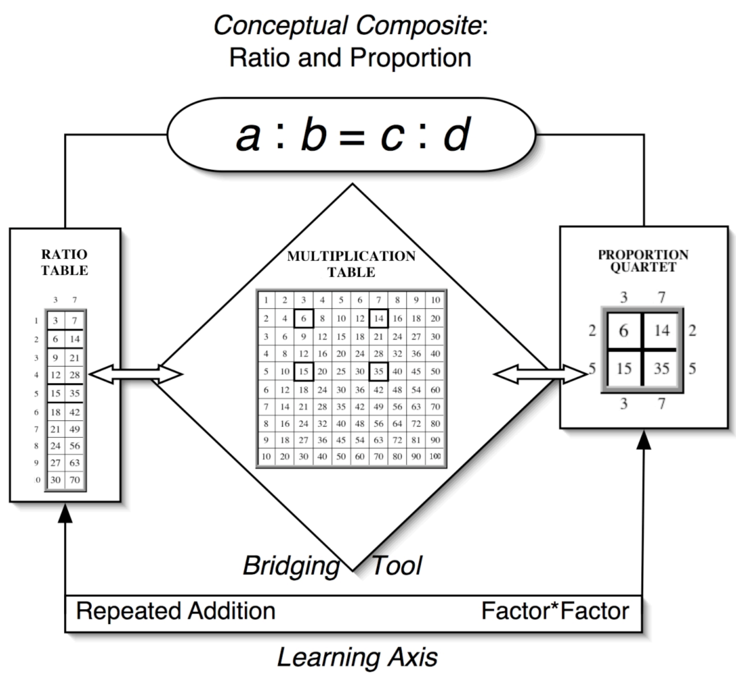


Figure 3. Decomposition of a proportional-equivalence diagrammatic composite.

Students solve unknown-value word problems using each of the three representations and investigate their relationships (see Abrahamson, 2003; Abrahamson & Cigan, 2003; Fuson & Abrahamson, 2005).

Probability

The domain of probability, along with statistics, is often portrayed as essential for informed citizenship, yet students' understanding of this material is wanting. We chose to focus on the classical problems of the binomial.

Figure 4 (across) shows the binomial distribution (top) decomposed into two constructs: (a) a combinatorial space of all possible events ('theoretical,' on the left); and (b) the outcome distribution ('empirical,' on the right). The stochastic device is a *4-block*, a 2-by-2 grid in which each cell is randomly either white or black (there are 16 unique patterns). Both the combinatorial space (on the left) and the outcome distribution (on the right) are organized by the exact number of white cells in the blocks: The 16 unique blocks are organized as 1-4-6-4-1 (on the left), and outcomes converge to those proportions (on the right; illustrating the Law of Large Numbers). The constructs are bridged by the *combinations tower*, an "itemized combinatorial space." This tower features all the unique events, yet it is shaped in the form of the binomial distribution. The probability experiments are run in: (1) the *marble scooper*, a manipulable 4-block device that is dipped into a box full of green and blue marbles to pull out samples; and (2) simulations created in NetLogo (Wilensky, 1999), a computer-based modeling-and-simulation environment. In the NetLogo simulations ('ProbLab,' see Abrahamson & Wilensky, 2002), outcome distributions accumulate samples in the samples' original form. That is, the samples themselves are stacked into bar-chart columns—"stalagmites"—enabling students to monitor specific outcomes *within* columns and not only relations *between* columns. Students construct the tower, run experiments, and discuss connections within and between theoretical and empirical activities (see Abrahamson & Cendak, 2006; Abrahamson & Wilensky, 2002, 2005).

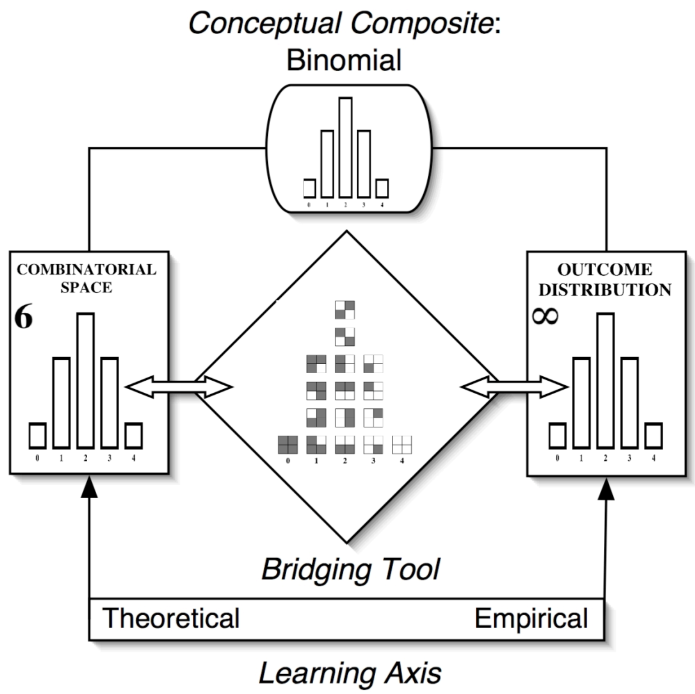
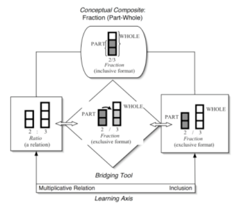
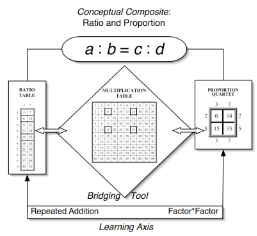
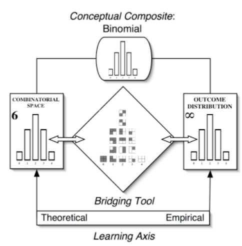


Figure 4. Decomposition of the binomial-distribution diagrammatic composite.

Summary

The paper introduced the design-theory construct 'conceptual composite' that frames domain analyses of mathematical concepts toward designing classroom learning artifacts. Standard mathematical representations are interpreted as implicitly enfolding ideas that students need to coordinate, thus "reinventing" the composite. Three designs were presented. For each, I interpreted students' learning challenge as difficulty in penetrating the canonical representation. Also, I presented clusters of new representations, including 'bridging tools,' which were created to enable students both to build on their mathematical understandings and to coordinate these understandings as a composite grounded in the standard mathematical representation (see Table 1, below, for a summary of these designs).

Table 1.
Decomposition of Mathematical Conceptual Composites and Bridging Tools for Recomposition

Concept	Diagrammatic Embodiment	Decomposition	Design Diagram	Bridging Tools, Media
Fractions, e.g., 2/3	Part-to-Whole Multiplicative Relation	Part-to-Part Ratio & Inclusion		The *Equistretch* (wood board; stretchable rubber ruler; set of laminated pictures; set of plastic strips; erasable markers; hook-and-loop fasteners; set of proportionately equivalent pictures for the *eye-trick* optical illusion, e.g., “wallet size,” “album size”).
Ratio and Proportion, e.g., 6:14 = 15:35	Proportional Equivalence (four values)	Repeated-Addition & Multiplication		*Multiplication Table, Ratio Table, and “Proportion Quartet”* (drawing material for situation stories; classroom-poster multiplication table; personal worksheets; multiplication-table column cutouts; situations describe two linked “multiplication stories,” e.g., “Robin saves $3 a day while Tim saves $7 a day”).
Probability: Binomial Function, e.g., $P_{.5}(2 \mid 4)$	Binomial Distribution	Combinatorial Space & Outcome Distribution		*4-Block* stochastic device in three media: *marble scooper* for sampling 4 marbles from box containing many of two colors; *combinations tower* combinatorial space of stock-paper and crayons; *NetLogo Stalagmite* computer simulation.

One of the teacher's roles is to demonstrate to the students how the mathematical representations within the cluster are related to each other. The teacher uses gestures and actual manipulations to highlight embedded properties and enact reversible transformation. For example, the teacher lifts a 'part' and places it on the 'whole' and then removes the 'part,' or the teacher pulls out of the multiplication table two or three columns that are needed to solve a proportion problem. Students practice these same actions in the context of solving problems.

Conceptual composites can be seen as analogous to a grain of salt—an opaque solid substance that only once re-presented as ‘NaCl,’ reveals its sodium and chlorine elements. That is, decomposing a conceptual composite can be a difficult task—it can be done by a designer or teacher who are well versed in the content, but hardly so by a student who does not know which elements to search for, let alone how these elements are coordinated within the composite.

Contribution

I have shared my research-based insights into the psychology of mathematics education. The nature of this research is interdisciplinary, in that the proposed framework is an attempt to articulate coherently a theory–practice reciprocity in the craft of designing for mathematical teaching and learning. This emergent framework may contribute to the work both of researchers and practitioners: (a) For design-based researchers, this framework may generate both analyses and design in further mathematical domains; and (b) the framework may inform guidelines for professional development and, specifically, it may sensitize teachers to the possible opacity of some taken-as-shared mathematical constructs—historically bridged composites that students receive "unabridged." Teachers informed by this framework may have new facilitation tools for seeing through the smokescreen of procedural fluency and helping students rebuild conceptual knowledge on their own robust understanding.

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