A KEY PROBLEM: PEDAGOGICAL TRADEOFFS ALONG FAMILIAR AND GENERIC DIMENSIONS

Dragan Trninic and Dor Abrahamson
University of California at Berkeley, United States of America
trninic@berkeley.edu

The design and selection of pedagogically effective problem situations is a critical yet undertheorized facet of reform-oriented mathematics education. Drawing on instructional-design frameworks and cognitive-science theory, we propose a situation taxonomy centered on the dimensions "familiar" and "generic" and hypothesize the contrasting learning affordances of exponents thereof. 51 undergraduate students solved either a familiar or a generic version of a compound-probability problem, and subgroups thereof then participated in semi-structured clinical interviews. Familiar problems evoked common sense, yet their treatment was liable to be mathematically non-normative; generic problems triggered mathematically suitable treatments, yet these treatments were liable to remain opaque. We discuss implications of these tradeoffs with an eye to fostering mathematical-probabilistic literacy that is both powerful and grounded.

BACKGROUND AND OBJECTIVES

The study reported in this paper was motivated by a concern that extant methodology for designing instructional mathematics problems is by-and-large undertheorized, resulting in some researchers' and practitioners' suboptimal interpretation of reform-based philosophy and recommendations, ultimately to the detriment of students. This paper was thus written with the intent of opening up a space for discussion of the potential pedagogical tradeoffs inherent in the selection of "everyday" situations perceived as embodying curricular content. In particular, we offer two dimensions—generic and familiar—by which we propose to characterize problem situations with respect to the resources that students leverage in initially solving them.

Our analysis both embarks from and elaborates on thematic constructs underlying the Realistic Mathematics Education (RME) pedagogical framework (Gravemeijer, 1994). RME traces its roots to the notion of mathematics as an activity (Freudenthal, 1973). In contrast to the then-prevalent educational view of mathematics as an organized, ready-made system, Freudenthal posited mathematical activity itself as central to the learning and doing of mathematics. This stance oriented RME towards the creation of hypothetical learning trajectories along which students would be guided towards "reinventing" formal mathematics through the activity of mathematizing realistic (read: realizable, imaginable) problems.

The starting point of RME learning trajectory is students' common sense; thus, "starting points for instructional sequences will often link up with everyday-life experience of students" (Gravemeijer & Doorman, 1999). It is these everyday-life experiences, and in particular their affordances as context for realistic problem-based mathematical probabilistic learning that, we submit, are undertheorized.

This paper departs from assumptions implicit to RME by putting forth the conjecture that not all realistic situations are created equal. Specifically, we conjecture that problems emerging from *familiar situations*—contextually rich situations learners recognize as typical of their lived experience—may paradoxically enfold a covert challenge for the process of mathematization. On the other hand, problems emerging from *generic situations*—minimalistic, canonical situations imagistically resonant with the underlying "conceptual metaphor" inherent to the embedded mathematical problem (Lakoff & Johnson, 1980)—are likely to cue the novice as requiring a particular well-versed mathematical treatment which, we posit, may prevent deeper understanding.

THEORETICAL PERSPECTIVES

It has been argued that all human reasoning, mathematical reasoning included, is embodied in the simulation of everyday experience or fragments or facets thereof (Barsalou, 2008). From this "grounded cognition" perspective, everyday experience and mathematics are intrinsically related through common cognitive mechanisms. However, students are liable to develop mathematical knowledge as epistemologically compartmentalized from their common sense, unless pedagogical

In C. Reading (Ed.), *Data and context in statistics education: Towards an evidence-based society. Proceedings of the Eighth International Conference on Teaching Statistics (ICOTS8, July, 2010), Ljubljana, Slovenia*. Voorburg, The Netherlands: International Statistical Institute. www.stat.auckland.ac.nz/~iase/publications.php

efforts are made to foster cognitive synthesis (Schön, 1981). Neurological research supports this view, in particular that "the system's connectivity becomes inseparable from its history… and related to the kind of task defined for the system" (Varela et al, 1991).

The paradoxical tradeoff of situating mathematical content could be partly explained by the *dual representation* hypothesis, by which learners must construct symbolical artifacts (e.g., a manipulative) both as objects onto themselves and as signs that are semiotically related to their referent (DeLoache, 2000; Gibson, 1979; Ittelson, 1996). Empirical findings (DeLoache, 2000) point to an inverse relationship between these two perceptions of an object: the more emphasis is placed on viewing a symbolical artifact as an object in its own right, the less likely is it to be perceived as a representation of something else, and vice versa.

We believe that the implications of this dual representation hypothesis are not limited to symbolical artifacts. In particular, any problem situation drawing on everyday contexts is: (a) an *a*mathematical, everyday-lived situation; and (b) a mathematical problem to be solved. Indeed, Fischbein (1987), commenting on the treatment of quantitative inferences of Tversky & Kahneman (1982), proposes that, amid conflict and ambiguity, "one tends not only to close the search for arguments, one tends to close the debate as early as possible". Interpretations and extrapolations are made in conformity with one's experiences: thus a familiar problem situation may be implicitly compartmentalized by the student as intrinsically *a*mathematical and thus resist interpretation as a mathematical problem to be solved. Consequently, we initially conjectured that a familiar setting may "close the debate" necessary for mathematical treatment of a problem situation by making salient enactive features germane to practical, but not mathematical, treatment.

In contrast to the familiar dimension, the generic dimension captures problem situations that are largely removed from the "mess" of everyday life—generic situations incorporate a visual presentation of the bare representational necessities without extraneous information. For example, consider the frequently used probabilistic thought experiment of drawing colored balls from an urn. We consider this exemplar to be generic as it captures the complex conceptual metaphor recruited in thinking about random events—that of selecting blindly from a collective of objects. Due to such cognitive and pragmatic affordances, we hypothesized, generic situations—in contrast to familiar situations—are more likely both to cue and afford normative mathematical treatments. As we shall see, the drawback of such cueing is that it may paradoxically impede deeper learning.

To evaluate this hypothesis, we compared students' behaviors as they attempted to solve a probability problem embedded in two mathematically commensurate situations, a familiar situation and a generic situation. We predicted that whereas the familiar situation would evoke non-normative yet robust common sense, the generic situation would trigger normative yet disconnected thinking.

PARTICIPANTS AND PROCEDURE

Fifty-one undergraduate students in an introductory statistics course at a large public university participated voluntarily in the study. The course itself is designed for students with little or no background in probability and statistics. Data were collected halfway through the semester, prior to any treatments of probabilistic thought experiments. No screening was conducted.

In order to test our conjectures with respect to generic and familiar settings, we presented students with two phenomenologically disparate yet mathematically isomorphic problem situations. For a generic problem situation, we created the following Urn Problem:

> "An urn contains four white marbles and one red marble. If you begin drawing the marbles from the urn (without placing them back in) what is the chance that the fourth marble drawn will be the red marble?"

For a familiar problem, we created the Key Problem:

> "You are attempting to open a door, but do not recall which key opens the lock. You have 5 keys and know that only one will open the lock. If you decide to try each key in turn, what is the chance that the fourth key you try opens the lock?"

The normative answer to both problems is 1/5, typically obtained by multiplying the probabilities of "not first, not second, not third, yes fourth": (4/5)(3/4)(2/3)(1/2) = 1/5.

To distribute potentially confounding variables as evenly as possible, participants were initially categorized by mathematical capability, then randomly assigned to either the "familiar problem" or "generic problem" condition and given up to thirty minutes to work individually on their assigned problem. They were then asked to write out their solution procedure and were encouraged to explain their reasoning in a brief paragraph. In addition, six students were subsequently interviewed as a means of probing the reasoning behind their responses.

RESULTS AND DISCUSSION

Whereas 22/25 Urn Problem participants provided the normative answer (through the normative computation or otherwise), only 17/26 Key Problem participants provided the normative answer. Analysis revealed a statistically ambiguous difference between the study groups with respect to the distribution of their responses between normative and non-normative ($p = 0.06$, Fisher's exact test, one tailed). However, we noticed that 6 Key Problem participants presented "1/2" as their answer, whereas only 1 Urn Problem participant did so as shown in Table 1.

Table 1. Students' Responses on the Urn and the Key Problems

	Answer	
	1/5	1/2
Urn	22	1
Key	17	6

While inconclusive, the quantitative analysis suggests a closer look at the answer "1/2." Indeed, our subsequent qualitative analysis of students' reasoning and interviews appears to support our hypothesis concerning the treatment of familiar and generic problems.

On the one hand, the tendency for students working on the Key Problem to answer "1/2" is, at least in part, a consequence of that problem's familiar setting recruiting students' practical, rather than mathematical, treatment. Consider the following statement, typical of students answering "1/2" on the Key Problem:

> Olivia: If I open the door on the fourth try, that means I only have two keys left… and if I just have two keys, then I have one-in-two chance of using the correct key.

On the other hand, when asked whether they prefer to imagine the key or the urn situation for the purposes of solving the problem, the interviewed participants indicated unanimous preference for the urn problem, identifying the urn situation as "easier" to model mathematically. Thus, as we had conjectured, familiar situations are liable to be treated practically rather than mathematically, whereas generic problems are more likely to invite and afford a mathematical treatment. Yet our quantitative analysis illuminated an additional pair of results.

First, our data support the reform-based notion that familiar problem situations tap into students' common sense in a productive manner. In particular, we believe that the experience of a breakdown between practically effective and mathematically normative treatments introduced by engaging familiar setting provides an important pedagogical opportunity. Despite initial difficulty treating the key problem mathematically, interviewed students reported gains in both confidence and ability after resolving this conflict. This finding identifies familiar settings as suitable and even advantageous candidates for problem settings in introductory probability, with the caveat that the instructor be aware of the aforementioned conflict between the practical and mathematical. Moreover, teachers should be aware of the socio-mathematical discursive norms underlying normative reading of verbal problems and aim to foster these reflectively.

Second, our data indicate that generic problems may present an unanticipated, covert challenge. Consider the following participant response, typical of Urn Problem interviewees:

> Mirko: Yeah, it's easy to think of this [the urn problem]… and so I was pretty sure I was right, but I guess it's kinda … I'm not sure why my answer works, basically.

Mirko seems to experience what Wilensky (1997) calls *epistemological anxiety*—a feeling that one does not comprehend "the meanings, purposes, sources or legitimacy of the mathematical objects one is manipulating and using". It thus seems that generic problem situations appear to a novice as belonging-to-mathematics, inviting a mathematical treatment, yet at the same time this treatment may be at the expense of students' attempts to make sense of the mathematical content. This result is surprising if one expects generic problems—being close to the conceptual metaphor—to recruit students' common sense. We conjecture that students' careers as solvers of generic problems of the school genre teach them to compartmentalize such problems from their everyday common sense (Lave, 1992; Verschaffel et al., 2009).

Thus the issue is one of tradeoffs. Classroom application of familiar and generic problem situations brandish the proverbial double-edged pedagogical sword (Mauks-Koepke et al., 2009). Whereas familiar problem situations may encourage students to enhance their common sense with mathematics, thus leading to deep understanding, familiarity with the situation may impede the very cognitive orientation requisite for solving the problem. Whereas generic problem situations may invite and afford mathematical treatments, they may be pre-compartmentalized by the student as belonging-to-school and thus fail to recruit students' common sense in a manner that could lead to deep learning.

Though our work is in its early stages, we believe that the theoretical constructs we introduce and questions our preliminary empirical findings raise are sufficient to engender productive scholarly discourse.

REFERENCES

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617-645.

DeLoache, J. S. (2000). Dual representation and young children's use of scale models. *Child Development, 71*(2), 329-338.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht: D. Reidel Publishing Company.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: D. Reidel Publishing Company.

Gibson, J. J. (1979). *The ecological approach to visual perception*. Boston: Houghton Mifflin.

Gravemeijer, K. (1994). *Developing Realistic Mathematics Education*. Utrecht: Freudenthal Institute.

Gravemeijer, K., & Doorman, M. (1999). Context problems in Realistic Mathematics Education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1), 111-129.

Ittelson, W. H. (1996). Visual perception of markings. *Psychonomic Bulletin and Review, 3*(2), 171-187.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago: University of Chicago Press.

Lave, J. (1992). Word problems: A microcosm of theories of learning. In P. Light & G. Butterworth (eds.), *Context and cognition: Ways of learning and knowing*. Hillsdale, NJ: Lawrence Erlbaum Associates, 74-92.

Mauks-Koepke, K. P., Buchanan, K., Relaford-Doyle, J., Souchkova, D., & Abrahamson, D. (2009). The double-edged sword of constructivist design. Paper presented at the *American Educational Research Association, San Diego, April 13-17.*

Schön, D. A. (1981). *Intuitive thinking? A metaphor underlying some ideas of educational reform (Working Paper 8).* Unpublished manuscript, Division for Study and Research, M.I.T.

Tversky, A., & Kahneman, D. (1982). Judgment under uncertainty: Heuristics and biases. In P. Slovic & A. Tversky (Eds.), *Judgment under uncertainty: Heuristics and biases* (pp. 3-20) Cambridge: Cambridge University Press.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge: The MIT Press.

Verschaffel, L., Van Dooren, W., Greer, B., & Mukhopadhyay, S. (2009). *Words and worlds: Modelling verbal descriptions of situations*. Rotterdam: Sense Publishers B.V.

Wilensky, U. (1997). What is normal anyway? Therapy for epistemological anxiety. *Educational Studies in Mathematics, 33*(2), 171-202.