

Virtual Mathematical Inquiry: Problem Solving at the Gestural–Symbolic Interface of Remote-Control Embodied-Interaction Design

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Abstract: What, if any, are unique affordances of embodied-interaction (EI) design for mathematics learning? How, in turn, might engineering and testing such design aid us in building theory of learning? We draw on media studies to argue that activities situated at the hybrid intersection of two media, for example physical-virtual interfaces, enable both learners and their researchers to objectify important notions. We evoke cultural-historical psychology theory, which implicates gesture as the ontogenesis of inscription, to interpret instructional EI as offering analogous enactive-semiotic tension and resolution. We support our view with selected empirical data from a design-based research project investigating the emergence of mathematical concepts from mediated engagement in EI activity. Specifically, we present, analyze, and juxtapose two case studies respectively illustrating student success or failure to grope toward a conceptual system (proportions) embedded in a technological system (EI).

The hybrid or the meeting of two media is a moment of truth and revelation from which new form is born....The moment of the meeting of media is a moment of freedom and release from the ordinary trance and numbness imposed....on our senses. (McLuhan, 1964, p. 63)

Objective: The Unique Affordances of Embodied-Interaction Pedagogical Design as Context for Research on Mathematics Learning

This is a theory-meets-design paper. The design in question is *embodied interaction* (EI), a form of technology-supported multimodal training activity. Through engaging in EI activities, users are expected to build schematic perceptuomotor structures consisting of mental connections between, on the one hand, physical actions they perform as they attempt to solve problems or respond to cues and, on the other hand, automated sensory feedback on these actions. Emblematic of EI activities, and what distinguishes EI from “hands on” educational activities in general, whether involving concrete or virtual objects, is that EI users’ physical actions are intrinsic, and not just logistically instrumental, to obtaining information (cf. Marshall, Cheng, & Luckin, 2010). That is, the learner is to some degree physically immersed in the microworld, so that finger, limb, torso, or even whole-body movements are not only in the service of acting upon objects but rather the motions themselves become part of the perceptuomotor structures learned. EI is not simply “hands on” but “hands *in*.” The theoretical issue we seek to address here is how the rising cognitive-science theory of embodied cognition (e.g., Barsalou, 2008; Hommel, Müsseler, Aschersleben, & Prinz, 2001) and related theoretical debates over the epistemic nature of enaction might influence design-based researchers’ conceptualization and implementation of EI designs. We contextualize this exploration in our recent efforts to create and research EI activities for mathematics learning.

As a point of departure, consider the following quotation from a recent *Interacting with Computers* special issue on “Enactive Interfaces,” where Chris Raymaekers characterizes research into enaction as “[striving] to create new types of interfaces, which make use of a user’s capacities to learn how a system works by using it... The computer is in this case a tool that is as invisible to the user as possible: the user interacts through the computer” (2009, p. 2).

In this paper, we—a research team investigating the nature of mathematics learning by developing and implementing innovative instructional technology—reflect on a recent EI study that availed heuristically of this alleged human capacity to “learn how a system works by using it.” In a sense, it is this very capacity that we design for, because we attempt to create artifacts that emulate principles of historically successful cultural mediations, such as children learning the numerical system through learning to count. In a recent study, we embedded a targeted mathematical system, proportions, as the normative interaction principle for operating a “mystery” technological device and studied whether and how students’ goal-oriented enactment in this EI microworld supports the guided development of commensurate conceptual systems. So doing, however, we encountered certain challenges in enabling students to “learn how a system works by using it.” As we will explain, the particular challenges in question here relate to very initial steps of getting “into” our EI media.

When instructional designers create artifacts, students may engage them in unexpected ways that lead them astray of the designers’ intended interaction trajectory. This is true of innovative electronic artifacts in general (Olive, 2000), possibly because these artifacts are historically new and so both designers and users are still developing an understanding of their affordances. For example, users of augmented reality technology who

manipulate tactile artifacts linked to virtual simulacra tacitly expect the virtual objects to interact in accord with their physical ontology, yet those anticipated affordances may not have been engineered into the interaction space (see Hornecker & Dünser, 2009). EI designs may preempt such false projection by incorporating generic remote controllers or even no controllers at all, such that no assumed ontology is evoked. Yet EI users nevertheless may still encounter challenges in negotiating the physical–virtual gulf. We will attempt, below, to illuminate these challenges of futuristic pedagogical tools by looking back to the works of Vygotsky and McLuhan as well as contemporary philosophers of Human–Computer Interaction (HCI). We thus expand on the CSSL 2011 call by connecting *outside* of the learning sciences to greater circles of inquiry. By looking back to move forward, we hope also to frame the enduring interdependences of educational design and theory.

Background: Learning as Appropriating Artifact-Bound Conceptual Systems

Much of research in education has been divided along a schism between “cognitive” and “sociocultural” perspectives. We posit that this split is a false dichotomy that has, to an extent, undermined our field’s efforts in developing and evaluating cogent theories of learning. Thus we are committed to a view of learning and instruction that sees the learner as a resourceful cognitive agent operating in a sociocultural sphere (cf. Karmiloff-Smith, 1988). Ultimately, we concur, “These are not optional perspectives, nor even strictly complementary ones. Each perspective completes the other, and in fact, they are probably not coherently construed independently” (diSessa, 2008, pp. 427-428). By investigating the ontological nature and pedagogical roles of educational artifacts, we hope to gain reliable purchase on the interplay of the cognitive and sociocultural, specifically because our designs are centered on creating opportunities for mediated discovery. In particular, EI design studies should be of interest to scholars who are developing models of learning informed by both cognitive and sociocultural theory, because the studies generate interaction data that bring out in relief mediation practices for the cultural signification of discipline-neutral schematic action. Through investigating EI-based mathematics learning, we are interested in theorizing how social interactions steer participants to leverage perceptuomotor competence in the appropriation of disciplinary forms of reasoning.

The intrinsically embedded, contextual nature of human artifacts, and specifically the ontologically requisite relations between artifacts and their engulfing normative application structures and activity systems, implies that these objects are never “objective.” Rather, artifacts might be better understood as the embodiments of certain cultural practices, condensed templates of theory-in-action of doing things as discovered in the history of human civilization (Stetsenko, 2002). As Leont’ev (1981) eloquently writes, the artifact “mediates activity and thus connects humans not only with the world of objects but also with other people. Because of this, humans’ activity *assimilates the experience of humankind*” (Leont’ev, 1981, p. 56, original italics).

Thus, adopting a sociocultural view, we espouse a position that learning is the residual effect of interacting with artifacts—or, more precisely, of engaging one’s own fluency-delimited mental construction of these artifacts—as means of accomplishing one’s goals (cf. Salomon, Perkins, & Globerson, 1991). However, in keeping with our dialectical commitment, we complement this view from a cognitivist perspective to promote a conceptualization of learning also as the iterative modification of existing goal-oriented schemes. That is, against the sociocultural backdrop view of *learning as imitating, internalizing, and appropriating* the elders’ artifact-mediated actions, we foreground a view of *learning as discovering* these artifacts’ horizons in the course of explorative problem solving, theory building, and creative inference making.

We propose that whereas the discovery of cultural knowledge entails coming to engage the world in newly instrumented ways, achieving these new ways depends on particular heuristics, such as pattern searching and cognitive offloading, that individuals draw on so as to empower their contextual grip on the world (Dreyfus & Dreyfus, 1999). That is, individuals do not see artifacts per se but rather tacitly *enframe* the world as *ad hoc* equipment for achieving their contextual goals (Heidegger, 1977). In turn, by virtue of learning to use this equipment, new views emerge for the learner in the form of utilization schemas inherent to disciplinary content (cf. Bartolini Bussi & Mariotti, 2008; Sfard, 2002; VÉrillon & Rabardel, 1995). Moreover, in the activity of applying these appropriated artifacts, people may reconfigure their strategies dramatically in ways that experts recognize as pedagogically desirable (Abrahamson, Gutiérrez, Lee, Reinholz, & Trninic, 2011). This conceptualization of mediated discovery as effortful yet somewhat inadvertent is epitomized in Vygotsky’s essay on the birth of inscription:

Gestures, it has been correctly said, are writing in air, and written signs frequently are simply gestures that have been fixed....children frequently switch to dramatization, depicting by gestures what they should show on the drawing; the pencil-marks are only a supplement to this gestural representation....Children do not draw, they indicate, and the pencil merely fixes the indicatory gesture. (Vygotsky, 1930/1978, pp. 107-108)

In this view, children’s inscriptional productions ought to be conceptualized as initially no more than fortuitous signatures of projected gestures onto an available canvas—not as deliberate, stylized expressive

depictions. Becoming literate thus consists of mastering the objectification of presymbolic notions in new semiotic systems embedded in a new media (cf. Radford, 2003; Saxe, 2004). We find this view of nascent literacy as transitional negotiations at the interface of semiotic systems particularly helpful in investigating technologically supported mathematics learning. Specifically, this paper scrutinizes empirical cases of students engaged in problem-solving activities who learn to appropriate virtual symbolic artifacts—a pair of crosshairs that mirror their hands on a computer display—as gestural extensions that amplify their inquiry. Building on sociocultural theory and its semiotic elaborations as well on cognitive and philosophical treatments of HCI, we ask, What interactive dimensions effectively characterize students' attempts to orient into virtual inquiry?

Data Source and Modes of Inquiry

The current study drew on data gathered in an ongoing design-based research study investigating the emergence of mathematical concepts from guided embodied interactional activities.

Our design conjecture, which built on the embodied/enactive approach (Lakoff & Núñez, 2000; Nemirovsky, 2003; Núñez, Edwards, & Matos, 1999), was that some mathematical concepts are difficult to learn because mundane life does not occasion opportunities to embody and rehearse the dynamic schemes that would form the requisite cognitive substrate for meaningfully appropriating the concepts' disciplinary analysis of situated phenomena. Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—"fixed difference" solutions (e.g., " $2/3 = 4/5$ " - Behr, Harel, Post, & Lesh, 1993)—indicate students' lack of multimodal action images to ground proportion-related concepts (Pirie & Kieren, 1994).

Accordingly, we engineered an EI inquiry activity for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the *Mathematical Imagery Trainer* (MIT; see Figure 1 and Figure 2; for detailed descriptions of the MIT's technical properties, see Howison, Trninic, Reinholz, & Abrahamson, 2011).

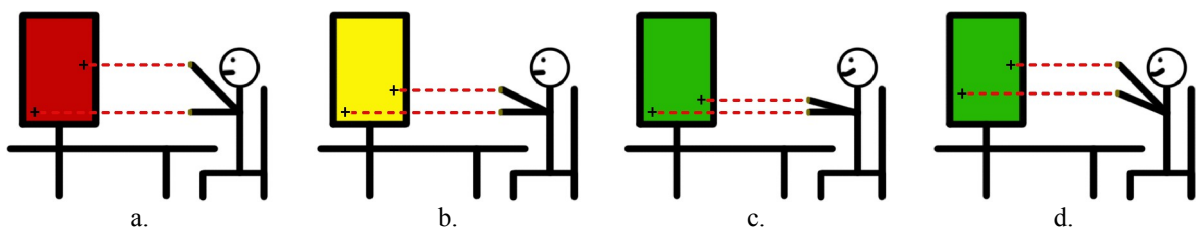


Figure 1. The Mathematical Imagery Trainer (MIT) set at a 1:2 ratio, so that the right hand needs to be twice as high along the monitor than the left hand: (a) incorrect performance (red feedback); (b) almost correct performance (yellow feedback); (c) correct performance (green feedback); and (d) another correct performance.



Figure 2. MIT in action: (a) a student's "incorrect" performance (raising both hands up away from a previous "green spot" at equal increments, i.e. fixed distance) turns the screen red; (b) "correct" performance (raising both hands up away from green at proportionate increments, i.e. different distances) keeps the screen green.

The MIT measures the height of the users' hands above the desk. When these heights (e.g., 10" & 20") match the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. So if the user then raises her hands in front of this "What's-my-rule?" phenomenon by proportionate increments (e.g., to 15" & 30"), the screen will remain green but will otherwise turn red (e.g., to 15" & 25"); note that this pair of heights is 5" higher than both the left and right hands, respectively, in the 10" & 20" case). Study participants were initially

tasked to “make the screen green.” Once the participants successfully effected a green screen, they were tasked to “find another green”; then eventually tasked with keeping the screen green *while* moving their hands.

The initial condition for green was set as a 1:2 ratio, and no feedback other than the background color was given (see Figure 3b; this challenging condition was used only in the last six interviews). Once the students displayed a degree of competence with “finding green,” the protocol introduced incremental layering of supplementary mathematical instruments onto this microworld. First, crosshairs were introduced that “mirrored” the location of participants’ hands (see Figure 3c). Next, a grid was overlain on the display monitor to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions (see Figure 3d). In time, the numerical labels “1, 2, 3,…” were overlain on the grid’s vertical axis to help students construct further meanings by more readily recruiting arithmetic knowledge and skills and more efficiently distributing the problem-solving task (see Figure 3e).

Participants included 22 students from a private K–8 suburban school in the greater San Francisco area (33% on financial aid; 10% minority students). Care was taken to include students of both genders from low-, middle-, and high-achieving groups as ranked by their teachers. Students participated either individually or paired in a semi-structured clinical interview (duration of mean 70 min.; SD 20 min.).

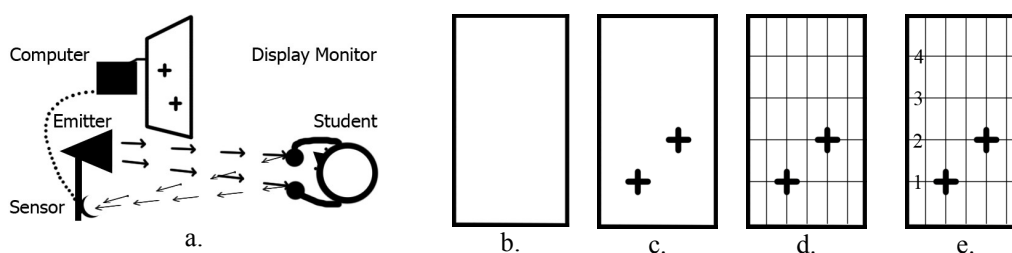


Figure 3. The Mathematical Imagery Trainer: (a) overview of the system featuring an earlier MIT version, in which students held tennis balls with reflective tape. Figures 3b – 3e are schematic representations of different display configurations, beginning with (b) a blank screen, and then featuring a set of symbolical objects that are incrementally overlain onto the display: (c) crosshairs; (d) a grid; and (e) numerals along the y-axis of the grid (in the actual design, the default range of the flexible grid and corresponding numerals was from 1 to 10).

Our investigation of the empirical data was conducted as collaborative, intensive microgenetic analyses (Schoenfeld, Smith, & Arcavi, 1991), and we applied general principles of grounded theory (Glaser & Strauss, 1967) to identify and articulate unfamiliar behavioral patterns in the entire data corpus. For this paper, we focus on two episodes, both of students navigating the early transition from a blank screen to a screen with interactive crosshairs (see Figures 3b&c, above; see our other cited publications for analyses of students’ later behaviors). We view these episodes as paradigmatic cases of learning environments wherein crossing the physical-to-virtual divide is required for students to orient into a symbolically enhanced domain of scrutiny with vastly augmented inquiry and representational affordances. In particular, these data enable us to examine how students are able to render visible artifacts “invisibles” so that they can immerse themselves in thinking “through the computer.”

Results: Two Data Episodes at the Gestural–Symbolic Interface

In this section, we present and discuss empirical data gathered around the introduction of the crosshairs. Whereas crosshairs per se are arguably *a*mathematical objects, their instrumentalization as virtual manipulation devices is a vital step towards progressive mathematization of the problem space, and, indeed, towards orienting into the problem space at all. As such, we view the introduction of crosshairs as instantiating a phenomenological breach of two dimensions, the embodied/physical/here (“I”) and the inscriptional/virtual/projected (“it”). As we observe in the data, this breach and how the student and instructor attempt to ford it may render visible some of students’ implicit framings of the situation. We now present two brief episodes: whereas the first showcases a user’s capacity to learn a system by using it, the second illustrates a “lower boundary” of the same capacity.

Fording the Gestural–Symbolic Divide

Asa is a 5th-grade male student indicated by his teachers as low achieving. In this study, Asa was paired with Kaylen, another 5th grader likewise indicated. Sitting side by side in front of the technology, they each hold and manipulate one of the two tracker devices and attempt to co-produce a green screen. Prior to the layering of the crosshairs into their interaction space, the boys were observed to alternate their perceptual attention back and forth between the hand-held devices and the computer display. They looked at their devices so as to note their relative positions (e.g., to maintain or change the distance between them), and they looked at the screen so as to note the color feedback (whether it is green or red). Even when they lifted their hands to the level of their eyes,

such that the devices were in the same line of vision as the screen, we still note the participants adjusting their gaze back and forth. JFG, a graduate-student researcher apprenticing through this study, facilitated this activity. The following transcript captures Asa's utterance *prior* to the introduction of the crosshairs.

Asa: <04:59> Oh I see. So I think what's going on is that they [referring to hand-held devices] have to be the same far... [RH thumb and index gestures a vertical interval] the same distance away from each other [RH, still holding the interval, gestures toward Kaylen's device].

Above, we note that Asa conjectured what we have called a "fixed distance" interaction conjecture, namely Asa believes that the distance between the hands should be invariant (see, below, Asa's "theory"). We also note Asa's use of the pronoun "it" in "...keep it the same distance," which suggests Asa has objectified the vertical interval between the two devices as the thing to be controlled. JFG now revealed the crosshairs on the screen.

JFG: <12:26> Can you see that?
 A/K: Yeah. [Some technology recalibration ensues for a few seconds]
 JFG: So let's see if these help us. [...] See if you can make the screen green.
 Asa: Well... [gazing at screen] So let's try to find green again. There. Stop! Yeah. I think my theory's right. Oh wait, but we both have to be *moving*, 'cause when we stop...
 JFG: You both have to be moving? What do you... What do you mean by that?
 Asa: Seems so... Oh wait, no, that can't be possible. [to Kaylen] Stop! [screen is still green] Yeah it's not true. Go up.

Asa temporarily considered a new theory, by which the crosshairs must be in continuous motion. He then quickly refuted this theory in light of the empirical data they gathered.

JFG: <13:55> You think you can find another green [gestures higher on screen] kind of up here...?
 Asa: My theory is obviously wrong.
 JFG: What was your theory before?
 Asa: My theory was that there was a specific [RH thumb/index again gestures vertical distance] height that they have to be from each other but [waves hand] that's wrong [shakes head].
 JFG: So what do you... What do you think now?
 Asa: Uhhm, I think that the height slowly increases. [Kaylen concurs]

In contrast to their alternating gazes back and forth between the hands and screen prior to overlaying the crosshairs on the screen, once the crosshairs appeared the children instantaneously gazed at them and thereafter never looked at their handheld devices while operating them. This transition from *here* to *there* led Asa to re-examine his previous fixed-distance conjecture; in turn, the increased precision afforded by this transition enabled Asa to gather data that he interpreted as rendering his conjecture "obviously wrong." Furthermore, the gestural-symbolic transition revealed one of Asa's implicit framings; namely, that devices/crosshairs "both have to be *moving*" so as to effect a green screen. At the point in the interview when the crosshairs were introduced, Asa had interacted with the artifact for approximately ten minutes with only modest progress along the dyad's inquiry into the mystery device. Then, over a very brief period of time *and without any expert modeling*, Asa was able not only to disprove two of his conjectures (fixed distances, moving), but also to produce a new conjecture, correctly noticing that the distance between the crosshairs should increase as they move up. In sum, Asa rapidly and seamlessly forded the gestural-symbolic divide and immediately availed successfully of the supplemental inquiry affordances in this virtually extended interactive problem space.

Stuck in the Gestural-Symbolic Divide

Next, we discuss the case of Boaz, a 5th-grade male student indicated by his teachers as "low achieving." The text below is a transcription from the early part of the interview, before the crosshairs had been revealed on the display. Boaz has been attempting to make the screen green. At the beginning of the transcript, he locates a "green pair" on the bottom of the screen. DR—another graduate-student research apprentice acting as interviewer—asks Boaz to explain his green-making strategy. In his response, Boaz will be referring to a "camera." This is not the video camera recording the session, but a sensor mounted on a tripod, part of the remote-action technology that picks up the infrared signal in the LED beam emitted by the handheld device the child operates. As we shall see, Boaz's idiosyncratic fixation on this device impeded his incorporation of the virtual objects and thus delimited the scope and efficacy of his inquiry into the mystery device.

- DR: <14:02> Well, then, *how* did you make it green? [gestures toward computer display]
- Boaz: *Oh!* By... uhhm...
- DR: Are you figuring the green?
- Boaz: By, uhhhm... [swerves body to the left, orienting torso away from the large monitor and toward the sensor; points toward sensor with both palms] By... So, I guess these two things right here [points to the devices that he placed on the desk] are showing the, uhhm [points to sensor] camera. Like [points back at devices] they're looking at the camera, and, I guess, those light things on it [refers to an obsolete LED array that is still mounted to the sensor, a relic of an earlier design cycle] each have a different color in each column. And so when they move it up or down or equal [mimes accordingly], each one, like, I'm, it's pinpointing....
- DR: <15:23> So what would you do to make it green?
- Boaz: I would... [operating in the lower range of the interaction space, just above the desk, Boaz lifts and lowers the two devices independently, monitoring for color on the large display; he stops when the screen turns green and holds his hands fixed at that position] ...try pinpointing it by moving them up and down 'til I made green.
- DR: Ok. So...[DR takes a mimed cue from DA] What if you tried to lift your hands up like this [lifts both hands *beside each other* into the upper ranges of the interaction space—a “neutral” gesture meant to mask actual locations] and keep it green? Do you know any way to do that?
- Boaz: [Boaz imitates by lifting his hands up *beside each other* above his shoulder height, still staying in the interaction space. The screen turns red] Oh! It would go out of the motion [gestures toward DR's laptop]. It would go out of the camera [gestures toward sensor].

In the course of explaining to Boaz the boundaries of the physical interaction space, DR mimed contiguous hand motions, which would not effect a green screen. Boaz, however, does not share DR's framing, and so he cannot differentiate between DR's intended phenomenal element of vertical up-and-down motion and epiphenomenal or tactical performance element of contiguous hands. Boaz thus literally imitates the contiguous motions—he assumes he was supposed to do just so, so that when he effects a red screen rather than the anticipated green screen, he concludes the approach must be *technically* in error, i.e., that he is out of detectable range. Surely, thinks Boaz, the teacher showed him precisely how to make the screen green, so that the red screen could not have resulted from the teacher's error but from something else. This is incorrect, we note, as in fact the devices remained well within the interaction space: the screen turned red because DR's instructive gestures deviated from the desired 1:2 ratio. Thus Boaz inadvertently conflated the system's technical and mathematical aspects—a legitimate, rational inference, but one that impeded his performance for the remainder of the interview.

Later on in the interview, when the crosshairs were eventually revealed, Boaz remained physically and perceptually oriented toward the sensor. Albeit he began tracking the crosshairs in his peripheral vision, his alternating “I”/“it” utterances suggest he wavered between his hands and projection thereof, between a gesture and its inscription, and thus between the situation's technical and mathematical aspects. In contrast to Asa, Boaz remained “stuck” between the two media, greatly impeding his chances to discover the embedded mathematics.

Discussion

In opening this paper, we stated that it was about theory-meets-design. It is particularly about embodied-cognition theory meets design-based research. Although both the theoretical models of embodied cognition and the methodological paradigms of design-based research are on the rise in the mathematics education community, critical questions remain unanswered regarding the interaction of the two—about implementation of embodied-cognition theory in practice and the use of design to evaluate and produce embodiment theory. We hold that one promising approach to understanding how these theory and design relate involves examining how users learn to extend their inquiry into virtual realms through appropriating the affordances of new EI systems.

Having introduced our design, we turned our attention to two episodes drawn from a recent design-based research study that investigated the emergence of proportional reasoning from EI activity. The first episode showcased how the introduction of crosshairs on the screen enabled a student to learn how a system works by using it. The second episode, in contradistinction, presented the case of a student who remained stuck between what we deem technical and mathematical dimensions of the situation to the detriment of his inquiry and learning. Whereas this latter student was the only such case in our data, this exception affords greater insight into the rule. How might we make sense of Boaz's inability to learn how a system works by using it?

The introduction of the crosshairs marks a critical incipience in the hybridization of the embodied and inscriptional spaces (cf. McLuhan, 1964)—when users first realize that their gestures are tracked in real time, they experience the crosshairs as functional extensions of their manipulations (cf. Gangopadhyay & Kiverstein, 2009). We believe that, akin to children in Vygotsky's study who, whilst drawing, transition from gesture to inscription, the introduction of crosshairs presents EI users with an opportunity to transition to a new medium. Children's drawings both mark their gesture and serve to create images; crosshairs likewise bear *two* potential

affordances: (1) interaction feedback for monitoring remote-interaction traction (i.e., “you are here”); and (2) strategic manipulation of the environment in accord with task demands (i.e., maintaining green). As designers, we rank these two maintenance goals respectively as: (1) peripheral (technical); and (2) central (e.g., mathematical). However, the designer’s discerning perspective does not necessarily transfer to the child, who perceives the entire system as one new whole. Indeed, our data suggest that whereas Asa reconciled the two maintenance goals, Boaz did not. On the contrary, Boaz’s actions indicate that, while utilizing the crosshairs’ affordance to indicate remote-control traction, he misses out entirely on their pivotal *strategic* relevance in effecting green. This inability to differentiate between central and peripheral utility is not unlike that of children in Vygotsky’s study, who “[w]hen asked to draw good weather...indicate the bottom of the page by making a horizontal motion of the hand, explaining, ‘This is the earth,’ and then, after a number of confused upward hatchwise motions, ‘And this is good weather’” (Vygotsky, 1930/1978, p. 108). To these children, as to Boaz, the central activity is in the gesture, not the projected inscription.

Vygotsky’s example of a child representing good weather suggests that Boaz’s behavior, an exception in our particular study, may in fact be learners’ *modus operandi* at the interface of physical and virtual media. More generally, it appears that users cannot learn how a system works by using it if they do not have some initial sense of what the system *is*—if they have not even begun to access its intended envelope of interactivity. And yet, systems such as the MIT interactive device afford as interactive potential not only designer-intended schemes, but also a host of unintended schemes, such as fixating on peripheral interaction (cf. Olive, 2000).

This, of course, is not meant to discourage EI design. Indeed, the remote-control technology giving rise to EI design bears an implicit yet profound affordance for instantiating progressive mathematization. In particular, it enables students to cross the embodied–inscriptional divide that is vital for making mathematical sense of immersive body-based extended functionalism. For example, crosshairs as virtual objects that mirror students’ hand positions bear an ambiguous semiotic status: students can experience these objects either as functional extension of the body (tips of “invisible canes”) or as data points expressing their embodied inquiry in a virtual reference field of measurement. Boaz is like a blindfolded person who has been offered a cane but has learned to use it only as a means of avoiding obstacles and not yet to “see” let alone engage the obstacles. In other words, whereas Asa incorporated crosshairs into his reasoning and, thus, reasons *through* them, to Boaz the crosshairs remain opaque, merely extensional location markers.

We are now able to provide some preliminary answers to the research questions guiding this work. First, what are the unique affordances of EI design for mathematical learning? By phenomenizing proportion into an interactive mystery device, we rendered mathematics learning as akin to empirical scientific inquiry. That is, in order to make the screen green, students need to figure out empirically the principles governing this interactive phenomenon that comes to exist only by doing. Second, how can EI design aid us in developing and articulating theory? EI design demands that users hybridize the embodied and inscriptional, and the consequent interaction transition lays open to scrutiny hitherto tacit cognitive activity. Namely, this hybridity presents itself as a double-edged sword: on the one hand, it provides users with an opportunity to link their inscription to gesture; on the other hand, the divide between the embodied and the inscriptional may prove too great—it may bog a student in the doldrums of the intended system’s peripheral features.

Yet our analysis offers insight toward answering another, third question: How might theory influence EI design? With respect to the issue of crossing the embodied–inscriptional divide, we advocate a markedly low-tech approach. Namely, we recommend looking back at the seminal work of Lev Vygotsky and his collaborators. They have already articulated the historical human achievement of fording the gestural–inscriptional media divide in the case of drawing on paper. This theoretical precedent, we submit, still holds for future studies into implementing equally successful media artifacts in the electronic age.

Whereas the empirical work explored here involved a relatively small group of children and treated brief interactions, the results suggest not only the methodological value of embodied-interaction design, but also future directions towards the development of effective computer environments for mathematics learning.

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