Running head: GENERIC MANIPULATIVES

A Better Story: An Embodied-Design Argument for Generic Manipulatives

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Abstract

Mathematics education practitioners and researchers have long debated best pedagogical practices for introducing to students new concepts. We report on results from analyzing the behaviors of 25 Grade 4 – 6 students who participated individually in tutorial activities designed to compare the pedagogical effect of manipulating objects that are either generic (non-representational, not signifying specific contexts, e.g., a circle) or situated (representational, signifying specific contexts, e.g., a hot-air balloon). The situated objects gave rise to richer stories than the generic objects, presumably because the students could bring to bear their everyday knowledge of these objects' properties, scenarios, and typical behaviors. However, in so doing, the students treated the objects' only as framed by those particular stories rather than considering other possible interpretations. Consequently, these students did not experience key struggles and insights that the designers believe to be pivotal to their conceptual development in this particular content (proportionality). Drawing on enactivist theory, we analyze several case studies qualitatively to explicate how rich situativity filters out critical opportunities for conceptually pivotal sensorimotor engagement. We caution that designers and teachers should be aware of the double-edged sword of rich situativity: Familiar objects are perhaps more engaging but can also limit the scope of learning. We advocate for our instructional methodology of entering mathematical concepts through the action level.

Keywords: attentional anchor, educational design, embodiment, formalization, manipulatives, mathematics, sensorimotor scheme, technology

Index: acontextual; affordance; attentional anchor; clinical interview; concrete, concept, contextual; constraint; Constructivism; coordination; cue; educational design; design; embodied design; embodiment; Enactivism; figural features; formalism; formalization; generalize; generic; ground; instructional methodology; Mathematics Imagery Trainer; mathematics; manipulation, manipulative; movement; multiplicative; narrative; pedagogical approach; progressive formalization; proportion; sensorimotor scheme; situated; symbolic; technology; tradeoff; virtual

A Better Story: An Embodied-Design Argument for Generic Manipulatives

1. Introduction: Dilemmas of Designing Manipulatives for Mathematics Learning

Imagine you are designing a mathematics lesson to introduce the notion of proportional equivalence, such as 2:3 = 4:6. Now further imagine that you wished for your students to understand proportionality as a multiplicative concept that builds on yet departs from additive forms of reasoning, so that the students can draw on what they already know yet expand this knowledge. In particular, you wished your students would consider a proportional equivalence sequence, such as 2:3 = 4:6 = 6:9 = 8:12, and so on, by focusing on the arithmetic difference between numbers in each ratio pair, that is, a difference of 1 in 2:3, a difference of 2 in 4:6, a difference of 3 in 6:9, a difference of 4 in 8:12, and so on. You would like the students to be surprised that this difference keeps changing, because you believe that this experience of surprise—of seeing a new type of equivalence—would precipitate meaningful conceptual development from additive to multiplicative reasoning.

Once satisfied with your educational design rationale, you set off to realize it in the form of some activity in which your students would engage. That is, you attempt to create for your students the interaction conditions that would give rise to an experience that you view as pivotal for learning the target content you have set for the lesson. Working either in a digital or material environment, you decide to create a scenario, materials, and task drawing on Aesop's parable of the Tortoise and the Hare. You will place images of the two protagonists at the starting point of a number-line racetrack, and you will guide students to advance the tortoise 2 units for every 3 units they advance the hare.

You try out this activity with young mathematics students. They find the activity engaging and eventually infer that the hare's lead over the tortoise keeps growing every go (1, 2, 3, 4, etc.). And yet the students are never surprised by this fact—it appears obvious to them due to their familiarity with the story. It would appear that the key learning experience you attempted to elicit was obviated by the scenario, materials, and task you had chosen.

You wish to improve the activity, and so you consider changing the appearance of the two objects from a tortoise and hare to some two other figures, such as nondescript circles. However in so doing, you realize, the very notion of two objects moving in parallel at different speeds toward a common target would potentially be lost, because the modified display would lack any familiar context that immediately prompts the desired scenario of a running competition between two agents of disparate athletic prowess.

You conclude, along with many other researchers in the past, that bringing familiar context into the process of learning mathematical concepts is not unproblematic; it is in fact riddled with tradeoffs (e.g., Uttal, Scudder, & DeLoache, 1997). A familiar scenario can instantly orient students toward relevant elements of an instructional activity as well as the elements' anticipated behaviors, but this very familiarity with the situated context might deprive the students of critical opportunities to struggle with inferring these behaviors and coordinating them with other knowledge they bring to the situation. On the other hand, one could begin with textbook definitions and solution procedures for proportional equivalence and only later apply these acontextual routines to everyday situations, and yet those routines would initially bear scant meaning for the students. It seems as though both approaches—from the situated to the symbolic, or from the symbolic to the situated—can be problematic. Is there a third option?

In this chapter, we will make the case for a third option. And yet this third option might appear quite different from the other two, because it highlights the educational role of the *physical actions* students perform as they manipulate objects in mathematics lessons. That is, teachers and researchers usually focus on how students select, arrange, and transform manipulatives in the working space (i.e., the *planning* and *product* of action) and what that could mean conceptually; few focus on how students coordinate their hands so as to move the manipulatives per the task objectives (i.e., the *process* of action; but see Abrahamson, 2004; de Freitas & Sinclair, 2012; Kim, Roth, & Thom, 2011; Nemirovsky, Kelton, & Rhodehamel, 2013).

In the activities we will discuss, students first learn to enact a new movement form and only later they ground that form in particular contexts as well as generalize it as mathematical rules. The students initially learn the movement form by way of solving an interactive manipulation problem involving two virtual objects, one per each hand. This instructional methodology draws on theories from the cognitive sciences that depict mathematical reasoning as the mental simulation of sensorimotor activity (Barsalou, 2010; Hutto, Kirchhoff, & Abrahamson, 2015; Landy & Goldstone, 2007; Vygotsky, 1926). Through their efforts to solve a new two-hand movement coordination, students may come to perceive the world in a new way. Educational designers can create conditions where this new moving/perceiving pattern is the meaning we experience and sustain for a particular mathematical concept that we are studying, even before we use formal symbolic notation (Abrahamson & Bakker, 2016).

The chapter will focus on comparing generic vs. situated objects with respect to how students interact with the objects and what they infer from these interactions. By *generic* we mean that the objects deployed in these activities are not contextualized as representing or referring to anything outside of the activity. They may afford goal-oriented interaction as tools for accomplishing a task, but the designer does not intend for them to signify or symbolize for all students any particular meanings from some other domain, at least not initially. The term “generic” (non-specific) also alludes to its cognates “genus” (a class of things), “generative” (bearing potential for growth and application), and “generalize” (produce inference from a case), all perceived as potential attributes of these instructional materials. We think of generic objects as less likely than *situated* objects to evoke rich experiential contexts or narrative content—they are means of engaging in an activity without drawing on associations with what they resemble, denote, or connote. Clearly any choice of terminology comes with its ineluctable philosophical and theoretical baggage from the cognitive sciences literature, such as epistemological, ontological, and phenomenological assumptions about human perception and reasoning (e.g., Wilensky, 1991), so that perhaps an example will cut to the chase: With “*generic*” we are attempting to characterize the difference between a circle and a hot-air balloon. We wish to understand how this difference bears on the processes and consequences of learning.

Working with technological media, the objects employed in our pedagogical activities will be virtual. The situated objects will be iconic images of hot-air balloons, whereas the generic objects will be stark circles. The students will manipulate these virtual objects in their attempts to solve the interaction problem of making a screen green. As shall be reported, students respond to manipulation problems involving familiar objects by bringing to bear what they know about these objects, such as how hot-air balloons typically behave in particular contexts. The students thus perceive and manipulate the familiar objects to enact imaginary micro-scenarios that would be plausible with these objects in those contexts. For example, students may engage a virtual hot-air balloon by launching it vertically from the ground upward, but they are less likely to rotate it. By contrast, generic objects do not constrain the scope of potential perceptions and movements as much, because they conjure for the student less immediate sense of what might be plausible and implausible to do with them. For example, one would not be inhibited in rotating a simple circle as one would an icon of a hot-air balloon, and one would be less inclined to construe the circle as necessarily launching from the screen base as one would the hot-air balloon. We conjecture that students are likely to perceive and move stark objects in more ways than they would iconic objects and, consequently, potentially infer a greater range of mathematical rules.

Understanding the effects of objects on actions is important for teaching mathematics with manipulatives. If we hope to elicit from students particular ways of moving, because we see these ways of moving as critical for the learning process, then we should choose or create our manipulatives wisely with those movements in mind. This principle holds both for digital and material instructional resources (see Sarama & Clements, 2009, on “concrete” virtual manipulatives).

Below we present a technical section that will expand on the theories of learning that have motivated our research, focusing on literature that treats the relation between interactive objects and the forms of reasoning they enable (Section 2). We then detail the methods used in this study (Section 3). Results and findings then follow (Section 4), and we end with conclusions as well as implications for design and teaching (Section 5).

2. Theoretical Background

2.1 Framing the Debate

Scholars of mathematics education tend to hold two diametrically opposed positions on best pedagogical practices for introducing new mathematical concepts (Abrahamson & Kapur, 2018; Nathan, 2012). The debate centers on the question of whether and when situated contexts should be employed in cultivating students' understanding of mathematical concepts. In particular, researchers debate on the optimal ontological nature of the objects that students are to consider as they solve instructional problems: Should these objects evoke specific, elaborate situations with rich contextual meanings, or should they be non-contextual "situation-agnostic" generic symbols and shapes? The formalisms-first approach (see Figure 1a; e.g., Kaminski, Sloutsky, & Heckler, 2008; Sloutsky, Kaminski, & Heckler, 2005; Stokes, 1997) posits that students should first work with abstract representations, such as mathematical symbols and geometrical shapes, to enact and understand solution procedures; only then should they extend and practice these formal strategies by applying them to specific situated contexts. The progressive-formalization approach (see Figure 1b; e.g., Goldstone, Landy, & Son, 2008; Gravemeijer, 1999; Noss & Hoyles, 1996; Ottmar & Landy, 2017), on the other hand, posits that students should begin from concrete situations and then progressively generalize, abstract, and formalize their understandings of the situations by creating, adopting, and using normative symbolical representations. In the course of adopting these mathematical visualizations and forms of discourse, cultural agents (such as designers and teachers) play key mediating roles in providing students with selected semiotic means of objectifying their emerging notions (Bartolini Bussi & Mariotti, 2008; Newman, Griffin, & Cole, 1989; Radford, 2013; Sfard, 2002). Each of these positions, we believe, holds merit, and yet each also suffers from the very shortcomings implicated by its critics. It could be that a third option exists that draws on the merits of each.

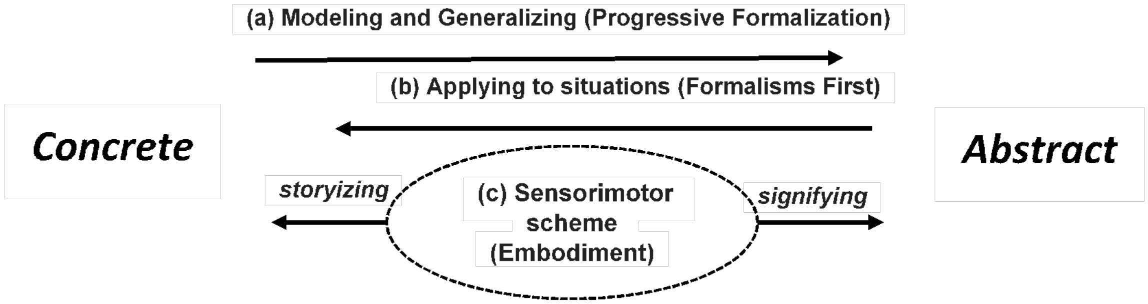


Figure 1. Positioning the (c) Embodiment approach with respect to the (a) Progressive-Formalization approach and the (b) Formalisms-First approach.

Inspired by the embodiment approach (Campbell, 2003; Chemero, 2009; Clark, 2013; Nemirovsky, 2003; Varela, Thompson, &, Rosch, 1991), the educational approach portrayed in Figure 1c positions sensorimotor schemes as the hypothetical epistemological core of mathematical learning and knowing. This approach responds also to calls (Allen & Bickhard, 2013; Arsalidou & Pascual-Leone, 2016; Varela, 1999) for renewed interest in Piaget's systemic theory of genetic epistemology (Piaget, 1968) as providing a viable alternative to the dominant paradigm of cognition as information processing. In line with our embodiment approach, we conjectured that students could encounter new mathematical concepts by first developing sensorimotor schemes and then both grounding these schemes in concrete situations (storyizing) and articulating the schemes in mathematical formalism (signifying; Howison, Trninic, Reinholz, & Abrahamson, 2011; see also Fuson & Abrahamson, 2005). Thus whereas we embrace the proposal to ground mathematical meaning in "our direct physical and perceptual experiences" (Nathan, 2012, p. 139), we decompose this idea by foregrounding and differentiating what we view as its two inherent phenomenological dimensions: sensorimotor schemes (goal-oriented movement) and situatedness (contextuality). We argue that these two dimensions have been conflated in historical debates (e.g., Barab et al., 2007; Bruner, 1986; Burton, 1999). That is, we maintain that learning activities can be created such that sensorimotor schemes are fostered either in contextual or acontextual situations, and we are interested in understanding the processes and consequences of these two instructional options.

To evaluate this embodiment approach to mathematics learning as it bears on pedagogical design, we began by formulating the hypothesis that different levels of contextuality have different effects on learning, and we assumed that sensorimotor schemes would mediate this effect. We believed more specifically that students would develop different sensorimotor schemes in low- vs. high-context activities and that the low-context condition would prove advantageous.

To operationalize this hypothesis, we designed and implemented a learning activity complete with materials, tasks, and facilitation techniques based on the embodied-design framework (Abrahamson, 2006, 2009, 2014). In the empirical study reported in the later sections of this chapter, we varied the contextuality of a manipulation problem by either incorporating or not incorporating iconic information that would potentially cue particular narrative framings of the situation, and we measured for effects of this experimental variation on content-relevant qualities of students' behaviors as they engaged in solving the problem. Our study thus aimed to empirically evaluate the in-between embodiment position with respect to the ongoing debate of formalisms first vs. progressive formalization, a devate which we now further detail.

2.2 Contrasting Approaches to Formalization

Summarizing a rich research literature, Nathan (2012) has characterized two opposing approaches to mathematics education as follows:

- *formalism first* proposes that students should encounter new concepts through abstract procedures and then map formalisms to concrete situations via application problems. For instance, a student might first learn the symbolic formula for adding fractions (finding a common denominator, etc.) and only later manipulate objects that serve to explain and demonstrate this algorithm; whereas,
- *progressive formalization* proposes that students should encounter new concepts in the context of meaningful concrete situations and then abstract toward formal models of these situations by progressively adopting mathematical forms and nomenclature. In this case, per the prior example, a student would first manipulate objects to discover principles for adding fractions and only later learn the symbolic formula that represents this procedure.

Among the studies supporting the formalism-first approach, the work of Kaminski et al. (2008) and Sloutsky et al. (2005) are of particular relevance to this discussion. In their experiments, undergraduate students participated in pattern-learning mathematics activities, where the elements composing the patterns were either generic and acontextual (non-descript geometrical shapes) or situated and contextual (readily identifiable objects). In a subsequent transfer task in a novel yet structurally identical domain, the acontextual participants outperformed the contextual participants. Based on these results, the researchers concluded that generic instantiations of mathematical structures are pedagogically superior to their concrete correlates. Concreteness, they argue, necessarily bears irrelevant contextual features, and these negatively influence both learning and transfer. First, learners may miss cross-domain structural alignment as a result of perceptual dissimilarity between rich representations. For example, they would not see how both the situation of two interlocking gears and the situation of two buildings and their shadows instantiate the concept of proportionality. Second, irrelevant aspects of concrete representations are liable to draw the focus of learners' attention away from conceptually critical information. For example, a demonstration of proportionality with interlocking gears might distract students to note the circles' counter-rotation at the expense of noting the multiplicative relations between the circles' circumferences. The logic of this argument is that students learning from an example cannot yet know what this will be an example *of*, and so they cannot in principle separate the conceptual wheat from the contextual chaff. Finally, concrete objects are more likely to be interpreted as ontologically intact entities rather than as symbolizing something else and thus may have limited referential flexibility, which is vital for the transfer. For example, students who use a printed 10-by-10 grid as an organizational scheme to build an elaborate construction out of a set of 1-by-1-by-1 wooden cubes would be less likely to later use that same grid as a topographical map with numerical values in each cell standing in for the height of the column of cubes towering up in that cell. The concrete object (the grid) takes on functional fixedness as a thing onto its own rather than as a potential representation of something else, so that the students miss out completely on a key learning objective.

In contrast, the research of Goldstone and Son (2005) supports progressive formalization. In their experiments, undergraduates worked with computer simulations to learn about complex adaptive systems. The simulation featured visual elements of varying perceptual concreteness, for example foraging ants were represented either by dots or by iconic images of ants. Students' performance was compared in both the initial and transfer tasks. Students were divided into four groups: abstract then concrete; abstract then abstract; concrete then concrete; and concrete then abstract. The best performance on both the learning and transfer tasks was obtained in the concrete-then-abstract group (i.e., the progressive formalization approach). The authors interpreted their findings to suggest that progressive formalization helps learners by enabling them first to enter a specific domain with the aid of concrete cues and then abstract and generalize principles as this concreteness fades out (see Ottmar & Landy, 2017, for a mathematics example).

Table 1: *Comparison of Three Pedagogical Frameworks According to Symbolic and Contextual Attributes of Learning Resources*

		Contextual	
		No	Yes
Symbolic	No	Embodied Design	Progressive Formalization
	Yes	Formalism First	–

The embodiment approach put forth in this article: (a) borrows the Progressive-Formalization epistemological position that abstract notions are grounded in activity with asymbolic objects; yet (b) also partially subscribes to the Formalism-First ontological position that mathematical concepts should be grounded in acontextual entities. Thus on the one hand, as per Progressive Formalization, embodied-design learning materials are asymbolic. Yet on the other hand, per Formalisms First, embodied-design materials are acontextual (see Table 1). These asymbolic acontextual learning materials are thus designed so as to avoid evoking students' knowledge about a narrow set of situations, that is, to avoid cueing particular narratives that might circumscribe the range of meanings students bring to bear in solving our tasks. Similar to generic construction materials, such as sand, play dough, or building blocks, inherent qualities of these virtual resources are designed so as to enable specific interactions and combinations yet without pre-constraining what meanings students bring to bear as they use these resources. We explain what the objects can do but not what they are.

2.3 Affordances and Constraints of Asymbolic vs. Symbolic Manipulatives

Pedagogical approaches inspired by constructivism and embodiment theory have highlighted the role of sensorimotor integration in students' cognition of mathematical concepts (Abrahamson, 2006; Gray & Tall, 1994; Nemirovsky, 2003; Steffe & Kieren, 1994; Thompson, 2013; von Glasersfeld, 1983). Our study considered from an embodiment perspective the effect of situatedness on the development of sensorimotor schemes prior to signifying the schemes in a discipline's semiotic register. We thus sought a theory of situated perception and action that would enable us to model, anticipate, and analyze for effects of experimentally varying an activity's situatedness.

Our focus on the relationship between the properties of objects that students manipulate and their actions on these objects led us to consider the theoretical notions of affordances and constraints as relevant to the goals of this study, bearing in mind the critical social role of cultural agents in creating and providing these objects and mediating their functions and forms of use. *Ecological psychology* (Gibson, 1977) theorizes an agent's potential actions on the environment as contingent on the agent–environment relations. An agent (e.g., a mathematics student) engaged in a particular task (e.g., solving an interaction problem) perceives opportunities for acting on objects in the environment (e.g., classroom manipulatives) in accord with these objects' subjective cues; the agent tacitly perceives the object as *affording* particular actions, that is, privileging certain forms of goal-oriented engagement (see Vérillon & Rabardel, 1995, for a complementary theorization of instrumental genesis). If you are attempting to exit a room, a door handle affords grabbing and rotating. Importing Gibson's interactionist views into educational research, Greeno (1994) modeled student learning as the process of attuning to constraints and affordances in recurring situations. Araújo and Davids (2004) further offer that an instructor can "channel" students' engagement in goal-oriented activity by controlling environmental constraints. That is, a teacher can organize a classroom space in which she steers students to engage manipulatives in particular ways she believes are conducive to learning targeted mathematical content (see Mariotti, 2009, for a complementary sociocultural view on semiotic mediation).

Still, to the extent that one subscribes to the constructivist thesis underlying this research, namely that sensorimotor learning grounds conceptual learning, *why might different degrees of the learning materials' contextuality afford different sensorimotor learning?* The answer, we believe, lies in the nature of these sensorimotor schemes vis-à-vis the particular features of the learning materials that the students mentally construct in the course of developing the materials' new perceived affordances. That is, a given situation may lend itself to different goal-oriented sensorimotor schemes. And whereas a variety of schemes may accomplish the prescribed task, some of these schemes may be more important than others for the pedagogical purposes of the activity. We hypothesize that *the situatedness (contextuality) of learning materials constrains which sensorimotor schemes the materials might come to afford*. Where particular contextual cues unwittingly preclude student development of pedagogically desirable affordances, the students' conceptual learning will thus be delimited.

In evaluating this hypothesis pertaining to the nature and quality of situated learning, we needed a theoretical construct that would both cohere with the embodiment perspective and enable us to implicate in our data which sensorimotor schemes students were developing. We realized we were searching for a means of determining how the students are mentally constructing the materials; what specifically they were looking at that mediated their successful manipulation. Such a theoretical construct already existed: an attentional anchor (see below).

An *attentional anchor* is a dynamical structure or pattern of real and/or projected features that an agent perceives in the environment as their means of facilitating the enactment of motor-action coordination (Hutto & Sánchez–García, 2015). Abrahamson and Sánchez–García (2016) demonstrated the utility of the construct, which originated in sports science, in the context of mathematics educational research. Abrahamson, Shayan, Bakker, and van der Schaaf (2016) studied the role that visual attention plays in the emergence of new sensorimotor schemes underlying the concept of proportion. They overlaid data of participants' eye-movement patterns onto concurrent data of their hand-movements. It was found that the participants' enactment of a new bimanual coordination coincided with a shift from unstructured gazing at salient figural contours to structured gazing at new *non*-salient figural features (even at blank screen locations that bore no contours at all). The participants' speech and gesture confirmed that they had just constructed a new attentional anchor as mediating their control of the environment (see also Duijzer et al., 2017).

For this study, we adopted the construct of an attentional anchor as a key component of our methods. We sought to characterize what attentional anchors students developed during their attempts to solve a motor-action manipulation task. By so doing we hoped to gauge for effects of varying the contextuality of learning materials (situated vs. generic) on student development of the sensorimotor scheme mediating an activity's learning goal. We hypothesized that the more situated manipulatives would constrain the scope of attentional anchors students develop, with the detrimental consequence of students missing out on interaction opportunities that the designer considered as pivotal for learning the target content.

3. Methods: Designing Constraints on Students' Sensorimotor Engagement of Manipulatable Elements in a Technological System

The Mathematics Imagery Trainer for Proportion (MITp; see Figure 2) sets the empirical context for this study. Students working with the MITp are asked to move two cursors so as to make the screen green and keep it green. Unknown to the students, the screen will become green only if the cursors' respective heights along the screen relate by a particular ratio. The color of the screen can change along a gradient from red through orange toward green, with the feedback for the correct ratio being a distinct base-color green. For instance, for a ratio of 1:2, the screen will be green only when one hand is twice as high along the monitor as the other hand. Students develop a variety of motor-action strategies to satisfy the task demand (Howison et al., 2011).

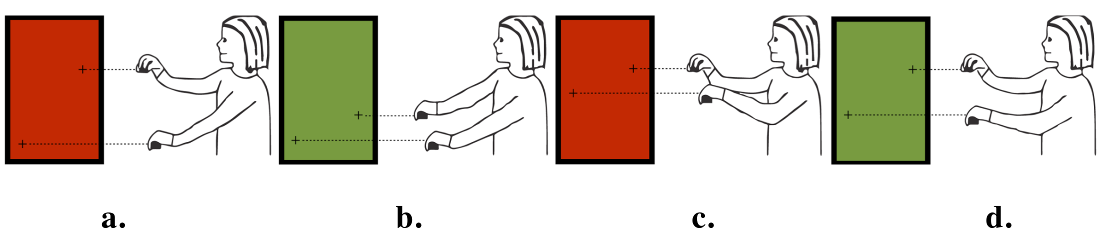
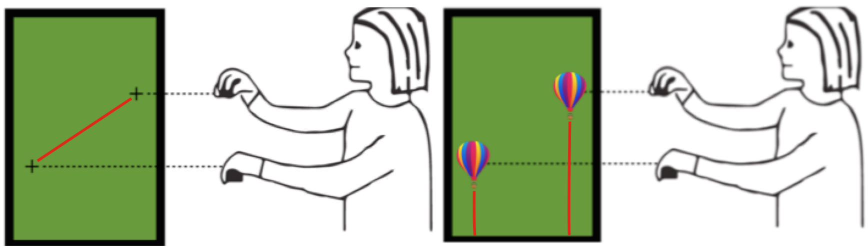


Figure 2. The Mathematical Imagery Trainer for Proportion (MITp) set at a 1:2 Ratio. Compare 2b and 2d to note the different vertical intervals between the hands and, correspondingly, the different vertical (or diagonal) intervals between the virtual objects. Noticing this difference is presumed to be crucial for experiencing, then resolving a key

cognitive conflict in expanding additive reasoning into multiplicative concepts.

In the current study, images appear at students' fingertips when they touch the screen. These images are either generic crosshair targets (see Figure 3a) or situated images (e.g., hot-air balloons; Figure 3b).



a. **b.**

Figure 3: Experimental conditions and hypothesized attentional anchors: (a) generic crosshair targets cue the vertical or diagonal interval between the hands; and (b) situated images (hot-air balloons) cue the interval between each object and the bottom of the screen directly below it. In the actual experiments we used large touchscreens where the hands are on the interface.

We selected hot-air balloons as exemplars of situated images, because presumably they evoke a schematic spatial–temporal narrative—a script that includes normative (default) topological plotting on the immediately available frame of reference (begin from screen base), an orientation and destination (upward toward the top of the screen), and a displacement vector and schedule (steady pace of motion along a linear vertical trajectory). Moreover, different balloons could conceivably rise at different rates due to their idiosyncratic payload, fueling, and navigation, so that two balloons might rise side by side, in parallel, each at its own speed. When two hot-air balloons launch together from the same location, presumably also the script of competition is evoked, because the balloons' respective motion then express human presence and agenda—the balloon moving at a greater speed might distinguish its pilot as more skillful and victorious. Consequently, students' multimodal attention to the objects they are manipulating would manifest as a tacit contradistinction between two individual entities, each with its particular identity, animacy, and effort. As such, students might wield the bimanual operation by rapidly alternating their attention between the two objects, ensuring in turn that *each* is moving correctly, rather than perhaps seeking a sensorimotor means of integrating the movements as a *relation* between the objects, such as by focusing on the spatial interval *between* the objects as it changes. Consequently, we assumed that students working with the hot-air balloons, as compared to those working with the generic objects, would be less likely to select the spatial interval between the objects as an attention anchor facilitating their task-oriented manipulation.

While hot-air balloon icons thus presumably constrain the range of potential interactions with the virtual objects (the "enactive landscape, Kirsh, 2013), students' tacit knowledge pertaining to how hot-air balloons behave also implicitly constrains them to manipulate the virtual objects along parameters relevant to the interaction. Namely, the software is programmed to respond only to the relative *vertical* location of these virtual objects (the *y* axis), not their horizontal locations (the *x* axis): The screen color is a function of the objects' relative distance from the bottom of the screen not its sides. The situated objects may thus better afford task-relevant manipulation as compared to the generic objects, thus minimizing exploration operations ("instrumenting," per Vérillon & Rabardel, 1995), just as a regular household wall-mounted light switch imposes vertical actions and precludes horizontal actions. Another set of situated objects designed for this activity were a pair of cars moving from the screen bottom to the top as per a birds-eye view of a racing track.

In both experimental conditions (generic and situated) students are led through a task-based semi-structured clinical interview. Following an unstructured orientation phase, in which the participants find several green locations, they are asked to maintain green while moving both hands from the bottom of the screen to the top. The interviewer then directly facilitates a coordination challenge, where the interviewer manipulates the left image and the student manipulates the right image. The student is asked to predict the green locations. This being a *semi*-structured interview, participants may experience additional opportunities to engage in tasks of finding green, maintaining green, and other unstructured exploration, either spontaneously or per the interviewer's suggestion. All along, the students are prompted to articulate rules for making the screen green. The interview was designed to last approximately 30 minutes, which included brief introductions and conclusions, with the core time equally divided between the two conditions, generic and situated.

We wished to investigate for attentional anchors that emerge during children's interactions with the technology. We reasoned that the attentional anchors would indicate what sensorimotor schemes the students developed. More specifically, we explored for an effect of experimental condition (generic vs. situated cursors) on the types of attentional anchors students construct and articulate (via speech and/or gesture). We also looked at the effect of condition sequence on the development of attentional anchors.

Twenty-five Grade 4 – 6 students participated individually in the interviews, 14 in the "generic-then-situated" condition and 11 in the "situated-then-generic" condition. In this study, we exclusively interviewed students around the numerical item of a 1:2 ratio, so as to minimize interview duration (see Abrahamson, Lee, Negrete, & Gutiérrez, 2014, for a study that explored other ratio items). These sessions were audio–video recorded for subsequent analysis. As our primary methodological approach, the laboratory researchers engaged in micro-analysis of selected episodes from the data corpus, focusing on the study participants' range of physical actions and multimodal utterance around the available media. Our working hypothesis, to iterate, was that the virtual objects' figural elements may cue (afford) particular sensorimotor orientations and thus "filter" the child's potential scope of interactions with the device. Namely, we analyzed for effects of the manipulatives' perceived affordances on participants' scope of interaction, bearing in mind that some interactions are more important than others for learning particular mathematical content.

4. Results: Implicit Affordances of Manipulation Objects Mediate Student Strategies

A main effect was found. Below, we report our findings in each experimental condition by first describing participants' typical strategies and then illustrating these behaviors through brief vignettes. The section ends with comparing observed student strategies under the two conditions.

4.1 Generic Targets Afford the "Distance Between the Hands" Attentional Anchor

In the trials where participants interacted with generic targets first, they began the activity by placing their left-hand- and right-hand fingertips on a blank touchscreen. Immediately they noticed crosshairs appear at the locations of their fingertips. In an attempt to make the screen green, the participants began moving their hands all over the screen with no apparent strategy, "freezing" their fingers as soon as the screen turned green. Eventually, participants oriented toward the spatial interval between their fingers, soon discovering that their fingers have to be a certain distance from each other at different heights along the screen. Finally they determined a dynamical covariation between the interval's size and height: the higher the hands, the bigger the interval must be (and vice versa). We turn to several vignettes (all names are pseudonyms). As we shall see, both participants will refer to an imaginary diagonal line connecting the cursors.

Luke (age 10). As he found various green-generating screen location, Luke commented about the space between his hands at these various locations: "It's the same angle. Well, I mean the line connecting them is the same direction" [4:53]. Later, he noted that the "[angle] is changing because my right hand is getting faster, so when this goes up that much (moves left hand approximately 2 inches on the screen) this one goes up at this much (moves right hand approximately 4 inches on the screen)" [11:10].

Amy (age 9). Amy reported her observation: "The diagonal [between the hands] at the top is different than [at] the bottom" [7:15]. Then later during the situated challenge, she said: "You have to make them different diagonally from each other to make it change color" [7:42].

Thus during the generic-target trials the participants not only noticed that the interval between their hands was changing in size, they came to see this interval as an imaginary line between their hands. In turn, this imaginary line—its size, angularity, and elevation along the screen—apparently served the participants in finding and keeping green, ultimately enabling them to articulate a strategy for doing so. This imaginary line along with attributed properties is an attentional anchor: It is crafted out of negative space to mediate the situated coordination of motor intentionality; subsequently this mentally constructed object serves to craft proto-proportional logico–mathematical propositions. This spontaneous appearance of a self-constraint that facilitated the enactment of a challenging motor-action coordination is in line with dynamical-systems theory (Kelso & Engstrøm, 2006).

Of the 14 students in this generic-then-situated experimental condition, 10 spoke about the interval between the hands still within the "generic" phase of their interview, and 8 of these 10 referred explicitly to its magnitude. Then during the "situated" phase of the interview, only 2 of these 10 students began to speak about the balloons as separate entities, focusing on the speed of each respective balloon, or reverting to a focus on the color feedback of the screen to determine where to place the hands. The remaining 8 of these 10 students continued to use the interval line between their hands as a guide for making the screen green. These students' attention to the diagonal line was consistent, suggesting that this imaginary "steering wheel" had become perceptually stable in their sensorimotor engagement with this technological system.

4.2 Situated Images Afford the "Distance From the Bottom" Attentional Anchor

Similar to the generic-then-situated condition, in the trials where students interacted with situated icons first, they began the activity by placing their left-hand- and right-hand fingers on a blank touchscreen. However in this condition they immediately saw hot-air balloons (not generic targets) appear on the screen. Thus, the virtual manipulatives in this condition are situational, even as the tasks are otherwise identical. Recall that these students worked first with the balloons and then with the crosshairs. As we will now explain, beginning with the balloons cued a narrative-based strategy, alluding to a frame of reference, that did not attend to the interval between the images but instead to each of these hot-air balloons' respective vertical distance above the "earth" (the bottom of the screen). This alternative sensorimotor orientation was so strong that it carried over to the crosshairs condition, so that by-and-large these participants were less likely to attend to the interval between the objects and thus were less likely to benefit from its potential contribution to their problem-solving strategy.

Leah (age 11). Having generated green for the first time, Leah noticed that when she moves one hand, the greenness dulls out toward red. Later, she described her strategy for making the screen green referring gesturally to the hand's distance from bottom of the screen: "I would say what I said before, where one hand chooses a place and the other hand chooses a color based on where the hand is, and you can adjust it to keep it green. Once you find that, you just need to keep it the same height [from the bottom]" [8:40]. Then in the next task, she maintains her strategy, saying: "When you move one hand up you need to move the other hand up so it's the same distance [from the bottom], but higher" [12:22].

Jake (age 11). Jake described his initial strategy in the form of a prescriptive rule, using the imperative grammatical mode, as though teaching another person how to accomplish the task:

> Try putting your hands together in the middle and then try moving one down or the other one up. One of the balloons should stay in the middle while the other moves [4:47].

Note how "middle" refers to that balloon's location along a vertical axis irrespective of the other balloon. Jake perseverated with this strategy throughout the set of challenges, moving his hands up along the screen sequentially rather than simultaneously. When later tasked to make the screen green with the stark targets, he appeared disoriented, noting, "This is harder because I don't have a starting point" [24:12]. Jake referred to the apparent absence of an "earth" as a grounding frame of reference for the cursors' vertical motion.

Of the 11 students who encountered the situated images first, 4 began to speak about the interval between the hands still during the situated condition, however these students did not elaborate about the line between the hands, and rather focused on each hand as a separate entity (e.g., stating that one hand controls color and the other controls brightness). During the second phase, in which they encountered the situated images, 2 of these 4 students as well as 3 of the 7 who had not attended to the interval demonstrated the emergence of this attentional anchor. The remaining students treated each of the two icons as separate entities throughout the entire interview, and hardly spoke about the interval between the hands. Collectively, these students were more inclined to treat the two objects on the screen as separate entities, focusing on the changing height of each object and the different speeds of the two objects as they move upward. Additional phenomena were encountered only in the iconic-then-stark condition. For example, one of the students (*Kate, age 11*), who spoke about the interval between the hands, used the length of iconic cursor itself to measure the interval. Kate explained her strategy for making green. It begins with placing the icons near each other at the bottom of the screen. Then, "in the middle there is one balloon between them, and at the top, two balloons between them. So it grows by one at a time" [06:45]. She accompanied this explanation with three quick demonstrations: at the bottom of the screen, in the middle, and on the top. When the icons were changed to the cars, Kate repeated her explanation:

> It's the same. They're right on top of each other at the bottom, and then in the middle it is like one car between them, and at the top—two cars. [08:32]

Later, Kate transferred this quantification approach to the generic condition, as follows:

> Um, let's say, move one of them [cursor], like, one length above the other, and then move the bottom one up until it's with another one, and then move the next like two lengths above, and then move the other one, and then here—four. [18:22]

Kate was well aware of the interval between her hands and in fact utilized it as an ad hoc unit of measurement so as to pace her bimanual ascent up along the screen (see Palatnik & Abrahamson, 2017, under review). Thus rather than negatively constrain her solution, as per our thesis, Kate's vignette provides a contrasting, if unique, non-protocol and idiosyncratic case of concreteness productively supporting progressive formalization.

4.3 Summary

Participants who began the activity in the generic condition oriented toward the distance between their hands as their attentional anchor, whereas participants who began in the situated condition tended to treat the manipulatives as independent, untethered entities. It would appear that participants who began in the generic condition generated the interval as their attentional anchor because no other frame of reference was cued. Participants who began in the situated condition, on the other hand, followed the cued narrative implicit to the familiar images and therefore tended rather to visualize the two balloons as launching up from the ground.

It thus appears that objects bearing rich associative content introduce a new layer of baggage onto an interaction task, including forms, dynamics, hierarchies, and social conventions that guide the students' perception of the action space (on "framing," see Fillmore, 1968; Fillmore & Atkins, 1992). For instance, we typically think of hot-air balloons as "starting" at a point, such as the ground, at takeoff, and these evoked frames implicitly constrain the scope of possible attentional orientations to a situation, for example, by privileging the interval from each object down to the bottom of the screen at the expense of attending to the interval between the objects. In contrast, when manipulating stark cursors, there is no "starting point" as such, making it more likely that students attend to the interval between the hands. Presumably one could design icons that would draw students' attention explicitly to the relation between the two objects rather than viewing the objects as independent. Doing so, however, might come at the expense of two design goals: (1) enabling students to *discover* the target parameters (the behavior of a varying spatial interval would be evoked by the script rather than through exploration and would thus prevent eliciting students' inappropriate schemes, which in turn would prevent their experience of cognitive conflict that leads to reflection); and (2) opening up the scope of polysemous sensorimotor schemes (see Abrahamson et al., 2014).

Supporting our study's hypothesis, the results suggest an effect of situatedness on the construction of sensorimotor schemes. This finding is relevant to mathematics pedagogy, because sensorimotor schemes are theorized as mediating conceptual learning. It follows that situatedness of instructional materials is liable to impede mathematical learning by precluding the emergence of sensorimotor schemes pertinent to a cognitive sequence toward the generalization of rules. Whereas situatedness could, in turn, orient students precisely to the key parameters of the instructional design, doing so is liable on the other hand to narrow the manipulatives' enactive landscape and thus the scope of meanings that students bring to bear and develop through the interaction. Future iterations of this intervention would avail of eye-tracking (e.g., Abrahamson et al., 2016; Duijzer et al., 2017) and other multimodal learning analytics (Worsley et al., 2016) to corroborate students' oral and gestural report of attentional anchors and to expand our understanding of relations between situatedness and learning.

5. Conclusion

Mathematics education researchers have long debated the question of pedagogical practices for introducing new mathematical concepts. The Formalism-First and Progressive-Formalization approaches offer diametrically contrasting positions on the question of whether concepts best develop from situated or generic learning materials. We tend to agree with the now-tempered view asserted by Day, Motz, and Goldstone (2015) that the "question of contextualization in instruction is neither simple nor settled" (p. 11; see also Goldstone & Sakamoto, 2003). Per their results, rich contextualization may encumber students' subsequent transfer of their understanding (see also McNeil, Uttal, Jarvin, & Sternberg, 2009). We, in turn, have contributed to the debate by offering that *the focus of situatedness research should be not on properties of the learning materials per se but on the sensorimotor schemes the materials may afford*. Thus, whereas Kaminsky et al. (2008) offer that "the difficulty of transferring knowledge acquired from concrete instantiations may stem from extraneous information diverting attention from the relevant mathematical structure" (p. 455), we refine that students' attention is diverted from the mathematically relevant *actions*. Richer materials, we have demonstrated, may unproductively constrain the scope of sensorimotor schemes students develop through engaging with the materials. In particular, richer materials may diminish opportunities for conceptual development, because they draw students' attention toward ways of thinking about the situations that, per the design, are less mathematically relevant. Students are liable thus to miss out on opportunities to think about the situation in ways that are critical for the educational success of an instructional sequence. On the other hand, where rich situated materials are designed so as to orient students explicitly on parameters that *are* relevant to the mathematical content, doing so would likely narrow the scope of meanings students bring to bear. For example, though we want students to attend primarily to the interval between the virtual objects, we wish for them to consider also the objects' relative speeds (see Abrahamson et al., 2014).

Students, that is to say children, are highly imaginative. They readily engage in pretense with generic objects, visualizing them one way and then another way. It is the *low* situativity of generic manipulatives that lends them to a greater variety of narratives and consequently a greater variety of sensorimotor orientations (see also Healy & Sinclair, 2007; Tahta, 1998). And so we agree with Uttal, Scudder, and DeLoache (1997) that sensory richness of manipulatives may derail certain forms of mathematics learning. But we stress that the issue here is not so much about sensory overload distracting from intended forms of engaging the objects. It is not about manipulatives but about *manipulating*—it is about task-oriented sensorimotor schemes students should develop in solving challenging bimanual motor-action problems. So the issue at hand is the hands' movements.

Goldstone and Son (2005) maintain that manipulatives combining concrete and abstract features facilitate students' learning and transfer better than those using uniform (e.g., only abstract) features. Similar, the tasks we used also combine elements of variable appearance. However, one might wish to bring into question the very dichotomy of concrete and abstract features. Per the constructivist approaches, concreteness is not an ontological trait but a phenomenological marker (q.v., Wilensky, 1991)—concreteness is the *result* of each student's inferential reflection on the movements of their own body, where action thus provides vital entry into the learning situation. We differentiate this sense of phenomenological concreteness from the concreteness of the icon per se, which in this case may constitute a source of superficial situatedness.

Our work bears implications for designing technologically enhanced embodied learning environments (see Lindgren & Johnson-Glenberg, 2013). Abrahamson and Lindgren (2014) called for further research to ascertain best principles governing designers' engineering of interactive materials, and in particular virtual manipulatives. The results of our study point to contextual advantages of generic manipulatives for the facilitation of anticipated learning outcomes toward conceptual understanding, at least in the realm of proportional thinking. Future work could examine how best to harness the affordances of situated manipulatives without interfering with the development of desired sensorimotor schemes. The field needs a deeper understanding also of cases where situatedness orients students toward *productive* engagement of instructional materials yet in so doing also narrows the scope of meanings students bring to bear (Abrahamson et al., 2014). Further research is necessary to understand how best to implement in classrooms technological media that enable students to enter conceptual domains by developing new coordinations toward new objects (e.g., see Negrete, Lee, & Abrahamson, 2013).

Learning is moving in new ways, and we should ensure that the tasks we create facilitate this moving. The perfunctory layering of contextual cues onto the objects learners are to manipulate might hit the 'engagement' goal yet in so doing quash the 'learning' goal (see also Abrahamson, 2015). In fact, sometimes the objects children manipulate might be so perceptually impoverished that there are no objects at all—just imagined objects. One might speak of mathematics students' right to bare arms.

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