THE EVOCATION AND ENACTMENT OF CONCEPTUAL SCHEMES: UNDERSTANDING THE MICROGENESIS OF MATHEMATICAL COGNITION THROUGH EMBODIED, ARTIFACT-MEDIATED ACTIVITY

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When instructional designers develop content-targeted pedagogical situations, their practice can be theorized as engineering students' development of conceptual schemes. To account for the contributions of students' prior schemes and situated experiences towards their development of conceptual schemes, I suggest a distinction between the evocation of schemes and enactment of situations. The former suggests that the design of an instructional situation can activate students' prior schemes. The latter suggests that the structure of students' activity in an instructional situation determines the nature of newly constructed schemes. I contextualize my views using empirical data from a case study of early fraction instruction.

Background and Objectives

An established principle of reform mathematics education is to foster students' understanding of mathematical concepts and reasoning skills through problem-solving activities (e.g., NCTM, 2000). These activities are often designed as scaffolded interactions between teacher and student, whereby the teacher presents students with an instructional situation and attempts to guide them to realize a particular instructional goal (e.g., Newman, Griffin & Cole, 1989). In many cases, instructional artifacts are also introduced to support students' appropriation of a particular mathematical concept/scheme (e.g., Dienes, 1964).

For better or worse, the measure of a successful instructional activity is often defined by whether or not the interaction leads students to successfully appropriate the targeted learning objective. While this statement should appear self-evident, various national performance indicators (e.g. National Mathematics Advisory Panel, 2008), as well as the research literature on students' 'misconceptions' (e.g., Smith, diSessa, & Roschelle, 1993) can be interpreted as indirect evidence that our collective understanding of design practice remains underdeveloped.

Indeed, the challenges that students have learning fraction concepts have been well documented (see Lamon, 2007), and a major concern in early rational-number instruction is that many students cope with the topic's inherent challenge by progressively learning to rely primarily on their ability to perform procedural algorithm, enacted as symbol manipulation, which they apply as a means of demonstrating their competence in this mathematical subfield (e.g., Freudenthal, 1983, Chapter 5). The pitfalls and limitations of relying on purely procedural fraction knowledge are poignantly demonstrated in the case of Liping Ma's (1999) elementary school *teachers*: While perfectly capable of performing the appropriate arithmetic procedures for solving fraction-division problems, these teachers were unable to explicate their solution procedures in the form of mathematically accurate scenarios.

Consequentially, more work is needed to explicate the rationales underlying the otherwise tacit design choices that inform instructional practice, as well as to understand the effects that these choices can have upon students' cognition and learning outcomes. A primary objective of this paper will be to explore how the design of an instructional situation can influence students' mathematical cognition, so as to inform a general framework for both fostering and analyzing students' interactions with instructional artifacts and/or activities.

The claim to be advanced is that an instructional activity may be designed so as to purposefully and productively *evoke* students' existing funds of knowledge, and/or to *enact* situations that support students' development of targeted concepts/schemes.

This thesis emerged from, and will be supported by empirical data from a number of case studies involving the implementation of an experimental unit that leverages multimodal activity to support students' learning of fraction concepts (Charoenying, 2010). The specific pedagogical content area involves fundamental notions of rational numbers (specifically, part-to-whole conceptualizations of the *a/b* fraction form). My analysis will explore how the material embodiment of a mathematical task (in the form of an instructional activity) contributes to students' subjective cognitive schema. The functional unit of analysis guiding this examination is the interplay between students' *schemes*, and the instructional *situations* that both influence and give rise to them (Vergnaud, 2009).

Theoretical Framework

Expanding our Notions of Inter-Subjectivity

A socio-cultural perspective of learning is that instructors mediate the speech, actions and perceptions of learners (Vygotsky, 1978; see also Newman, Griffin, & Cole, 1989). Through this process, the learner appropriates and internalizes the socially established meanings of cultural forms such as mathematical signs and concepts.

Bartolini Bussi and Mariotti (2008) have elaborated upon the notion of mediation by characterizing the artifact mediated interactions between instructor and learner as a 'didactic cycle,' whereby the instructor progressively guides the learner to understand the link between the artifact used to complete a task and the formal mathematical meanings.

Although the inter-subjective dialectic between instructor and learner is clearly a vital component of teaching and learning, this framing essentially constrains any analyses along two dimensions—the knowledge, goals, and beliefs of the instructor (Schoenfeld, 1998; see also Ma, 1999); and the prior knowledge and cognition of the learner. Given the fact that not all instructional interactions are successful in supporting students' development of a targeted concept/scheme, a purely socio-cultural analyses would imply some failing on either the part of instructor or learner.

Rather than continue to problematize unsuccessful teaching and learning interactions in terms of the participants, I believe it may be more productive to problematize our existing theoretical frameworks for analysis (Smith, diSessa, Roschelle, 1993). Here, I suggest that a potential gap in socio-cultural accounts of learning is that they do not adequately address the contributions of learners' tacit multimodal activity underlying action and expression to cognition, or their prior knowledge to learning (cf., Abrahamson, 2009). A potentially more productive approach towards designing and understanding learning interactions, would be to incorporate a more nuanced examination of how the *intra*-subjective contributions of students' multi-modal perceptions and prior-funds of knowledge interact within a designed, instructional situation.

Evoking Schemes and Enacting Situations

I would begin by highlighting that whereas learners may experience some activities as conceptually "meaningless" and others as "meaningful," what can be common to these activities are the material objects students encounter, the operations they conduct upon and with them, and the discernable results of these actions.

Here, I propose utilizing the verb "to *evoke*" to refer to the activation of students' prior schemes. It is widely accepted that familiar artifacts, symbols, and signs can evoke an individual's pre-existing schemes. Similarly, students' schemes may be evoked upon their discovery of familiar features present in an instructional situation. Understanding and anticipating the schemes that are likely to be evoked by particular design features would position designers and educators to better interpret and address students' responses as they guide them towards the desired learning objective.

In tandem, I propose utilizing the verb "to *enact*" to describe how students' experience of a situation can be purposefully designed so as to support a particular learning objective. Bruner (1966) had originally used the term "enactive representation" to elucidate how children's actions and activities in the world contribute to their development of iconic and/or symbolic forms of representation. Bruner had suggested that only after "something" is first acted upon and experienced in the world, can it be referenced; first as an object of thought, and later as an icon and/or symbol (see also Hutchins, 2006). Following this reasoning, I argue that the educator/designer can guide students to enact a situation— to act upon, observe, and/or produce situated phenomena —the experience of which can form the basis from which students can subsequently "reverse-engineer" the targeted instructional schemes/concepts the educator wishes for them to develop.

In summary, given the prominent and established role that artifacts play in supporting early fraction instruction, it is important to understand how the design and utilization of these objects can influence students' mathematical cognition. I have proposed the constructs of evocation and enactment in order to further elucidate the relationship between the schemes that are activated when students' notice particular features of a design, the forms of interactions that the design supports, and ultimately, the contributions of situated activity towards mathematical learning. Finally, the instructional design presented herein is influenced in part by theoretical claims that cognition in general (Barsalou, 1999), and mathematical reasoning in particular (Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999), are grounded and tacitly instantiated in real-world experiences. Therefore, an explicit design consideration was to provide students with activities that leveraged their multi-modal perceptions and embodied actions in situ to help mediate and modulate the appropriation of mathematical forms and modes of reasoning (Abrahamson, 2009).

Methods

Given the dual objectives of this project: to better understand student learning, and improve instructional materials, design-based research (Brown, 1992; Confrey, 2005) was selected as an appropriate investigative approach.

This work draws on a series of case studies conducted during an ongoing design-based research study examining how students learn through interacting with tangible mathematical objects. I acted as teacher–researcher-designer. Students from two grade general-education classes (n=54) and from a Grade 3-5 self-contained special-education class (n=7) participated in a series of small group and one-on-one, video-taped teaching interactions.

My design rationale for building the instructional artifacts was as follows: First, I sought a familiar physical object rather than a diagrammatic or computational illustration. I felt it would be important for students to be able to interact with the objects so as to experimentally create, confirm, and reverse mathematical conjectures. A second criterion aimed to encourage broad dissemination was that the design be inexpensive, easily procured, and simple to use.

The resulting instructional design is a tutorial session that engages students in problem solving activities involving water and standard kitchen measuring cups (see Figure 2a).

Through a premeditated sequence of problem posing and facilitated hands-on inquiry, students are guided to perform and report perceptual judgments, measure quantitative aspects of the situation, uncover patterns, and so doing reformulate their initial judgments of part-to whole relationships as aligned with mathematical algorithms and vocabulary. Along the way, the teacher challenges students to warrant/evaluate their claims using the available materials.

As with traditional instructional representations, such as area models and number lines, a key affordance of measuring cups is that they enable students to harness their visual modality to support their mathematical judgments. Thus, to compare the unit fractions 1/3 and 1/4, one could place the respective measuring cups side-by-side (see Figure 2b). More importantly, the measuring cup activities provide a tangible means for teachers to guide students to physically enact situations they believe are analogous to the mathematical schemes they wish students to develop. For example, to help students develop a part-to-whole scheme, students could be guided to enact a situation in which they iterated the same volume from a unit-fraction cup into a one-whole cup measurer until it was filled (see Figure 2c). These activities also provide opportunities for teachers to observe and assess student reasoning.

Figure 2a. Standard kitchen measuring cups.

Figure 2b. A side by side visual comparison of two unit fraction measures.

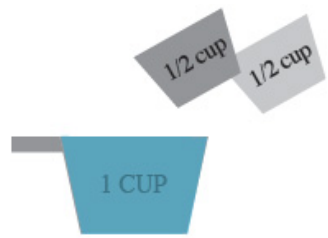

Figure 2c. Enacting a part-to whole scheme by iterating ½ cup measures into 1 cup.

The specific case study presented in this paper is from a one-on-one tutorial session with a Grade-3 special education student, hereafter referred to by the pseudonym "Manny." Pre-assessment data and consultation with the classroom teacher revealed that Manny was performing far below grade level in mathematics.

Results and Discussions

The pedagogical objective of the tutorial session was to guide Manny (age=8) to formulate a *qualitative* understanding of individual unit fractions and the part-to-whole scheme by using the measuring cups to physically enact mathematically analogous situations.

Enacting Mathematically Analogous Situations

After allowing him to familiarize himself with the measuring cups, I ask him how many scoops from the 1/2 cup measure are needed to fill the 1 cup measure. Manny correctly guesses two scoops. I then instruct him to enact his conjecture by iterating 1/2 cup measures of water into a one cup measure. While this instructional sequence would seemingly suggest some understanding of the part-to-whole scheme, his haphazard response to a similar question posed with the 1/4 cup measure reveals otherwise.

R: How many scoops do you need with this one [the 1/4 cup measure]?

M: Three, No four.
R: You need four of these? Why?
M: [Manny looks at the symbol for 1/4] It's one-half

His utterances suggest he has yet to understand the mathematical meaning signified by the fractional notations inscribed on the measures. I write 1/4 onto a slate.

R: No. Can you read one over four. Or you can say one fourth.

I then instruct Manny to iterate four scoops of water from the 1/4 measure directly into the one whole cup measure. He then reverses the operation by scooping water directly out of the one whole cup measure (into a separate reservoir of water) using the 1/4 measure.

R: How many of these guys? [I show him the 1/3 measure]
M: Three.
R: What do you call this one [the one third measure]?
M: I don't know.
R: One over three. Wait. Before you scoop, you have to write it. [Manny writes 1/3, and iterates the 1/3 cup measure into the one whole cup (figure 3)].

Figure 3. Manny iterating water iterating from the 1/3 measure into one cup

A Conflict Between Prior Knowledge and Perceptions

Having guided Manny to enact two situations I believe will support his understanding of the part-to-whole relationship of unit fractions, I purposefully attempt to dis-equilibrate his existing number scheme. I write 1/3 and 1/4 onto a slate.

R: Alright, question Manny. Which fraction is bigger? One third or one fourth?
M: That one [indicating the 1/3 cup measure].
R: You are correct, but why is this one bigger? Can you explain it to me. This is a one third cup [I hold the 1/3 cup next to its numerical inscription] this is a one fourth cup [I hold up the 1/4 cup next to its numerical inscription]. Why is it bigger? [I point again to the 1/3 cup] Why is one-third bigger?
M: Wait. [He matches the cups to the inscription. So this one's bigger! [M points back to the 1/4th cup measure]! No. This is!? [points again to the 1/3 cup measure] I don't know!
R: What do you mean you don't know? You can see with your eyes!

Interestingly, the prior number schemes that the numerical inscriptions evoke for Manny overrule his perceptual judgment. He picks up the 1/4 cup measure, physically nests it into the 1/3 cup, and again tentatively concludes that 1/3 is "bigger."

Evoking a Recently Enacted Situation

To reinforce Manny's intuitions and help him to appropriate a part-to-whole scheme, I guide Manny to reflect upon his earlier enactment of the mathematically analogous situations.

R: Okay. How many times do you need to scoop this one [How many times must the 1/3 measure be iterated to fill 1 cup?]

M: Three

R: And how many times do you need to scoop this one to fill this one.[How many times must the 1/4 measure be iterated to fill 1 cup?]

M: Four.

R: Do you understand why one third is bigger than one fourth yet? Let's see if you can explain why this fraction [inscription for 1/3 written on a whiteboard] is bigger than this one [inscription for 1/4 written on a whiteboard]

M: Because this one doesn't put too much water in it. [simulates scooping of water using the 1/4 measure into the 1 cup measure] And this one [points to the 1/3] can scoop a lot of water [more than the 1/4 measure].

Manny then examines each measurer in turn again, including the 1/2 measure. He physically simulates the pouring action with each into the one whole cup measure. I remove the measures and inscribe the fractional notation for 1/5 and 1/10 onto a board.

R: Which one would be bigger? [1/5 versus 1/10] Which one is going to be a bigger fraction? If these were cups, which would be bigger? [Manny attempts to locate a 1/5 cup and 1/10 cup measure] We don't have these cups...

M: Oh this one [he points to the 1/5 inscription]

R: Why? If this were a scooper…

Manny is initially unable to articulate an answer. He looks to the inscription of 1/10 and utters 11, which could suggest that the arrangement of numbers evoked his addition scheme. He looks again to the inscription of 1/5 before suddenly exclaiming:

M: Wait now I know! This one [the inscription for 1/5] only needs 5 cups, the small one [the inscription for 1/10] needs 10 cups!

Strikingly, the prior enactment of the mathematically analogous situation—which is arguably a function of the concrete, physical properties of the instructional artifact— provided an embodied point of reference for Manny's generalization (see also Pratt & Noss, 2002 on 'situated abstraction'). Manny's statement suggests that he has begun to build an experiential resource commensurate with the mathematical law that $a * 1/a = 1$. A generalized scheme has been abstracted from the situation. One is left to infer that the utility of an instructional situation is contingent on its capacity to support students' enactment of situations that are analogous to the scheme to be learned.

In many respects, Manny's initial interpretation of the 1/*a* form for fractions is typical of students his age, his learning disabilities notwithstanding. What differentiates this particular instructional activity from more traditional approaches involving drawn representations such as area models or number lines for example, is that Manny is able to physically enact both the composition and decomposition of fractional parts into wholes, and so doing, construct an understanding of fractions that is directly grounded in experience. Additionally, the physical medium allows me as teacher to design an activity I believe corresponds to the mathematical concepts/schemes I wish for him to appropriate. Just as importantly, it allows me to indirectly infer the changes in his conceptual understanding by directly observing his actions.

Summary and Conclusions

It is widely accepted that mathematical concepts can emerge through guided acts of representation, as students attempt to articulate emerging schemes in language and any other available means of objectification. While the contributions of a more knowledgeable other are instrumental to this process, I would argue that an analysis of an instructional interaction is incomplete unless one also accounts for the interplay between the instructional situation, and the cognitive and multi-modal perceptual resources that learner student brings to bear.

I have proposed two mechanisms for characterizing learners' interactions with an instructional artifact or activity. First, I highlight the fact that instructional activities are situations that can evoke pre-existing schemes in the mind of the learner. Awareness of this mechanism is arguably of vital importance for instructional designers. As a design heuristic, this notion of evocation leads the designer to consider how students might perceive and experience an instructional situation. Vitally, it prepares designers to better account for the otherwise unexpected ways that a student might make sense of a novel instructional situation. Knowing a priori that a particular design feature is likely to evoke a particular scheme for students (e.g., how the discrete units of an area model evoke counting schemes), would help to inform how an educator/designer chooses to mediate students' learning activities.

Second, I use the term "enact" to describe how students' situated activity can be structured so as to help facilitate their construction of a given scheme. Schemes are not constructed *ex nihilo,* but from students' encounters with, and assimilation of, new situations.

It stands to reason then that the manner in which students' situated activity is organized and orchestrated is central to their conceptual development. Instructional designers can selectively determine the experiences students have, and thereby influence the trajectory of students' cognitive development. Students' enactment of an instructional situation furnishes them with a set of experiences that may form the cognitive bases for the instructor's targeted learning objective.

Although evidence from only a single, brief case study episode is provided, the full corpus of data suggest that the Water Works design appeared to support students' articulation of their mathematical ideas by allowing them to physically enact a variety of representative situations. On the other hand, situations involving fractions for which there was no corresponding measuring cup such as 1/5, 1/100 etc., could not be directly modeled with the set of measurers. The students in the study did not appear to immediately transfer their newly constructed schemes (e.g., four scoops from the 1/4 measure to fill one cup) to the purely symbolic representations for which there had been no situation-specific analog. This is a clear limitation of the present design. In such cases, an abstract, drawn or computationally modeled representation of the scheme/situation might have been much more practical.

In conclusion, the argument advanced by introducing the constructs of evocation and enactment is this: Educator/designers effectively determine the actions and outcomes that arise from students' interactions with a particular situation. Therefore they influence in large part the schemes students construct. Conceptualizing instructional practice in terms of structuring learning situations may provide educators, designers, and researchers alike with new productive insights for anticipating and evaluating the effectiveness of a proposed instructional design.

Finally, the instructional design presented in this study is a low-cost, low-tech example of an embodied representational context that can be adapted by teachers in *any* classroom.

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ENGENDERING MULTIPLICATIVE REASONING IN STUDENTS WITH LEARNING DISABILITIES IN MATHEMATICS[1]: SAM'S COMPUTER-ASSISTED TRANSITION TO ANTICIPATORY UNIT DIFFERENTIATION-AND-SELECTION

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We examined how a student with learning disabilities (SLD) in mathematics constructed a scheme for differentiating, selecting, and properly operating on/with units that constitute a multiplicative situation, namely, singletons ('1s') and composite units (abbreviated UDS). Conducted as part of a larger teaching experiment in a learning environment that synergizes human and computer-assisted teaching, this study included 12 videotaped teaching episodes with a 5th grader (pseudonym-Sam), analyzed qualitatively. Our data provide a window onto the conceptual transformation involved in advancing from absence, through a participatory, to an anticipatory stage of a UDS scheme—a cognitive root for the distributive property. We postulate this scheme as a fundamental step in SLDs' learning to reason multiplicatively, and highlight the transfer-empowering nature of constructing it at the anticipatory stage.

Introduction

This study examined how students with learning disabilities (SLD) may construct a scheme for reasoning about multiplicative situations. Learning to reason multiplicatively is a major feat for elementary age children (Harel & Confrey, 1994; Kamii & Clark, 1996; Sowder, et al., 1998;