IT'S NOT EASY BEING GREEN: EMBODIED ARTIFACTS AND THE GUIDED EMERGENCE OF MATHEMATICAL MEANING

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This on-going design-based research study focuses on Grade 4-6 students' guided task-based interaction with a novel computer-based hand-tracking system built to suggest the limitations of naïve additive schemes and create opportunities to develop core notions of proportionality as elaborations on these schemes, even before engaging numerical semiotic forms. Study participants struggled with canonical issues inherent to rational numbers. They formulated a string of insights leading up to a new type of equivalence class. Reported as a case study of Itamar, a 5th-grade middle-achieving student, our analyses reveal emergence of conceptually critical mathematical meanings in an activity that initially bears little mathematical significance.

Introduction

This paper draws on empirical data elicited in the context of an ongoing design-based research project that aims to foster and investigate Grade 4-6 student learning of the fundamental notions of ratio and proportion (Abrahamson & Howison, 2010). Drawing on grounded cognition theory (Barsalou, 2008), we conjectured that the enduring conceptual difficulty students experience with proportional reasoning (e.g., Lamon, 2007) may arise in part because everyday experience rarely affords opportunities to engage these quantitative notions sensomotorically; thus, students lack appropriate embodied experiences from which to construct the key concepts upon simulated dynamic imagery. Accordingly, we sought to engineer for students an embodied-proportion experience conductive to constructing these targeted concepts.

The particular experience designed for the study involved a non-routine activity—the student used two handheld devices, one in each hand, to remotely manipulate the vertical positions of two virtual objects displayed on a large monitor (see Figure 1). Unbeknownst to the student, the apparatus compared the ratio preset by the experimenter on a hidden monitor, e.g., a 1:2 ratio, to the ratio between the measured heights of each of the two handheld devices above the desk, e.g., 20 cm and 30 cm, respectively—a 2:3 ratio. The display's background color provided feedback to the student on how closely the performed ratio matched the unknown ratio, with red indicating "incorrect," yellow indicating "almost correct," and green indicating "correct." Students were tasked to "make the screen green" but were given no direction as to how to accomplish this task.

Students' guided inquiry into this mystery apparatus was designed to give rise to a succession of insights into the mathematical principles that covertly governed the automated feedback pattern they experienced as they manually explored the problem space. So doing, we expected, students would initially bring to bear naive "additive" reasoning and then cope with cognitive conflicts engendered by such contextually inappropriate reasoning. For example, students might expect the within-pair vertical distance to be constant across all "green" location pairs; therefore, once they found their first green location, they would tend to "lock" the vertical distance between

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their hands and raise this fixed-difference hand pair, only to see the green screen turn red; surprised, the students would then attempt to account for this anomaly by formulating and testing rules and strategies to enable the determination and enactment of further "green" locations. Thus, students were to reinvent core notions of proportionality by attempting to master an unknown computational function embodied in emergent features of an interactive apparatus.

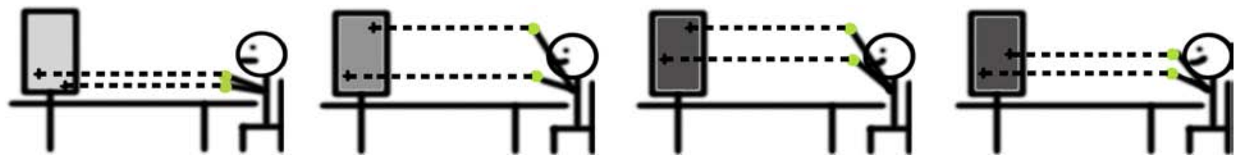

Figure 1. (a) Incorrect performance; (b) almost correct performance; (c) correct performance (1:2 ratio between hand heights); (d) another instance of correct performance (again, a 1:2 ratio)

Our rationale was that the color green, an arbitrary perceptual stimulus, would initially constitute for the student an objective, then serve as the student's source of performance feedback while attempting to achieve that objective, and would ultimately give rise to an equivalence class—a collection of hand-location pairs which the student perceives as "the same" by virtue of their common effect on the screen. By subsequently introducing discursive prompts, measurement overlays, and representation templates, we hoped to guide students toward articulating mathematically principles governing this initially opaque yet gradually emerging class.

The above design-based research study serves in the current paper as an empirical context for examining the following pair of related questions on the nature of mathematical learning: (1) Can students develop mathematical meanings, signs, and concepts through engaging in a task that initially does not cue mathematical treatment as relevant to the solution of a problem, and if so, how?; (2) From the perspective of pedagogy and instructional design, what personal resources, technological artifacts, and discursive mechanisms could possibly give rise to mathematical meanings under circumstances of ostensibly *a*mathematical problems? These questions appear germane to the PME-NA 32 theme, Optimizing Student Understanding in Mathematics.

We sought to explore our research questions by analyzing videographed footage from a set of individual-student task-based clinical interviews, in which a protocol structured their interactions with the mystery device. As we report below, analyses of these sessions suggest that students constructed the targeted mathematical principles through engaging in the guided activity. In the conclusion we will advance a tentative assertion that, given appropriate materials, activities, and facilitation, students can tackle canonically difficult aspects of mathematical concepts while still engaged in *pre*symbolic quantitative reasoning. Specifically, we submit, students learning the subject matter of proportions can become aware of their own inclination to reason in terms of additive-only rather than multiplicative relations and, moreover, begin to juxtapose and reconcile these two basic types of quantitative reasoning even before inscribing mathematical expressions.

Background

The ongoing investigation reported in this paper draws on two theoretical perspectives of current interest in the mathematics-education research literature—embodied cognition and semiotic mediation—and seeks to explore relations between them. Our attention to the role of embodied action was inspired by grounded cognition theory (Barsalou, 2008), a rising cognitive-sciences paradigm that resonates well with genetic epistemology (Piaget, 1968), phenomenology (e.g., Husserl, 2000), and the construct of tacit knowledge (Polanyi, 1967; see also Varela et al., 1991, on enactive cognition). Grounded cognition rejects earlier notions of the mind as an information processing machine operating separately from the body's sensorimotor systems. Rather, perception, kinesthesia, and cognition are viewed as functionally linked, because reasoning consists of simulating fragments of embodied experiences. Quantitative reasoning, too, is thus necessarily grounded in sensorimotor experience (c.f. Lakoff & Núñez, 2000). It follows, we argue, that mathematical notions may differ with respect to the learning challenges they present to students as contingent on the everyday availability of embodied experiences that form the concepts' simulation substrate, so much so that in the absence of appropriate embodied experiences, students may be disadvantaged in developing grounded understandings of certain mathematical content. Specifically, we view the notion of ratio as "conceptually ambidextrous"; that is, the embodied substrate of a ratio, interpreted as isomorphism of measures (Vergnaud, 1983), could simulate the simultaneous enactment of two changes, such as the dynamic co-production of two hand gestures, a slower and a faster one. We conjecture that proportionality is challenging to students *conceptually* (see in Davis, 2003) in part because everyday experiences rarely if ever afford opportunities to perform and practice such actions *physically* as requisite of developing an embodied artifact that could then be leveraged to mathematize proportion.

Accordingly, the Kinemathics activity has been designed with the explicit goal that students construct what we call an *embodied artifact as an object-to-think-with*—an externally induced coordinated physical action that, similar to the embodied cultural legacy of martial arts, dance, or instrumented musical performance, students learn to perform even before, yet as a condition for, developing disciplinary meanings. Indeed, the semiotic potentials (Mariotti, 2009) of physical discursive action performed in the context of math instruction have been broadly demonstrated in gesture studies (e.g., Goldin–Meadow, Wagner Cook, & Mitchell, 2009). Our interest in such *action-before-concept* learning (ABC) is part of our larger inquiry into cultural precedence for pedagogical practice within explicitly embodied domains, wherein procedures are initially learned on trust yet subsequently—only toward perfecting the procedures toward mastery and further dissemination—are interpreted by experts as encoding emergent disciplinary knowledge.

The materials and protocol designed for this study, and in particular the initial absence of explicit mathematical signs in the core activity, appear to create an amenable empirical arena for studying artifacts and teachers' inter-connected roles in the semiotic mediation of mathematical knowledge (cf. Mariotti, 2009; Radford, 2009; Wertsch, 1979), because the gradual introduction of mathematical tools helps researchers disentangle these roles. That is, by initially refraining from enlisting mathematical signs to explain the task yet subsequently overlaying the situation with a set of graphical–numerical elements familiar to the students, we hoped to create data of students negotiating between naïve and scientific meanings for the variety of spatial magnitudes embedded in the situation. Specifically, we expected students to alternate between two types of mathematical reasoning that draw on different epistemological resources: presymbolic quantitative reasoning evoked by the hands-only activity and mathematical schemas evoked by symbolical mediation of this same activity (Abrahamson, 2009; Bamberger & diSessa, 2003).

We maintain that these two approaches—the embodied and the semiotic—can and should be modeled as pedagogically complementary resources for mathematics instruction aiming for deep understanding, precisely because they are co-instrumental in the learning–teaching dialectic (see diSessa, 2008, on dialectic theory; see Cole, 2009, on obuchenie). Researching student behavior around the Kinemathics task, wherein the embodied and semiotic are differentiable, could constitute a step toward further illuminating this dialectic. Accordingly, our study sought not so much to evaluate the effect of this brief intervention on students' understanding of the targeted content as much as to model the emergence of mathematical meaning in this non-routine design.

Methods

The device described above is a type of *Mathematical Image Trainer* (*MIT*, Abrahamson & Howison, 2010), which tracks the vertical positions of each of the student's hands. Infrared rays emanate from the device, reflect off special tape covering tennis balls held by the student, and are then sensed, interpreted, and visually represented on a display in the form of two crosshair symbols (trackers). The display is calibrated so as to continuously position the crosshairs at the actual physical height of each hand, in an attempt to enhance the embodied experience of virtual remote manipulation. Also, the researcher can control the error tolerance of students' hand positions. Finally, this MIT has a mode in which the trackers are controlled by inputting numbers into a ratio table rather than by remote handling. The color green, students' "correct" feedback, is a pedagogical scaffold for mathematizing proportional relations: this computationally fabricated *product* of measure indicates as equivalent a set of *isomorphism*-of-measure relations (cf. Vergnaud, 1983), thus supporting students' experience of embodied ratios as perceptual gestalts.

This paper draws on 20 interviews (total n = 24) conducted in an ongoing study with Grade 4-6 students from a private K-8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students). These participants were selected from a larger pool of student volunteers, all prior to formal instruction in ratio and proportion, in an attempt to achieve equal representation in terms of grade, gender, and teacher-ranked mathematical capability. Each student participated in a semi-structured interview (duration: mean 69 min.; SD 20.35 min.). Elsewhere, we report on results from pre/post-interview card-sorting tasks—one pictorial (a pair of hot-air balloons), the other numerical. In each of these tasks, students could assemble and narrate card sequences depicting either a "fixed-difference" or a "different difference" story.

The heart of the interview consisted of working with the MIT (see Figure 2a). At first, the condition for green was set as a 1:2 ratio, and no feedback other than the background color was given (see Figure 2b). Next, a grid was overlaid on the display monitor to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions (see Figure 2c). In time, the numerical labels "1, 2, 3,…" were overlaid on the grid's vertical y-axis to help students construct further meanings by more readily recruiting arithmetic knowledge and skills and better distributing the problem-solving task (see Fig. 2d, partial view). Next, we asked students whether it was possible to keep the screen *continuously* green while moving the hands up and down. If students could not perform this task, the interviewer would do so by literally manipulating their hands up and down, enabling them gradually to assume agency.

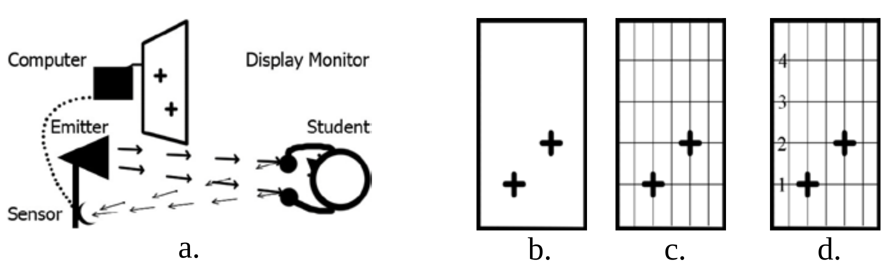

Figure 2. MIT: (a) system; the display in (b) free mode; (c) with grid overlay; (d) also numerals

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After working with the 1:2 ratio, students were told that the “rule for green” would change, and ratios of 1:3 and then 2:3 were used. Finally, the technology was reset from manual to numerical control. The interviewer introduced students to the numerical mode by inputting and “running” a set of green ordered pairs the student had just discovered in the previous activity. The interviewer then cleared the ratio table, typed a new pair of numbers in the top row, and asked the student to fill into the table numbers that would generate a green screen throughout the run. While working in the numerical mode, students were allowed to revisit the manual mode.

Our research team has been analyzing the videographed sessions collaboratively by applying microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1991). Whereas variability in students’ mathematical competence was manifest in their performance, we discerned family resemblance among their trajectories. For this brief report we collapsed the variability and focused on the resemblance. General results will lead to a case study of Itamar, a G5 middle-level male student. An edited video of Itamar’s interview accompanies the results (http://www.tinyurl.com/edrl-mit).

Results and Analyses

Recall that students were initially instructed to “make the screen green.” As expected, after stumbling upon their first “green location,” students attempted to create another by moving both hands while maintaining a fixed distance between them. They were surprised when the screen consequently turned red and moved their hands about until they found green again, whether in the same or a new location (see Figure 3; beams and sensors are three feet left of the monitor).

a. b. c.

Figure 3. Searching for green: (a) Itamar finds his first “green pair”; (b) his fixed-distance upward motion turns the screen red; (c) he lowers his left hand and finds a new green location

As students began to find more green pairs, they were inclined to determine principles underlying this emerging equivalence set of hand positions that each caused the screen to be green. Students tended to assert that as the hands rise, the greater the distance between the hands must be so as to sustain green. Students’ gestures and utterances suggested they were surprised by this equivalence set, in which magnitudes are not fixed but, rather, grow across instances (different differences). Thus students shifted from viewing the within-pair relation as constant to viewing it as covariant with (vertical) location. Yet, prior to the Cartesian reticulation, they still conceptualized *individual* hand locations irrespective of their height above the desk.

The introduction of the grid appears to have imposed a pedagogically effective tradeoff en route to reinventing proportion, because students adopted a “snap-to-grid” discretization of the continuous Cartesian space. Namely, whereas students now confined their manual search for new green locations to integer-unit locations (upon the gridlines), so doing they discovered that as the left hand rose 1 grid unit up from a green location, the right hand must rise 2 grid units in order

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to re-green the screen, what we call the "per change" solution strategy. Thus the grid afforded or enhanced opportunities to engage in the mathematical activities of measurement, comparison, and counting, cueing students to objectify their hand locations and hand motions *numerically*.

Introducing numerals along the grid's vertical axis further supported the emergence of multiplicative meanings for the activity. Initially, the numbers afforded succinct naming for the hand positions as a practical means of replicating these positions with precision and speed. Yet so doing, students became conscious of the particular number pairs and responded to them. Our selection of 1:2 as the protocol's first unknown ratio proved supportive—it enabled students to leverage their familiarity with simple multiplicative relations (e.g., double, half). Such naming of multiplicative relations, in turn, was critical to the emergence of proportion, as we now elaborate.

Prior to introducing the grid, students had focused on the distance between the hands. Equipped with the grid, students viewed hand locations individually, then re-focused on relations between different hand-*pair* positions, namely on "per change" recursive productions, but no longer on the relation *within* each pair (distance), even though this relation remain proportionally invariant amid transitions between pairs. Introducing numbers as new tools in the working space appears to have refocused student attention upon within-pair properties of green pairs, contingent on their multiplicative fluency. For example, students were generally able to correctly assert that one hand magnitude should be double the other so as to recreate green. In sum, whereas the pre-grid portion of the activity sequence launched students' construction of proportion by refuting their additive expectation and supporting a qualitative alternative, the grid and then numerals provided students with semiotic means of objectifying this embodied experience as additive, then multiplicative relations. We thus witness a compelling case where conceptual understanding emerges reflexively, contingent on the availability of notational systems supportive of inquiry.

Nevertheless, the meanings students built for proportion up to this point in the interview were still markedly discrete in nature. For example, asked if he could move his hands up the screen from bottom to top while keeping it continuously green, Itamar said he could not move his hands fast enough to prevent the screen from flashing red as he transitioned between two green pairs, in this case [2 4] and [3 6]. Itamar thus seems to have viewed his hands as transitioning between two discrete green states, rather than moving through a continuous set of green pairs. In other words, Itamar only attended to [2 4] and [3 6] while ignoring all hand location pairs in between.

With Itamar's permission, Dor held his hands and gently demonstrated the ambidextrous feat of keeping the screen green while moving both hands up simultaneously and then down again.

> **Dor**: So we start at you said 1 and 2. Here we go. [Moves hands up while maintaining green]
> **Itamar**: Then 2 and 4.
> **Dor**: Look, how am I doing it? [Refers to continuous production of green "despite" the grid]
> **Itamar**: Oh, because it's the same amount of space, and then this one's slowly going higher.
> **Dor**: Uh huh! This one's slowly going higher, and that one's going faster going higher?
> Itamar: Oh! It's like the air balloons! It's like the hot-air balloons. This one's going faster than the other one. They start at the same spot, and then one goes faster than the other.

Apparently, dynamic motion of two objects moving up at constant yet different speeds evoked for Itamar imagery of the balloons pictures used in the pre-interview card-sorting task, *despite his not having constructed a proportional story during that pretest*. He had only constructed a fixed-difference sequence, and when asked to compare the balloons' speeds had insisted that they rose at the same speed, as evidenced in the following, earlier exchange during that pretest:

Dor: Now is one of them going faster than the other one, and if so, which one?
Itamar: Well if one *was* going faster it would be the red one [the higher balloon, on the right], but I don't think one is going faster, I think one just started before the other.

We now return to the interview. Whereas the previous meanings constructed for green involved discrete extensive quantities—either the distance between the hands or the per-change amounts between pairs—this portion of the protocol introduced rate-based intensive-quantity meanings, namely speed (distance/time). Introducing alternative meanings for one and the same situation is not just a matter of accumulating meaning. Rather, with a teacher's guidance, multiple meanings set up opportunities to struggle with and synthesize disparate notions into more complex conceptual structures. Importantly, understanding relationships between amount and rate is a key learning issue in both ratio-based and non-ratio-based contexts (Stroup, 2002).

Having manipulated the MIT manually, students then turned to operating the device numerically, using the virtual ratio table. In this phase, some students regressed from their multiplicative insight to additive fixed-difference reasoning. For instance, after inputting the sequence [2 3], [4 6], [6 9], [8 12] that he had found in the hands portion of the activity, Itamar correctly noted that the left side was counting by 2 and the right side by 3. Yet, prompted to begin from [3 4], he input [5 6], [7 8], [9 10] so as to "add 2 for each one." He had thus reverted to fixed-difference reasoning. We believe, though, that through further reflective interaction with the MIT, students could learn to articulate their nascent hands-on insights in numerical forms.

In sum, setting off from haphazard hand waving yet progressively discovering the MIT's systematicity, students offered the following sequence of observations that address many learning issues of the targeted notion of proportionality, with the more advanced students tackling and connecting more of these ideas: (a) the locations of *both* hands are necessary to achieve green; (b) the critical quality for achieving green is a type of *relation* between the hands' respective locations; (c) these locations should be reinterpreted as magnitudes—the objects' heights above the base line; (d) the difference between the heights of the hands in correct pairs should not be constant—it will change between correct pairs; (e) this difference should increase as the pair's height increases; (f) moving from one correct position to another can be achieved by increasing these heights differentially, for example the left hand should rise 2 ad hoc units for every 3 units the right hand rises—a constant coordinated-change principle that can be used iteratively; (g) the multiplicative relation within each pair, too, is a constant that characterizes that particular problem; and (h) one and the same number pair expresses three aspects of the interaction—the lowest integer-pair location on the grid, the differential additive change pair, and the multiplicative relation within each and every correct pair: for example 2 and 3 units are the lowest correct integer pair of heights, raising the left hand by 2 units for every 3 raised by the right results in another correct location, and 2/3 or 3/2 is the constant within-pair multiplicative relation. Along the way, students also realized that there are infinitely many location pairs, so that in fact one can *simultaneously* raise both hands, thus reinterpreting the solution as two hands moving at different speeds that could be characterized as, for example, 2 and 3 units per beat.

Conclusion

Students' successful modeling of a problem situation even before engaging normative mathematical media, notation systems, and formats, suggests promise in instructional designs affording embodied, presymbolic quantitative reasoning. Such experiences enable students to struggle with qualitative aspects of a mathematical domain and discover its quantitative

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principles before they are burdened with the supplementary cognitive load of the disciplinarily requisite inscription and calculation procedures. Namely, as students are guided through a sequence of insights into the properties of a mathematical phenomenon, they can perform core conceptual work even prior to symbolic articulation. Through the appropriate introduction of canonical semiotic means, students can then be guided to formulate these insights normatively: as Itamar summed up, referring to the within-pair differences of 1, 2, and 3 in the ordered-pair sequence [2 3] [4 6] [6 9], "the difference doesn't have to stay the same." That is, by virtue of mastering a computational device that engaged yet challenged their additive reasoning, students constructed a new type of equivalence class articulated as a proto-proportional elaboration on robust pre-multiplicative schemes. For just one hour, we propose, these results are promising.

Though this is pioneering work and further research is underway to validate and substantiate our arguments, the empirical data thus far tend to support our conjecture concerning the potential pedagogical role of embodied artifacts as presymbolic objects-to-think-with. Furthermore, our action-before-concept design appears to resonate with the view that learners often engage in social activities prior to, yet as a condition of, interpreting their own behaviors in accordance with adult definitions of the situation (see Wertsch, 1979). Building on these emergent insights, we can now better guide students to optimize their development of personal meanings, invoked by engaging with the artifact, into normative mathematical meanings. Moreover, through studying student learning, we, in turn, are better positioned to optimize our theory of learning.

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