

EMERGENT ONTOLOGY IN EMBODIED INTERACTION: AUTOMATED FEEDBACK AS CONCEPTUAL PLACEHOLDER

Dragan Trninic
University of California at Berkeley
trninic@berkeley.edu

Dor Abrahamson
University of California at Berkeley
dor@berkeley.edu

Recent theories of cognition model human reasoning as tacit simulated action. Implications for the philosophy, design, and practice of mathematics instruction may be momentous. We report on findings from a pioneering design-based research study into the embodied roots of proportional reasoning that also explored the pedagogical potential of embodied-interaction (EI), a form of technology-enabled immersive activity. 22 Grade 4-6 individual/paired interviewees remote-controlled virtual objects in a non-symbolic space to solve a problem, then progressively mathematized their strategy using symbolic artifacts interpolated into the space. Drawing on qualitative analyses of the filmed work, we build a sociocognitive account of the role of automated feedback in the mediated construction of perceptuomotor schemes that undergird conceptual development, and we offer a heuristic EI design framework.

Introduction

Over the recent decades, cognitive scientists, psychologists, and philosophers have begun to increasingly question theories of cognition that model the mind as a symbol processor. Alternative “embodied” or “enactive” theories suggest that sensorimotor interaction in the natural and sociocultural ecology deeply shapes the mind—even thinking with or about “abstract” ideas is in fact the mental simulation and coordination of multimodal schematic image schemas (for a recent survey, see Barsalou, 2010). In particular, embodied cognition has been presented as a useful framework for both theorizing mathematical reasoning and designing pedagogically effective learning environments (Abrahamson, 2009; Nemirovsky & Ferrara, 2009; Núñez, Edwards, & Matos, 1999).

Parallel to the rise in popularity of theories of embodiment is the dramatic recent progress in technological affordances for embodied interaction (e.g., Nintendo Wii and Playstation Move, iPhone 4, and Xbox Kinect). Innovative designers tuned to this progress are constantly devising ways of utilizing this commercial technology in novel ways that serve a diverse audience of researchers and practitioners (e.g., Lee, 2008). As such, media that only recently appeared as esoteric instructional equipment will imminently be at the fingertips of billions of prospective learners. And yet, *What forms should “embodied” learning take? How should we theorize such learning? What are best design principles for fostering embodied interaction?*

In what follows, we discuss embodied interaction (EI) as bearing unique affordances for mathematics teaching and learning as well as research on this process. We then demonstrate these affordances by presenting an EI design for proportions as well as vignettes from implementing this design. The vignettes were selected so as to contextualize a proposed sociocognitive view on EI design: educators use EI first to foster student development of a targeted perceptuomotor scheme, then to guide student appropriation of mathematical forms as means of redescribing this scheme in accord with disciplinary practice and parlance. Finally, we offer an emerging heuristic design framework for EI mathematics instruction.

Embodied Interaction in the Research and Practice of Mathematical Learning

EI is a form of technology-supported training activity created, implemented, and researched by scholars interested in investigating multimodal learning. Through engaging in EI activities, users build schematic perceptuomotor structures consisting of mental connections between, on the one hand, physical actions they perform as they attempt to solve problems or respond to cues and, on the other hand, automated sensory feedback on these actions. One objective of EI design is for users to develop or enhance targeted schemes that undergird specialized forms of human practice, such as mathematical reasoning. As is true of all simulation-based training, EI is particularly powerful when everyday authentic opportunities to develop the targeted schemes are too infrequent, complex, expensive, or risky. Emblematic of EI activities, and what distinguishes EI from “hands on” educational activities in general, whether involving concrete or virtual objects, is that EI users’ physical actions are intrinsic, and not just logistically instrumental, to obtaining information (cf. Marshall, Cheng, & Luckin, 2010). That is, the learner is to some degree physically immersed in the microworld, so that finger, limb, torso, or even whole-body movements are not only in the service of acting *upon* objects but rather the motions themselves become part of the perceptuomotor structures learned. Thus, all EI gestures are perceived as epistemic actions, even if they are initiated as pragmatic actions (cf. Kirsh, 2006). In EI, the learner’s body—its structure and action—becomes concrete instructional material. EI is “hands in.”

EI activities typically emphasize explorative perceptuomotor tasks and draw less on propositional or domain-specific reasoning (e.g., Antle, Corness, & Droumeva, 2009). Notwithstanding, EI activities may include standard symbolic elements, such as alphanumeric notation, diagrams, and graphs (e.g., Cress, Fischer, Moeller, Sauter, & Nuerk, 2010; Nemirovsky, Tierney, & Wright, 1998). Indeed, content-oriented EI activities are often designed explicitly to foster the guided emergence of domain-specific conceptual structures from domain-neutral perceptuomotor schemes.

We consider EI activities as creating useful empirical settings for research on guided mathematical ontogenesis. In particular, because EI begins with *a* mathematical hands-in problem solving, data from these activities bring out in relief micro-phases of a learning trajectory that may simulate and generalize to all mathematical development: transitioning from unreflective orientation in a multimodal instrumented space to reflective mastery over the disciplinary re-description of this acquired competence. We thus propose to merge enactive and sociocultural theory to investigate how social interaction steers individuals to leverage perceptuomotor competence in appropriating mathematical forms of reasoning; more broadly, how learners come to embody, inhabit, and signify epistemic practice mediated through guided participation in cognitively demanding social activity (cf. Roth, 2009).

Our paper re-analyzes data from a recent study, in which we investigated an instructional methodology for scaffolding the emergence of proportional reasoning from EI problem-solving activities. Our analyses implicate the vital role of natural discursive modalities, such as verbal and gestural utterance, as well as mathematical semiotic artifacts, such as a virtual Cartesian grid and numerals, as the means by which students re-describe their entrained perceptuomotor enactment in disciplinary form (cf. Edwards, Radford, & Arzarello, 2009). Moreover, we found that these re-descriptions can take surprising, pedagogically useful directions, as students discover *in situ* and *ex tempore* better ways of using the symbolic artifacts so as to enact, explain, or evaluate their task strategy (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, in press). As such, we see our work as expanding on neo-Vygotskian conceptualizations of appropriation (e.g.,

Bartolini Bussi & Mariotti, 2008; Sfard, 2002). Namely, we implicate learners' creative *reappropriation* of symbolic artifacts in ways that are *not* modeled by the instructor yet students nonetheless discover as semiotic–enactive affordances.

Finally, as reflective designers we also wish to contribute to the theory and practice of EI-based mathematics instruction, which we view as bearing promise. We therefore conclude this report with a summary of our current heuristic design framework for mathematics learning activities in learning environments availing of affordances unique to EI technology.

Design and Implementation of the Mathematical Imagery Trainer

Our design conjecture, which built on the embodied/enactive approach discussed above, was that some mathematical concepts are difficult to learn because mundane life does not occasion opportunities to embody and rehearse particular schemes that constitute the requisite cognitive substrate for meaningfully appropriating these concepts' numerical procedures. Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—"fixed difference" solutions (e.g., " $2/3 = 4/5$ " - Lamon, 2007)—indicate students' lack of multimodal action images to ground proportion-related concepts (Pirie & Kieren, 1994). Accordingly, we engineered an EI inquiry activity for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the Mathematical Imagery Trainer, which we introduce below (MIT - see Figures 1&2, below, and for detailed descriptions of the device's rationale and technical properties, see Abrahamson et al., in press; and Howison, Trninic, Reinholz, & Abrahamson, 2011, respectively).

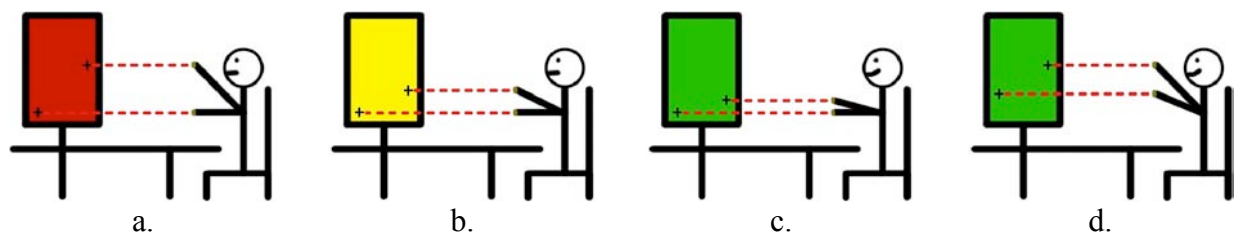


Figure 1. MIT interaction schematics, with the device set at a 1:2 ratio, so that the right hand needs to be twice as high than the left hand: (a) incorrect performance (red feedback on exploratory gestures); (b) almost correct performance (yellow feedback); (c) correct performance (green feedback); and (d) another correct performance.



a.

b.

Figure 2. MIT in action: (a) “incorrect” enactment turns the screen red; and (b) “correct” enactment turns the screen green. See www.tinyurl.com/edrl-mit for a 5 minute video clip showing the MIT in action.

The MIT measures the height of the users’ hands above the desk. When these heights (e.g., 10’’ & 20’’) match the unknown ratio set on the interviewer’s console (e.g., 1:2), the screen is green. So if the user then raises her hands proportionate distances (e.g., to 15’’ & 30’’), the screen will remain green. Otherwise, it will turn red (e.g., raising equal distances to 15’’ & 25’’). As such, this MIT is designed to hone pre-numerical struggle around the additive/multiplicative tension commonly implicated in the literature as underlying student challenges in moving into rational numbers (Lamon, 2007). Study participants were tasked first to find green then to maintain it while moving their hands. The protocol included layering a set of mathematical artifacts onto the display, such as an adaptable Cartesian grid (see Figure 3c, below), to stimulate progressive mathematization of emergent strategies.

Participants included 22 students from a private K–8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students). Care was taken to balance for students of both genders from low-, middle-, and high-achieving groups as ranked by their teachers. Students participated either individually or paired in a semi-structured clinical interview (duration of mean 70 min.; SD 20 min.). Interviews consisted primarily of working with the MIT. At first, the condition for green was set at a 1:2 ratio, and no feedback other than background color was given (see Figure 3a; we used this challenging condition only in the last six interviews). Then, crosshairs were introduced (see Figure 3b): these virtual objects mirrored the location of participants’ *hands* in space yet, so doing, became the objects users acted *on*, then *through*. Next, a grid was overlain on the display (see Figure 3c) to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions. In time, numerical labels “1, 2, 3,…” were overlain along the grid’s *y*-axis (see Figure 3d): these enabled students to construct further meanings by more readily recruiting arithmetic knowledge and skill so as to distribute the problem-solving task.

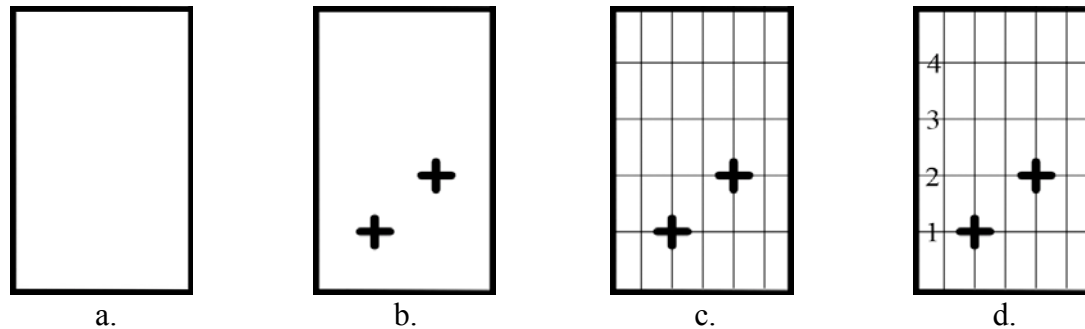


Figure 3. Display configuration sketches: (a) continuous-space mode; (b) continuous-space mode with crosshairs, i.e. virtual objects users manipulate; (c) crosshairs with grid overlay; (d) crosshairs with grid overlay and y-axis numerals.

For the purposes of this paper, we focus on two brief episodes demonstrating milestones in students' instrumenting their perceptuomotor solution strategies then redescribing them.

Results and Discussion

All students ultimately succeeded in devising and articulating strategies for making the screen green, and these strategies were aligned with the mathematical content of proportionality. We observed minor variation in individual participants' initial interpretation of the task as well as consequent variation in their subsequent trajectory through the protocol. However, by and large the students progressed through similar problem-solving stages, with the more mathematically competent students generating more strategies and coordinating more among quantitative properties, relations, and patterns they noticed.

Each student began either by working with only one hand at a time, waving both hands up and down in opposite directions, or lifting both hands up at the same pace, possibly in abrupt gestures. They soon realized that the actions of both hands are necessary to achieve green and that the vertical distance between their hands was a critical factor. Importantly, all children initially moved their hands at a fixed difference, certainly a legitimate, reasonable strategy.

The following data excerpts will sketch how we used the MIT-based design first to foster student development of a perceptuomotor scheme centered on obtaining "green" feedback and then to leverage their skill in mediating its mathematically instrumented re-descriptions.

We begin with a 6th-grade male student, Penuel, who took longer than others in realizing that the relation between the hands' respective positions is a critical task-relevant quality.

Penuel: So it looks like... they have to be a certain distance away from each other for it to turn green...and if it's not a certain distance, it's not green, it's yellow or red.

Penuel then identified that the positions of the hand should be reinterpreted as magnitudes. That is, he re-saw the location of each hand in space in terms of how high that hand is above the desk, so that empty space below each hand took on the palpability of virtual substance.

Penuel: Well, they obviously can't be at the same distance [above the desk]. But if I start here, and if the right one is moving, like, a little faster, and it's going farther and farther away from the left hand, it will still stay green.

Note references to velocities. Later still, once the grid and numerals were introduced, Penuel quantified his sense of "moving... a little faster" as a proto-ratio by noticing a distinctly mathematical pattern emerging from green locations. Prompted to sum his discovery, he said:

Penuel: You start from the ground [indicates desk], you try to get to the first green... you have to have one... left hand on “1,” right hand on “2.” Exactly. And you start from there, and you keep doubling it.

Like all students, Penuel was prompted to “make the screen green.” As he interacted with the MIT, and through our interview prompts, “green” transformed from an *objective* to *feedback*, as seen by his observations about the “correctness” of types of movements that elicit green. Finally, this feedback enabled him to discern a *mathematical* notion (“doubling”) from the set of hand locations eliciting green. This pattern of emergence of mathematical meaning was common to all the students interviewed. Here we can provide only one more illustrative case.

Liat, a 5th-grade female student, exhibited a telltale indication of conceptual transition: mismatch between gestured action and verbal explanation (Church & Goldin-Meadow, 1986). She consistently moved her hands up in a fixed-distance motion, received red feedback, and adjusted the left hand down for green, yet she stated a fixed-distance strategy.

Liat: I think if I keep them apart and keep going up, it stays the same...

Int: If you keep them apart and you keep going up it stays the same?

Liat: It's not becoming red, but...

Int: So... how are you thinking about keeping them apart?

Liat: Oh maybe it's more. If it's farther up, then it has to be...they have to be more apart.

Later, upon the introduction of the grid and numerals, Liat was asked to predict green locations without moving her hands. She noted “one row” in between the crosshairs when the left hand is at 1 and the right hand is at 2, making green. She extended the thought:

Liat: And if you go... 10... if you go up to 10, there's gonna be like 4 or 5 rows. [i.e., if the right hand is at 10, the left should be 4-5 rows lower so as to make green.]

Thus, Liat was able to instrumentalize the grid to enhance her previous qualitative strategy for green, namely that the “farther up” her hands are, the “more apart” they ought to be. However, the 1:2 ratio was yet to become articulated, as her guess indicated she was still thinking in terms of approximate magnitude, “4 or 5,” rather than relying upon more powerful mathematics (e.g., half of ten is five). Yet here precisely came the moment of guided transition to the more powerful mode of reasoning, multiplicative relations, as seen from the following exchange, where the interviewer asked her to decide between 4 and 5:

Liat: No... five!

Int: Five? How did you do that so quickly? How did you know it was five?

Liat: Half of ten is five.

Later, during the post-interview debriefing, Liat reflected on the activity of finding green.

Liat: It's not just moving hands... it's... [Liat moves her hands up and down, grasping for words]... it's... you're trying to do something and get the number.

In the ensuing discussion, Liat said that, at first, the activity was “not easy” yet that “actually, now it's easier” because she figured out how it works mathematically. Thus Liat, like Penuel, *comes to re-describe her newly developed skill of “green finding” via mathematics*, concurrently using green as an objective, feedback, and “conservation” for an ontogenesis of proportion. This pattern, common to all our study participants, suggests certain stable affordances of EI design for mathematics learning, as we discuss below in the closing section.

Conclusions and an Outline for a Heuristic Embodied-Interaction Design Framework

Even as learning scientists are increasingly accepting a view of mathematical reasoning as multimodal spatial-temporal activity, technological advances and free-market forces bode an impending ubiquity of personal devices capable of utilizing remote-embodied input. Poised

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between theory and industry, design-based researchers are only beginning to wrap their minds around the protean marriage of embodied cognition and remote action. Embodied interaction, a form of physically immersive instrumented activity, is geared to augment everyday perceptuomotor learning by fostering cognitive structures that leverage homo sapiens' evolved capacity to orient and navigate in a three-dimensional space, wherein the brain developed *by* and *for* action. We currently have far more questions than answers respecting both the prospects of EI and principles for best design and facilitation of these innovations. Our strategy has been to engage in conjecture-driven cycles of building, testing, and reflecting. In this spirit, the current paper aimed to share our excitement with EI and offer some early observations and caveats. Yet, as often occurs when new media are encountered, we learn as much about *pan*-media practice as about the new media per se. As such, the “ontological innovations” we have stumbled upon through our design-based research appear to bear more generally on how people do and could learn (cf. diSessa & Cobb, 2004).

Our analyses depicted learning as the evolution of users' subjective meaning for the automated feedback and, in particular, the role of perceptuomotor scheme as a vehicle or platform for the mediation of conceptual development. EI automated feedback (such as “green” in our design) evolved in the functional and cognitive roles it played. That is, green: (a) began as the task *objective*; (b) soon became the perceptuomotor *feedback*, as the users attempted to complete the task objective; and (c) came to hold together a set of otherwise unrelated hand-location pairs sharing a common effect and begging a name. Feedback on perceptuomotor performance thus came to demarcate the set of “green” number pairs as all belonging to an emerging *ontology*—a phenomenal class connecting seemingly disparate conceptions, observations, and hunches. To the expert, this emergence is centered around the concept of proportion; the student, however, is merely trying to “make the screen green” and explain exactly what makes it green. It is in this sense that green functioned as more than an objective or feedback—it served as an ontological scaffold or conceptual placeholder. Through appropriate facilitation, the scaffold ultimately collapses, or the placeholder is filled, once users determine the activity's mathematical rule and recognize the rule's power for anticipating, recording, and communicating the MIT's solution procedure. As one child gleefully quipped, upon determining the multiplicative relation of an unknown ratio, “I hacked the system!” Similar, when Penuel referred to “green” as the objective of his strategy (see his last excerpt, above), it is evident in his response that he has populated the notion of green with appropriate mathematical machinery needed to explain and predict “green.”

EI thus creates arenas for launching mathematical learning trajectories from body-based qualitative notions. As the students engage in problem-solving our MIT mystery device, their physical actions inscribe a “choreographed” form with increasing deftness—forms that are very difficult or perhaps impossible to mediate outside of EI design. Whereas the student views these forms as physical solution procedures, the educator—who views the forms from the vantage point of an expert's disciplinary perspective—conceptualizes these forms as the multimodal image schema underlying the cognition of the targeted concept. Using representational resources and discursive guidance, the instructor may then steer students to progressively signify these image schema into what become concept images of the emerging mathematical ideas. That is, even as the gestured forms lend meaning to mathematical propositions, they take on the epistemological role of metaphorical simulations (concept-specific “math kata,” if you will). In practice, the mathematical concept emerges when students utilize new mathematical symbolic artifacts, which the instructor introduces into the problem-solving space, as means of enacting,

explaining, enhancing, and/or evaluating their solution procedure. As such, mathematical knowledge emerges through recognizing a particular cultural form (e.g., the Cartesian grid) as contextually useful tools. Specifically, students utilize these available forms to articulate their physical solution procedures, first multi-modally (verbally and gesturally) and then also symbolically (by utilizing numerical inscriptions). Initially, these articulations are naive and qualitative, but they progressively adhere to mathematical forms via situated ascension from the physical to the mathematical.

In sum, we view EI as bearing the capacity of supporting transformative teaching and learning. Specifically, EI enhances the implementation of visionary design frameworks, by which students should begin inquiry into complex mathematical concepts from presymbolic action-based quantitative reasoning (Forman, 1988; Thompson, 1993). More generally, we submit, EI activities constitute rewarding empirical contexts for research aiming to deepen and expand our field's understanding of how instructors discipline learners' perception of a shared domain of scrutiny (Stevens & Hall, 1998). Whereas our work is in its early stages and our conclusions tentative, we hope to have conveyed some enthusiasm over EI's unique instructional and theoretical affordances. Our future work will compare our results with non-EI interventions and continue to seek improvements in both theory and design, availing of recent hands-free EI development.

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