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Using Learning Path Research to Balance Mathematics Education: Teaching/Learning for Understanding and Fluency

Karen C. Fuson, Aki Murata, and Dor Abrahamson

[−] **Abstract and Keywords**

This chapter is an overview of central research-based perspectives that support teaching-learning for understanding and for fluency. We summarize the Class Learning Path Model that integrates two theoretical foci – a Piagetian focus on learning and a Vygotskiian focus on teaching – and specifies phases in learning that reflect Vygotsky's assertion about the move from spontaneous to scientific concepts. Major aspects of the model were drawn from national research-based reports. This model connects understanding and fluency with a focus on mathematically important but also accessible methods in the middle and on maths drawings and other supports for understanding these methods. Such methods can be generated by students and can bridge from less-advanced student methods to formal methods that are unnecessarily complex. For three maths domains in Grades Kindergarten through Grade 6, we illustrate and discuss methods in the middle and drawings (diagrams) that support these methods: problem solving and especially the full range of word problem situations with each quantity the unknown; multidigit addition, subtraction, multiplication, and division; and ratio and proportion. Central features of the Common Core State Standards Mathematical Practices (CCSSO/NGA 2010) in these domains are identified, and how these can support understanding and fluency are briefly discussed. Further aspects of how the pedagogical supports help students move through the Class Learning Path in their own individual ways, and implications for research and for designing maths programmes are then discussed.

Keywords: Maths, education, learning path, problem solving, CCSS, understanding, fluency, proportion, Vygotsky, computation

The past 40 years have seen an explosion of research on numerical cognition and some research on how best to teach various numerical topics. It is now clear that children develop different methods for solving numerical problems and that these move from simple and slow to more advanced and rapid. Early on children worldwide go through a progression of levels of counting, adding, and subtracting. However, as the numerical topics become more advanced, and cultural symbol systems become more central, children's methods increasingly depend on what they are taught in school.

Research on children's thinking and on methods they develop has often been carried out within a Piagetian perspective (e.g. 1965/1941). Research focused more on cultural symbol systems has often been carried out within a Vygotskiian perspective (Vygotsky, 1934/1986, 1978). Fuson (2009) discussed extreme views of teaching that are sometimes drawn from a Piagetian view ('learning without teaching' that values only children's invented methods) and from a Vygotskiian view ('teaching without learning' with a traditional emphasis on fluency, rather than on meaning making). A balanced learning–teaching approach that relates these views was summarized in that paper. In this approach, called learning path teaching, mathematically desirable methods that are accessible to children, and may have been invented by children, are linked to and explained using maths drawings or other visual referents to support meaning making. This approach enables all children to use general methods with understanding and move to fluency.

In this paper, we describe this balanced middle in more detail as the Class Learning Path Model, provide examples

in three maths domains that show different aspects of such balanced learning-teaching, and then relate this view to past and present efforts to reform mathematics education in various countries. This approach allows us to exemplify major results on numerical cognition that affect mathematics education and to identify fruitful new directions for such research.

Teaching/Learning Within a Class Learning Path

The Class Learning Path Model integrates two theoretical foci – a Piagetian focus on learning and a Vygotskiian focus on teaching – and specifies phases in learning that reflect Vygotsky's assertion about children's move from spontaneous to scientific concepts (this model is discussed in more detail in Fuson & Murata 2007; Fuson, Murata, & Abrahamson, 2011; Murata & Fuson, 2006). Major aspects of the model were drawn from principles in two National Research Council reports on research on maths teaching and learning (Donovan & Bransford, 2005; Kilpatrick, Swafford, & Findell, 2001) and from the National Council of Teachers process standards (National Council of Teachers of Mathematics, 2000) in the United States. These reports and initiatives are based on intensive reviews of the research.

This model also draws from two other sources – research by the second author analysing Japanese approaches to teaching (Murata, 2008, 2013) and results of a 10-year research and curriculum development project, the Children's Math Worlds Project, directed by the first author. The project developed teaching materials, implemented them with teachers in a wide range of classrooms, and revised them in several cycles of revision. The materials sought to find and stimulate student methods in the middle that would relate to traditional methods, but be easier to understand and to carry out. This project drew from and contributed to on-going research and research reviews (e.g. Fuson, 1992, 2003). It was published as a Kindergarten through Grade 5 full maths programme *Math Expressions* (Fuson, 2006) and now includes Grade 6 (2012). This programme contains a coherent sequence of research-based visual models and methods in the middle that do connect children's invented methods with traditional methods along a learning path.

Mathematics education and cognitive research have many different terms. The lack of a shared language complicates communication efforts. However, considerable research has been reported as learning paths (trajectories, progressions) through which students move from basic and often informal understandings and methods to more formal, advanced, and fluent methods (e.g. Clements & Sarama, 2009, 2012). In Fuson et al. (2011), we sought to bring together various perspectives on understanding and fluency, provide a model of classroom teaching/learning that included this learning path research, and provide language that would communicate across different kinds of research literature. We found the word 'form' to be a central unifying term. We characterized Piaget's and Vygotsky's conceptual activity as involving three types of external maths forms: *situational (contextual), pedagogical*, and *cultural math forms* (Piaget, 1941/1965; Vygotsky, 1934/1986, 1978). Each learner continually forms and re-forms *individual internal forms* (IIFs) that are interpretations of the external forms. This parallel use of the word *forms* links the external and internal forms, but emphasizes that each individual internal form may vary from the external form because the internal form is an interpretation. Doing maths is using *individual-internal-forms-in-action* to form actions with *external forms.* Within a class learning path, each learner moves in a learning path from using informally-learned spontaneous forms to using explicitly-learned academic cultural maths forms (Vygotsky, 1934/1986). Such movement can be stimulated within the classroom by teaching/learning that is inter- or intra-semiotic mediation (re-forming forms) via instructional conversations within class learning zones. Teaching (by the Teacher and by all of the students at some times) leads learners' attention to aspects of the external forms and supports inter-forming them by:

   Teaching: inscribing, speaking and gesturing about, and inter-forming external forms.
   Learning: re-forming *individual internal forms* in response to teaching.

Such interactive cognizing over time leads to increasingly similar *individual internal forms* that can be taken-as-shared (Cobb & Bauersfeld, 1995). The individual forms become increasingly well-formed (correct and mathematically advanced), and they inter-form into networks of *individual internal forms* for the topic.

Vygotsky's *zone of proximal development* is what an individual can learn with assistance (1934/1986). We use the term *class learning zone* to mean what a given class can learn with the assistance of a teacher and of the pedagogical and situational external forms. Instructional conversations are possible because the external forms

and the means of assistance illuminating and inter-forming the external forms direct and constrain the possible *individual internal forms* that individuals create and use within the classroom forms-in-action. Each student's *individual internal forms* evolve within the class learning path toward a *well-formed network* of individual internal forms that can inter-form with other students' networks, but still have idiosyncratic differences. This movement of individuals within the class learning path can be visualized as paths intertwining and coming closer within a corridor (Confrey, 2005) that overall looks more like a truncated cone as all class members inter-form their *individual internal forms* while cognizing interactively with assistance (Murata, 2013).

Notice that the whole learning path of methods can be elicited in Phase 1 or introduced early in Phase 2 (these methods are labelled as at Level 0, 1, and 2 in Table 1). The classroom instructional conversations support individualized instruction within whole-class activity as the methods of all students appear and are discussed. Diversity can be accepted and used to increase understanding by all, but the Class Learning Path model also assumes and makes possible the realization of high academic expectations by the early introduction and support of visual models and methods in the middle (Murata, 2013).

Table 1 Four Phases in the Class Learning Path to Well-Formed Networks of Individual Internal Forms

Solution Method or Situation Form by Level at Each Phase
Phase 1 Guided Introducing: Eliciting Individual Internal Forms and Forming Initial Forms-in-Action

Level 0	Level 1	Level 2
unformed	basic & slow	**a)** mathematically-desirable & accessible (MD&A)
	typical errors	**b)** mathematically-desirable & not accessible (often current common[a] methods)

Phase 2 Learning Unfolding: Forming Well-Formed Individual Nets-For-Action (Major Meaning-Making Phase)

Level 0	Level 1	Level 2
		MD&A methods dominate errors decrease

Phase 3 Kneading Knowledge: Major Fluency Phase for Fast-Forms-in-Action

Level 0	Level 1	Level 2
		Each student fast-forms one Level 2 (mathematically desirable) method; many students inter-form ≥ 2 methods.

Phase 4 Maintaining Fluency and Relating to Later Topics: Remembering Fluent Methods and Re-Forming Individual Nets-For-Action

Level 0	Level 1	Level 2
		Each student remembers and maintains Phase 3 performance.

([a]) The current common method sometimes is mistermed "the standard algorithm" but should be considered as one variation of the standard algorithmic approach, which uses the major ideas of the method.

The four phases in the Class Learning Path model came from earlier work reported in Murata and Fuson (2006), Fuson and Murata (2007), and Murata (2008) that used research-based principles from NRC reports and the NCTM Process Standards to justify the parts of the model (see Box 1). The model itself was drawn from our on-going work describing common features of two kinds of classrooms that both used this model – those in the *Children's Math Worlds/Math Expressions* classrooms and in Japanese classrooms. Box 1 provides crucial detail that relates to student numerical cognizing, and provides a fuller view of classrooms in action that support and advance student cognizing.

An early version of these four phases was presented at a conference with Japanese maths educators (Lewis & Takahashi, 2006) to frame a requested discussion of the 'maths wars' in the United States. The response of the Japanese educators was that they use the same four phases in their elementary maths curricula. Japanese teachers' manuals that accompany elementary mathematics textbooks generally outline and describe these phases. We use here translations of these terms by Murata (2008) for the first three phases in Table 1 and Box 1 – *guided introducing, learning unfolding, and kneading knowledge*.

Box 1 NRC principles and NCTM Standards Summarize the Class Learning Path Model

Overall: create the year-long nurturing meaning-making maths-talk community

• The teacher orchestrates collaborative instructional conversations focused on the mathematical thinking of classroom members (*How Students Learn Principle 1* and *NCTM Process Standards: Problem Solving, Reasoning & Proof, Communication).*

• Students and the teacher use responsive means of assistance that facilitate learning and teaching by all: engaging and involving, managing, and coaching: modelling, clarifying, instructing/explaining, questioning, and feedback.

For each maths topic: use inquiry learning path teaching–learning

The teacher supports the meaning-making of all classroom members by using and assisting students to use and relate (inter-form) coherent mathematical situations, pedagogical forms, and cultural mathematical forms (*NCTM Process Standards: Connections & Representation*) as the class moves through four class learning zone teaching phases.

Phase 1 guided introducing

Supported by the coherent pedagogical forms, the teacher elicits and the class works with understandings that students bring to a topic (*How Students Learn Principle 1*).

(a) Teacher and students value and discuss student ideas and methods (they inter-form the individual internal forms-in-action using external forms).
(b) Teacher identifies different levels of solution methods used by students and typical errors and ensures that these are seen and discussed by the class.

Phase 2 Learning unfolding (major meaning-making phase)

The Teacher helps students form emergent networks of forms-in-action (*How Students Learn Principle 2*):

(a) Explanations of methods and of mathematical issues continue to use maths drawings and other pedagogical supports (external forms) to stimulate correct relating (inter-forming) of the forms.
(b) Teacher focuses on or introduces mathematically-desirable and accessible method(s).
(c) Erroneous methods are analysed and repaired with explanations.

(d) Advantages and disadvantages of various methods including the current common method are discussed so that central mathematical aspects of the topic become explicit.

Phase 3 kneading knowledge (fluency)

The Teacher helps students gain fluency with desired method(s):

- Students may choose a method.

- Fluency includes being able to explain the method.

- Some reflection and explaining still continue (kneading the individual internal forms)

- Students stop making maths drawings when they do not need them (*Adding It Up: fluency & understanding).*

Phase 4 maintaining fluency and relating to later topics

The teacher assists remembering by giving occasional problems and initiates and orchestrates instructional discussions to assist re-forming individual internal forms to support (form-under) and stimulate new individual internal-nets-for-action for related topics.

Result: together these achieve the overall high-level goal for all

Build resourceful self-regulating problem solvers (*How Students Learn Principle 3)* by continually intertwining the 5 strands of mathematical proficiency:

- Conceptual understanding.

- Procedural fluency.

- Strategic competence.

- Adaptive reasoning.

- Productive disposition (*Adding It Up).*

Aspects of Table 1 and Box 1 obviously relate to many other theoretical and research perspectives. Some of these are discussed in the reports from which aspects were drawn, and some in the longer papers about this model (e.g. Gal'perin's instructional framework is discussed in Murata (2008), and the internalizing and abbreviating movement from 'speech for others' to 'speech to self' is discussed in the Fuson/Murata papers). We will mention here the importance of Case's neo-Piagetian framework as it undergirds the notion of learning paths as building another step onto a method that has become more fluent and of Case's educational work identifying bridging contexts as meaning-making supports (e.g. Oxford, 1991; Case & Okamoto, 1996). Our framework highlights the need for coherence within all maths programmes so that they can build systematically and the need for sufficient work to prepare children to have understandings required for simple Level 1 solution methods for Phase 1 before methods are elicited.

The Importance of Maths Drawings, Mathematically-Desirable and Accessible Methods

MD&A methods, and extended Phase 2 instructional conversations

Phase 1 is emphasized in reform curricula – eliciting and discussing children's invented methods and focusing on understanding. Traditional curricula emphasize Phase 3 to focus on fluency. The new important part of the model is Phase 2 that connects Phase 1 and Phase 3, and provides deep and ambitious learning. As students compare, contrast, and analyse (inter-form) different methods in Phase 2, core maths concepts can be lifted up from the problem contexts or specific methods and connected (inter-formed within their *individual internal forms*). Coherent

pedagogical forms (and especially the mathematically-desirable and accessible methods and maths drawings) enable students to inter-form their external forms to foster the growth of increasingly well-formed (culturally adapted and taken-as-shared) *individual internal forms*.

The word *inquiry* in *inquiry learning-path teaching–(learning)* is used here primarily in the sense of *problematizing* all topics (Freudenthal, 1983; Hiebert, Carpenter, Fennema, Fuson, Murray, Olivier, et al. 1996). Students are to approach all topics with inquiring minds, seeking to understand and to share their own thinking with classmates. Inquiry learning path teaching–learning for a given topic requires coherent mathematical situations, pedagogical forms, and cultural maths forms for a given topic to assist students in attuning their emergent individual networks-in-action through learning paths to well-formed networks of *individual internal forms*.

Phase 2 is the heart of this process, as the class focuses on and discusses mathematically-desirable and accessible (MD&A) methods with the help of some kind of visual supports for meaning-making. A major research result of the 10 years of classroom research underlying *Math Expressions* was the importance of maths drawings (visual models, diagrams) as pedagogical forms to support individual thinking and problem solving and instructional conversations (such Maths Talk is discussed in more detail in Fuson, Atler, Roedel, & Zaccariello, 2009; Hufferd-Ackles, Fuson, & Sherin, 2004). Maths drawings facilitate problem solving because students can relate steps in the maths drawing to steps with maths symbols (cultural maths external forms) and can label the drawing to relate to the problem situation or to maths concepts (e.g. tens). These drawings can help bridge problem situations with mathematical solutions through mathematizing (Murata & Kattubadi, 2012). Maths drawings assist instructional conversations because they can be put on the board for all to see and leave a trace of all steps in the thinking, so each step can be explained. They are inexpensive, easy to manage, can be used on homework, and remain after the problem is solved to support reflection and further explanation. Teachers can collect pages containing them and reflect on these windows into the minds of students outside of the class time. Many East Asian elementary maths programmes also have a history of using diagrams, as do some other countries around the world. Maths drawings initially can show all of the objects and later they can be diagrams with numbers in them. An initial phase of concrete objects may be helpful for very young children or for some special needs children, but for many maths topics this can be very short or non-existent.

We exemplify now coherent sets of visual models and the mathematically-desirable and accessible methods for two crucial domains of the elementary school content – problem-solving and multidigit computation. These are drawn from the extensive classroom-based design research of *Math Expressions*. These examples permit the reader to get some sense of how student cognizing has become the centre of new visions of teaching and learning. The research on these domains is vast and cannot be summarized here, nor can the methods or models be discussed in detail (for more details, see the Clements and Sarama, the Fuson, and the NRC reports referenced above and the Fuson, 2013, professional development webcasts for various topics listed under projects on https://www.sesp.northwestern.edu/profile/?p=61).

Diagrams for representing problem situations

There is a large international research base about representing and solving word problem types as the bases for understanding of operations ($+$, $-$, $\times$, $\div$). Learning paths of difficulty have been identified that depend on the problem types and the particular unknown. Algebraic problems are those where the *situation equation*, such as $\leq$ $+ 4 = 9$, is not the same as the *solution equation*, $4 + \leq = 9$ or $9 - 4 = \leq$. Students can also work in kindergarten with forms of equations with one number on the left (e.g. $5 = 2 + 3$ and $5 = 4 + 1$) as they decompose a given number (here, 5) and record each decomposition by a drawing or equation. Experience with these various forms of equations can eliminate the typical difficulty many students have with equations in algebra, where their limited experience with one form of equation leads them to expect only equations with one number on the right.
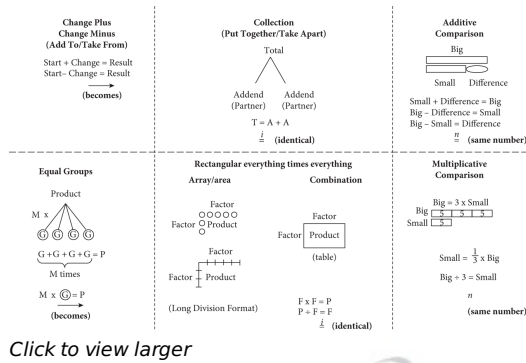
Click to view larger

Figure 1 The related MD&A diagrams for addition (top row) and multiplication (bottom row) situations.

Figure 1 shows the final pedagogical forms used in *Math Expressions* to represent (form) the situations (for more details about these drawings, see Fuson & Abrahamson, 2012). The diagrams support a student with algebraic problem-solving – represent the situation by making a diagram and then use the numerical relationships in that diagram to find the solution. The diagrams are the Phase 2 MD&A methods in the middle for algebraic problem-solving. They have moved beyond students' Level 1 maths drawings that show all of the objects, and they are not yet algebra, which uses only an equation to represent the situation. These diagrams bridge these two levels and give students extensive experience with writing, understanding, solving, and explaining/discussing situation equations like □ – 538 = 286 or 5/7 = 2/7 + □.

Seeing the diagrams together shows their coherence, e.g. equal group situations arise from add to/take from or put together/take apart situations when an addend is a group that is added repeatedly, and additive comparisons likewise become specially restricted as multiplicative comparison situations. Figure 1 shows how different situations actually involve different meanings of the equals sign, indicated at the bottom of each cell. This single set of diagrams can be used for all of the quantities students experience through Grade 6 (from single-digit numbers through fractions and decimals) and for many multi-step problems. They thus also support Phase 4 connections among problem situations as students move through the grades.

Maths drawings and MD&A methods for multidigit computation
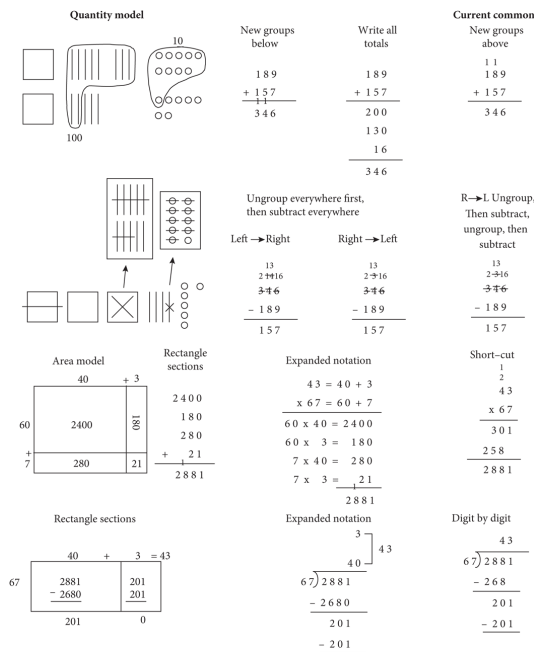

Click to view larger

Figure 2 Mathematically-desirable and accessible algorithms and diagrams.

Understanding multidigit computation requires understanding the nature of the numbers involved. There is considerable research on ways to show the meanings of multidigit numbers and on different methods of computing.

Continual reviewing of such research from around the world, and classroom-based research into supports and methods invented in our *Children's Math Worlds* classrooms, were carried out for many years. Mathematically-desirable methods were then tested with a range of students and teachers in other classrooms to find out how easy they were to understand and carry out. The final *Math Expressions* research-based maths drawings (in the left column) and MD&A methods for multidigit addition, subtraction, multiplication, and division are shown in the middle column of Figure 2. These methods were all invented in classrooms and are described in more detail in Fuson (2003), in National Council of Teachers of Mathematics (2010, 2011), and in the Number and Operations in Base Ten professional development webcasts listed under Projects on https://www.sesp.northwestern.edu/profile/?p=61. Related pedagogical forms that support the meaningful development and use of the quick-hundreds, quick-tens, and ones and of the area models are also discussed in these resources. Students inter-form the drawings with the written methods (link them step-by-step) so that the cultural maths place-value symbols take on meanings as hundreds, tens, and ones. Students use such place-value language in their explanations to support this inter-forming. We found that some students want the extra support within a written method of the expanded notation forms that show the place values separately. Three of the MD&A methods show such expanded notation.

The mathematically-desirable and accessible methods in the middle use the standard algorithmic approach, but write the steps in different places or ways than in the current common form of that approach. The term 'the standard algorithm' actually refers to the major mathematical features of the process and not to the details of how these are written. Thus, all of the methods in Figure 2 can be called 'the standard algorithm' for purposes of goals that require such use. Phase 2 instructional conversations focusing on how the methods are alike and different help students understand the big ideas involved, and that these ideas can be written in different ways.

The versions in the middle are more accessible than are the current common forms in the right column. For example, the addition and subtraction methods in the table all add/subtract like units (place values) and group/ungroup between adjacent place-value units where needed. However, New Groups Below is easier than New Groups Above because:

- The 2-digit totals can be seen more easily.
- The new one ten or one hundred waits below, so you add the two numbers you see and then add the new group if needed.
- You write the totals in the usual order (e.g. 1 ten 6 ones, not as 6 ones then 1 ten).
- and you do not change the problem by writing numbers up within it, instead of down at the bottom.

The subtraction method allows students to keep using one operation (ungrouping) and then change to subtracting, rather than alternating ungrouping and subtracting, which is more difficult. The area model organizes the multiplications, and the expanded notation method has supports to align like place values, see the places in the multiples, and remember which products one has done.

Expanding cultural maths forms to become pedagogical forms
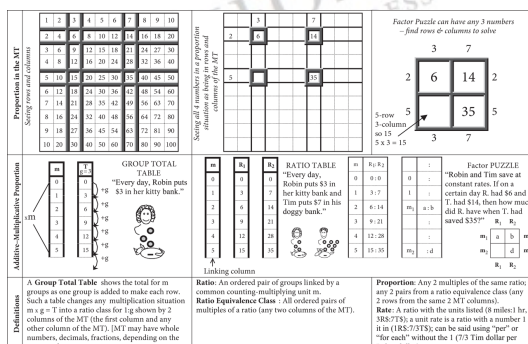
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Figure 3 Using the multiplication table for teaching and learning ratio and proportion.

A final example shows how cultural maths forms can take on meanings by being inter-formed with situations, and how they can be extended or abbreviated by students and by design researchers to become pedagogical forms

inter-formed with both situations and cultural maths forms. Figure 3 presents the major forms and situations used in a unit on ratio and proportion for fifth graders (Abrahamson, 2003; Abrahamson & Cigan, 2003; Fuson & Abrahamson, 2005). Because many proportion errors involve students adding instead of multiplying, we wanted to ground the topic firmly in multiplication. We did this by inter-forming ratio tables and 2 × 2 proportion forms with the multiplication table and with basic ratio incrementing stories, such as how the money in the banks of two siblings increased by $3 (for Robin) and by $7 (for Tim) a day. Students enacted this story by successive additions that they recorded in separate ratio tables and also in a joint ratio table with a connecting row on the left that showed the day (the unit that linked the successive ratios). All of these columns could be seen by students and were also discussed as columns from the multiplication table. A proportion was any two ratios from this story, which could be seen as (inter-formed as) two rows from the multiplication table and also from the linked ratio table. A Factor Puzzle was a proportion with one unknown value; factors of its rows and columns could be identified to find the unknown value.

Most of these forms were very close to the cultural maths forms, but could be considered as pedagogical in that they had something extra to support meaning making. Students also abbreviated forms in problem solving, for example, by skipping rows in a ratio table to fill in the second ratio in a proportion, thereby leading to the discovery that they could just multiply and did not have to write all of the intervening rows in the ratio/multiplication table. Students inter-formed all of the forms through language and gesture in instructional conversations (Fuson & Abrahamson, 2005). Such inter-forming helped students move from their Phase 1 repeated-addition solutions involving filled-in ratio tables to use of Factor Puzzles to solve proportions by multiplication and division inter-formed by finding rows and columns of the multiplication table. The pedagogical forms support MD&A methods for the whole-number problems given at this age level, and they provide a basis for extending to the general unit ratio and cross-multiplication methods needed for problems with fractions (some examples are in Abrahamson, 2004).

We have only shown examples of three major school maths topics. While the three-phase model unit can last over weeks for such major topics, some minor topics may require only one or two lessons for the three phases. Geometry, measurement, and data topics also can use maths drawings and MD&A methods. For example, sketching a rectangle and filling in the measures of all four sides can help students find perimeters. When students can remember to use all four sides, they can drop the measures of two adjacent sides to show the usual way perimeter problems are presented.

Learning Path Teaching–learning Takes Time and Support

As students invent support steps toward making sense of the cultural and pedagogical forms, they recruit *individual internal forms* that initially may be fragile and not involve any inscribed forms, but only verbal and gestural utterance, such as idiosyncratic metaphorical constructions (Abrahamson, Gutiérrez, & Baddorf, 2012). At this stage, attentive teachers should 'listen' very closely (Confrey, 1991; Davis, 1994) and support these fledgling formulations, because they may enable more students to evoke and inter-form similar *individual internal forms* and, thus, bring the whole class to bridge and adapt their respective *individual internal forms* to the external forms.

The extended inter-forming of coherent pedagogical and cultural maths forms, especially with the support of maths drawings within instructional conversations, allows students to build well-formed, but individual networks of *individual internal forms* that allow students to be adaptive. There are always creative student variations discussed in Phase 2 and even in Phase 3. Students create forms close enough to the mathematically-desirable and accessible methods and maths drawings in Figures 1 and 2 to be taken-as-shared by their classmates, but there is often individual creativity that adds interest and depth to the instructional conversation and to the continually inter-forming networks of *individual internal forms* for all class members including the teacher.

Some mathematically-desirable and accessible methods have extra support steps that can be dropped when they are no longer needed. For example, the partial products multiplication method in Figure 2 was invented by a class of low-achieving African–American students who wanted to put in all of the steps any student needed. Later many of them selectively dropped various supportive steps ('learner wheels'), just as students stop making maths drawings when they no longer need them. Pedagogical forms assist meaning-making for the cultural maths forms provided these are actively inter-formed within the student's network of *individual internal forms*. At some point for

each student, the pedagogical form no longer needs to be used because the student's *individual internal forms* stimulate that meaning-in-action for the cultural maths forms (e.g. Abrahamson, 2002).

The *class* learning path simplifies the teacher's task to something that teachers perceive as do-able, especially if supported by a learning path programme. The teacher's task is not the commonly-perceived reform task of celebrating every student's methods and continually eliciting more ways. This can be overwhelming to teachers who think there are as many 'different' methods as the number of students (Murata, 2013). We instead use a big picture of the three levels of solution methods shown in Table 1: Level 1 basic and slow, Level 2a MD&A, and Level 2b mathematically-desirable and not accessible. These levels help teachers see certain methods as minor variations of each other and to place these within the phases of teaching a topic.

Learning path teaching–learning requires coherent external forms

The teacher and (student) teachers 'tune' learners' *individual internal forms* toward the external forms and toward the more-advanced methods with the assistance of the pedagogical forms (similar to diSessa's tuning toward expertise in physics, 1993). Pedagogical forms need to be selected or designed to illuminate the central mathematical aspects of the cultural maths forms by their affordances and constraints (their *attunements*, Greeno, 1998). Such tuning takes time and much inter-forming by gesture and language by the teacher and students. Students' networks of *individual internal forms* have layers from less-advanced to more-advanced *individual-internal-forms-in-action*, and they may fold back (Martin, 2008; Pirie & Kieren, 1994) to a lower level to inter-form and make more meaningful a higher level. Because mathematics builds, the situational and pedagogical forms need to be coherent so that children can move among their layers of understanding easily. Our situational diagrams that work across all kinds of quantities is one such example. The use of the same quantity drawings for multidigit numbers in addition and in subtraction is another example. Methods in the middle that can extend from children's invented methods and relate to difficult formal maths methods require careful analysis and classroom research to reach coherence for teachers and for children. Pedagogical forms such as secret-code cards (layered cards that show 374, but with 300 under 70 under the 4) that compensate for difficult cultural forms such as English words for teens and tens are an important part of such analysis and research (see National Council of Teachers of Mathematics, 2011).

The extensive and excellent Dutch programme of research on Realistic Mathematics Education (e.g. Gravemeijer, 1994; Streefland, 1991) that extends Freudenthal's theory (e.g. Freudenthal, 1983) shares many features with the model proposed here. Our model does suggest a further examination of the coherence of that programme's situational and pedagogical 'models of' within and across topics, and raises the possibility that students might move to Phase 2 general methods in the middle more rapidly.

Using research on student cognition for mathematics education

Initiatives have arisen in many countries to adapt mathematics education to reflect the research on student learning and to prepare students for the demands of lives infused with technology. Many of these initiatives have reflected the long-term perceived conflict between understanding and fluency, with emphases either on student inventing of methods or on traditional teaching of formal methods by teacher telling and showing (this conflict has been termed the 'maths wars' in the USA and some of these over-emphases are discussed in Fuson, 2009). We have seen here that there is a research-based middle ground: students can understand general methods and do not have to be limited to special methods that arise from particular numbers or from particular situations. This does raise questions about what seems to be a strong emphasis on such special methods in the National Numeracy Strategy in English primary schools (Askew & Brown, 2001; Vollaard, Rabinovich, Bowman, & van Stolk, 2008). A related emphasis in some countries on mental computation likewise seems too strong, especially if the methods cannot be written down by students. Such methods are often restricted to smaller numbers and do not generalize easily (e.g. methods of adding on or back for multidigit numbers; for more discussion of limitations of such methods, see National Council of Teachers of Mathematics, 2011; Fuson & Beckmann, 2012/2013). Our experience is that children are empowered by general methods they can understand and explain. Recording methods using place value notation is a core aspect of mathematics and can be done as early as age 7 if approached in the ways outlined in our model. However, it is also important to examine the form of the written methods that are to be taught because there may be variations that are easier for students to understand and/or to carry out.

A recent such initiative in the United States reflects this middle ground and heavily uses the research on cognition. There has been a special difficulty in the United States because different teaching/learning standards are adopted by each of the 50 states. There has been huge variation in the standards across states that led to characterizing the maths goals in the United States as 'a mile wide and an inch deep' (Schmidt, McKnight, & Raizen, 1997). Textbooks were enormous, massive amounts of time and energy were spent on *what* to teach instead of *how* to teach it *well*. The new coherent teaching/learning standards, the Common Core State Standards (CCSSO/NGA, 2010), are based on research and were adopted by most states. The standards reflect research and curricula from around the world, and are the result of an intensive, prolonged feedback and revision period from many sources. Thus, they reflect a negotiated balance of views about how to fit together learning paths in various domains.

For example, the Common Core State Standards operations and algebraic thinking standards lay out an ambitious learning path with word problem types as the bases for understanding of operations (+, −, ×, ÷). The standards identify grade-appropriate levels at which students work with the various problem types and with unknowns for all three of the quantities. The standards appropriately specify that students use drawn models and equations with a symbol for the unknown number to represent the problem (*situation equations*, such as $\square + 6 = 14$). Thus, from grade 1 on students will have crucial experience with the more difficult algebraic problems (those in which the situation equation might vary a solution equation, such as $6 + \square = 14$ or $14 − 6 = \square$ for the situation equation $\square + 6 = 14$).

The Common Core State Standards drew from design-research and learning path research to include within standards the requirement that students are to use visual models, relate these to the problem situation or to the steps in a computation, and explain the reasoning used. For most numerical topics this meaning-making phase is one or two years ahead of the standard that calls for fluency, thus using phases that extend over years. These Phase 2 methods discussed above and shown in Figure 2 meet the more-advanced Common Core State Standards that students are to develop, discuss, and use efficient, accurate, and generalizable methods including the standard algorithm (for more see Fuson & Beckmann, 2012/2013). For more about features of these standards see Fuson (2012).

The Common Core State Standards emphasize the Maths Talk aspect of teaching-learning with eight Mathematical Practices that can be summarized in four pairs of practices:

- *Math Sense-Making (MP 1 and 6):* make sense and use appropriate precision.
- *Math Structure (MP 7 and 8):* see structure and generalize.
- *Math Drawings (MP 4 and 5):* model and use tools.
- *Math Talk (MP 2 and 3):* reason and explain.

This can be summarized as: Do math sense-making about math structure using math drawings to support math talk.

General Conclusion and Call for Future Research

In conclusion we assert that it is the responsibility of a research-based maths programme to provide:

(a) In Phase 1 the situations or pedagogical forms (especially maths drawings) to stimulate students' and Teacher's *individual internal forms* meaningfully toward the externals forms, and to have stimulated and practiced well-formed enough *individual internal forms* in earlier units before reaching Phase 1 for a topic.

(b) In Phase 2, the curriculum must provide to teachers research-based MD&A methods and pedagogical and/or situational forms to assist students to build well-formed *individual internal forms* that can assist them in using meaningful cultural maths forms.

(c) Pedagogical and formal maths external forms must be explained enough in the programme so that teachers can assist students to progress in their learning paths.

Point (c) is important because many teachers have not had sufficient opportunity to learn maths meaningfully themselves or to learn about student learning paths. They need the assistance of the coherent supports within Phase 1 and Phase 2 to form their own *individual internal forms* to teach with meaning. In our experience,

teachers enthusiastically welcome the opportunity to learn meaningful maths and to use a 'teach while learning' approach. One group of pilot teachers articulated such feelings by calling the Children's Math Worlds programme 'maths therapy for teachers.'

Teachers' experiences vary considerably on the whole continuum of experiences from very constructivist to traditional. The Class Learning Path Model enables any Teacher to begin from initial strengths s/he has and to build new teaching–learning competencies as s/he moves along her/his own learning path. All teachers find phases within which they initially feel comfortable. They all gain confidence and knowledge from the learning supports in Phase 2. As they experience the Class Learning Path Model of teaching, they build competencies and understandings (*individual internal forms* about maths and about teaching) that enable them to use a more balanced approach to teaching in the following year.

The cultural maths forms for a topic are fairly well defined for a given culture, and there is relatively little variation in these around the world. What is and can be varied to affect learning are the situations, pedagogical forms including especially maths drawings and MD&A methods, and the sequence of problems and activities. More research and dialogue about the external teaching forms (situations, pedagogical, and cultural maths forms) would be beneficial. This dialogue needs to focus on the mathematical aspects of the learning paths (e.g. which methods are mathematically desirable?), as well as on data about them (e.g. How did the pedagogical forms work? How could they be made more coherent across topics and grades?). In many countries, an initiative to modify difficult forms of standard algorithms and other solution methods to mathematically-desirable forms more adapted to student cognizing would make mathematics education more successful and help students and teachers believe that mathematics is understandable.

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Karen C. Fuson

Karen C. Fuson, Northwestern University

Aki Murata

Aki Murata, Stanford University

Dor Abrahamson

Dor Abrahamson, University of California-Berkeley