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BOTH RHYME AND REASON: TOWARD DESIGN THAT GOES BEYOND WHAT MEETS THE EYE

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Drawing on design-based studies where students worked with learning tools for proportionality, probability, and statistics, I appraise whether students had opportunities to construct conceptual understanding of the targeted mathematical content. I conclude that visualizations of perceptually privileged mathematical constructs support effective pedagogical activity only to the extent that they enable students to coordinate perceptual conviction with mathematical operations—intuiting that*, and not* how*, two representations are related constitutes perceptually powerful yet conceptually weak situatedness. In constructivist learning, as in empirical research, regularity apprehended in observed phenomena is measured, expressed, and schematized. Students should articulate or corroborate visual thinking with step-by-step procedures, e.g., synoptic views of multiplicative constructs should include tools for distributed-addition handles.*

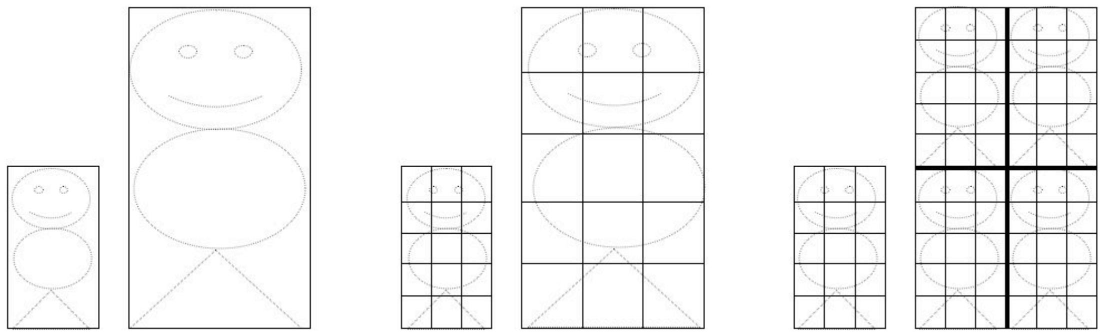
"Neither rhyme nor reason can express how much"
(Shakespeare, *As You Like It*, 3:2)

This theory-of-design paper builds on empirical studies of mathematical cognition and cognitive psychology and is inspired by phenomenological and analytic philosophy. I argue that visualizations of perceptually privileged mathematical constructs, e.g., proportionality (Gelman & Williams, 1998), are effective teaching/learning activities only to the extent that they enable students to coordinate intuitive and explicit knowings (Schön, 1981)—apprehending the *that* but not the *how* of quantitative relations embedded in images constitutes perceptually powerful yet conceptually problematic situatedness (e.g., Davis, 1993). Worse, students' false sense of understanding may hamper personal reflection or formative assessment. To demonstrate epistemological tensions and pedagogical promise inherent to *visual thinking* (Arnheim, 1969), I discuss four data episodes pertai ng to the domains of proportionality and probability. In each study, participant students can be said to have grounded the mathematical content in the situation only to the extent that they were equipped to articulate their multiplicative judgments additively.

Theoretical Background

A critique proverbially inveighed against traditionalist instruction is that students who demonstrate procedural skill often have not developed deep understanding of the mathematical concept. Yet, could such procedural–conceptual hiatus result from participating in activities that purport to embody constructivist pedagogical philosophy? This paper's point of departure is that some situational contexts intended to help students ground the meaning of a mathematical concept may not do so, even when they appear as though they might. These pedagogical situations have the trappings of powerful learning environments: interactive tools representing mathematical quantities, symbols, and relations are laid out for manipulation, and rules of situated–symbolic translation are either empirically apparent or provided by a facilitator. Yet, I contend, these rules of translation themselves may not be transparent. Namely, I argue, designers are liable to inadvertently confound perceptual constancies with operatory conservation (Piaget, 1952); psychology with epistemology (Papert, 2000); embodied schemas with mathematical fluency (Abrahamson, 2004). Therefore, sorting out phenomenologically entangled roles of perception and reasoning in mathematical learning is pivotal for constructivist design. Learning tools purporting to make manifest quantitative relations underlying mathematical concepts, I demonstrate, do so only if learners are supported in going beyond what meets the eye.

Rhetorical Case-Study Exposition: Epistemological Nuances of Perceived Equivalence



a. b. c.

Figure 1. Toward a critique of pictures that purport to ground equivalence: (a) equivalence by perceptual judgment; (b) unit-stretching complementing the phenomenology of geometrical similitude; and (c) grounding geometrical similitude with a fixed unit.

Figure 1a invites the learner to evaluate the geometrical similitude of two rectangles. An intuition of sameness results from the activation of automatic perceptual mechanisms for judging the identity of two percepts (for references, see Abrahamson, 2002). Namely, I contend, *the phenomenology of geometrical similitude is one of identity*—by virtue of judging for geometrical similitude one necessarily considers that the smaller and larger images are the very same object.

Whereas geometrical similitude is first apprehended perceptually, holistically, the multiplicativity of the proportional relations between these objects' corresponding dimensions needs to be learned—whether prescribed or discovered empirically, it is a rule initially dissociated from the phenomenology of perception. That is, the perceptuality per se of geometrical similitude is not multiplicative, additive, or even logarithmic, for that matter—it is a prereflective knowing embedded in our everyday sensory comportment, our being-in-the-world, and is not given to explicitization (Piaget, 1952).

In Figure 1a, above, no tools are provided for a learner to analytically elaborate and verify the intuitive judgment of sameness. Figure 1b presents the larger of the two objects as a zoom-in of the smaller object, as though the two objects are in fact one and the same object as seen from different distances. Figure 1c provides tools for coordinating the perceptual apprehension of similitude with an empirical rule: The two dimensions of the smaller rectangle—height and width—are each multiplied by the same factor so the smaller rectangle fit or become the larger.

The initial apprehension of identity (in Figure 1a) is visually seductive—it subtly carries over as compelling positive affect, through a quantitative lens on geometrical similitude (stretched grids in Figure 1b), to the empirical rule of applying a constant scalar factor to both dimensions (uniform unit grids in Figure 1c), vesting a numerical proportional equivalence (4:3 = 8:6) with truth value. Yet, note that Figure 1c does not elaborate Figure 1b—whereas Figure 1b engages an opaque multiplicative transformation (Kaput & West, 1994), Figure 1c engages the accessible repeated-addition model, an analytic model that deviates from the identity synopsis yet provides semiotic equipment to ground multiplicative constructs as phenomenal–conceptual syntheses.

Examples From Empirical Studies of Design for Diverse Mathematical Content

1. Stretch/Shrink vs. Additive Construction of Proportional Progression

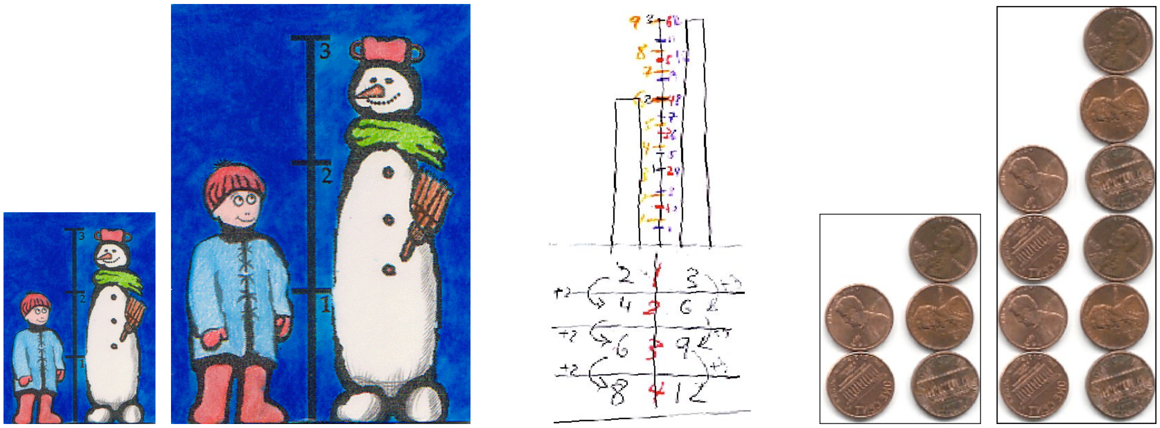


Figure 2. The 'eye-trick' design for proportion: Learning materials and student artifact

In the *eye-trick* design (Abrahamson, 2002), students work with proportionately equivalent card pictures, each displaying a pair of personas, such as Danny & Snowy (see Figure 2, on the left). Closing one eye and holding the smaller card nearer to their open eye, students experience an optical illusion as though the cards are identical, due to the similar retinal prints seen through monoscopic vision. Students use a ruler to measure the actual heights of the persons, 2'' & 3'' and 4'' & 6'', then mark these values in a table, under a pair of schematic rectangles (Figure 2, center). Additional cards, for 6 & 9 and 8 & 12, are judged as similar, measured, and tabulated.

The design's objective is for students to ground proportional equivalence in perceptual identity and thus interpret corresponding measurement pairs as equivalent. Two high-achieving Grade 3 participants determined the arithmetic sequence in each column and iteratively added the constant addend down each (see '+2' and '+3' arrows). Yet, the students did not construct the situation as multiplicative. In fact, they were surprised that differences between the values grew (see numerals 1, 2, 3, & 4 between the columns). Moreover, when subsequently building coin towers (Figure 2, on right), where the iterative rule was "+2 coins here, +3 coins there," neither student perceived any relation to the earlier eye-trick activity, until they had tabulated the coin quantities and recognized the table as identical to the table they had previously built. I concluded that apprehension of induced identity followed by tabulated measurement could be a compelling extension activity yet is problematic as conceptual entry to proportional equivalence.

2. Global Color Density vs. Local Sampling in Statistical Investigation

S.A.M.P.L.E.R., Statistics as Multi-Participant Learning-Environment Resource, is a collaborative activity designed for networked middle-school classrooms (Abrahamson & Wilensky, 2002, 2007). The design introduces students to fundamental statistical constructs, such as sampling. In the first activity, students are asked to estimate the percentage of light-colored squares in an array of thousands of light and dark squares (Figure 3, on the left). Typically, students begin by eyeballing the entire array and offering estimates. Yet, to warrant their estimates, students spontaneously sample localized color densities, which they count (e.g., 4 light squares in a 3-by-3 sample of 9 squares), compare, and compile (Figure 3, center and right). Thus, the array affords either proportional reasoning or enumeration, and facilitated classroom discussion explores relations between these perspectives so as to support their coordination.

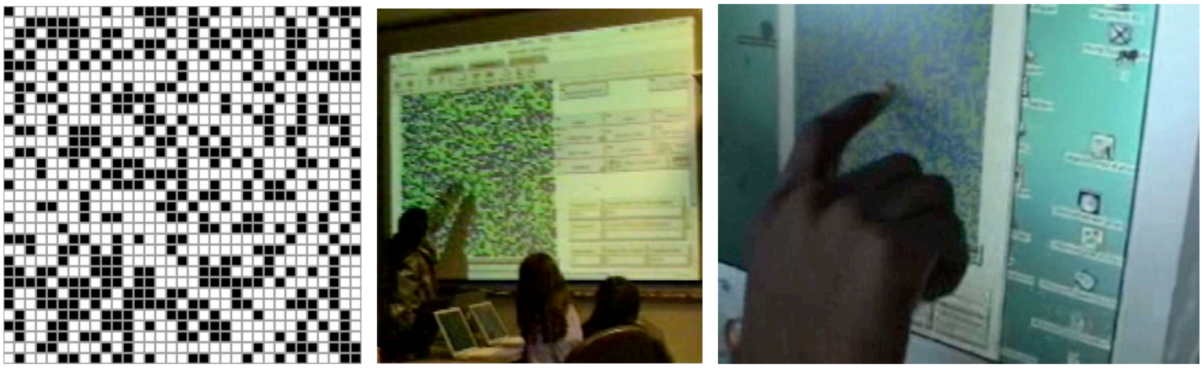


Figure 3. S.A.M.P.L.E.R. activity: coordinating proportional judgment with sampling.

3. Perceptual- vs. Analysis-Based Frequency Expectation

Twenty-eight Grade 4 – 6 students and 25 college students were given a box with equal numbers of mixed green and blue marbles—a few hundred in total—and a utensil for scooping out an ordered sample with exactly 4 marbles (see Figure 4, below, on the left). We asked them to predict what will happen when they scoop. By and large, all students expressed that whereas they cannot know what they will receive on particular trials, their long-run expectation for the greatest relative frequency is of a sample with two green and two blue marbles. When prompted to further warrant their claim, students typically said, "I don't know the reasoning behind it, but it seems kind of obvious to me" or "I just saw it," and many articulated that the mode sample should reflect the green-to-blue ratio in the box (Abrahamson, 2007; Abrahamson & Cendak, 2006; see Tversky & Kahneman, 1974, on the 'representativeness heuristic'). Whereas such synoptic reasoning is accurate, it is delimited in its trajectories as grounds for methodically appropriating normative mathematical strategies, such as calculations pertaining to expected value, the binomial function, or combinatorial analysis. Moreover, such reasoning cannot readily accommodate situations in which the green–blue mix is unequal (such that p is not equal to .5).

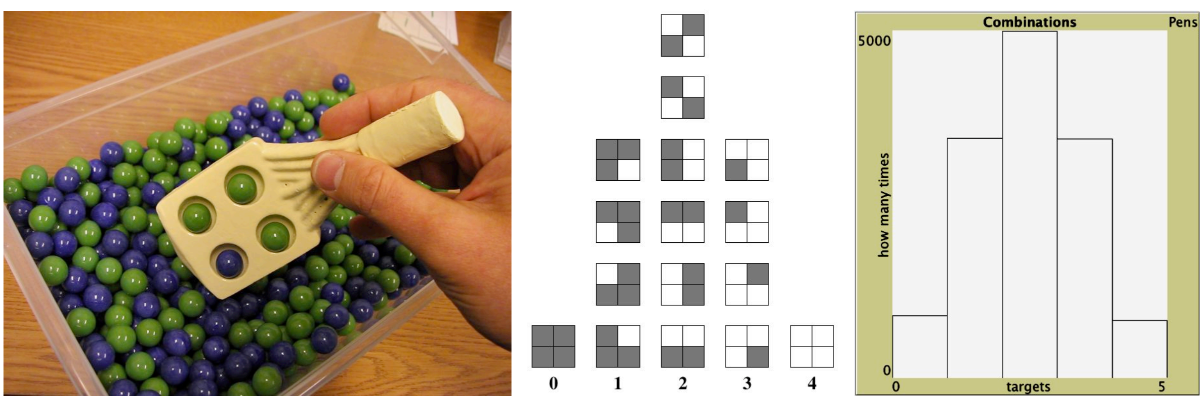


Figure 4. Selected materials used in studies of probabilistic cognition. From left: The *4-block marble scooper* samples from a green/blue mix of marbles; the *combinations tower*—the 4-block's distributed sample space; a computer-based empirical outcome distribution.

We guide students to build the 4-block's sample space and assemble it in accord with the statistic in question, the number of green marbles (Figure 4, above, center). Upon beholding this configured set, all the college students and all but one of the Grade 4 – 6 students immediately recognized an analytic warrant for their intuitive assertion that 2-green outcomes would be the most common. Thus, the participants coordinated one intuitive perceptual judgment, which they could not directly articulate, to another perceptual judgment—phenomenologically disparate yet mathematically commensurate—that is more conducive to appropriating normative mathematics.

4. Event- vs. Outcome-Based Expectation for Binomial Distribution

The combinations tower (Figure 4, previous page, center) is designed to facilitate comparison of the marble-scooping sample space (the 16 unique configurations) and distributions of empirical outcomes from numerous simulated trials with this device (Figure 4, on the right; note similar shapes). Whereas they appropriated this perceptual–conceptual relation, students' explanations revealed a lacuna in understanding the emergence of distribution as contingent on sample space, probability, and random selection. Specifically, students initially failed to realize that each of the 16 equiprobable outcomes was expected to occur an equal number of times in the experiment and that therefore the 1-4-6-4-1 groups would converge to 1:4:6:4:1 distribution.

Figure 5 (across) shows new interactive features added to the computer-based simulation. Outcomes from experiments with the 4-block device accumulate in their respective columns. Thus, empirical distributions are constructed as "stochastic stretches" of the sample-space distribution and enhanced by re-ordering and color-coding. Middle-school and college students working with these tools were more likely to perceive the empirical outcome distribution as 16 more-or-less equal groups arranged in 5 columns in correspondence with the arbitrary criterion that had designated 4-block events in the sample space by their number of light-colored squares. Such perceptions mark avenues for bridging classicist and frequentist approaches to stochasm.

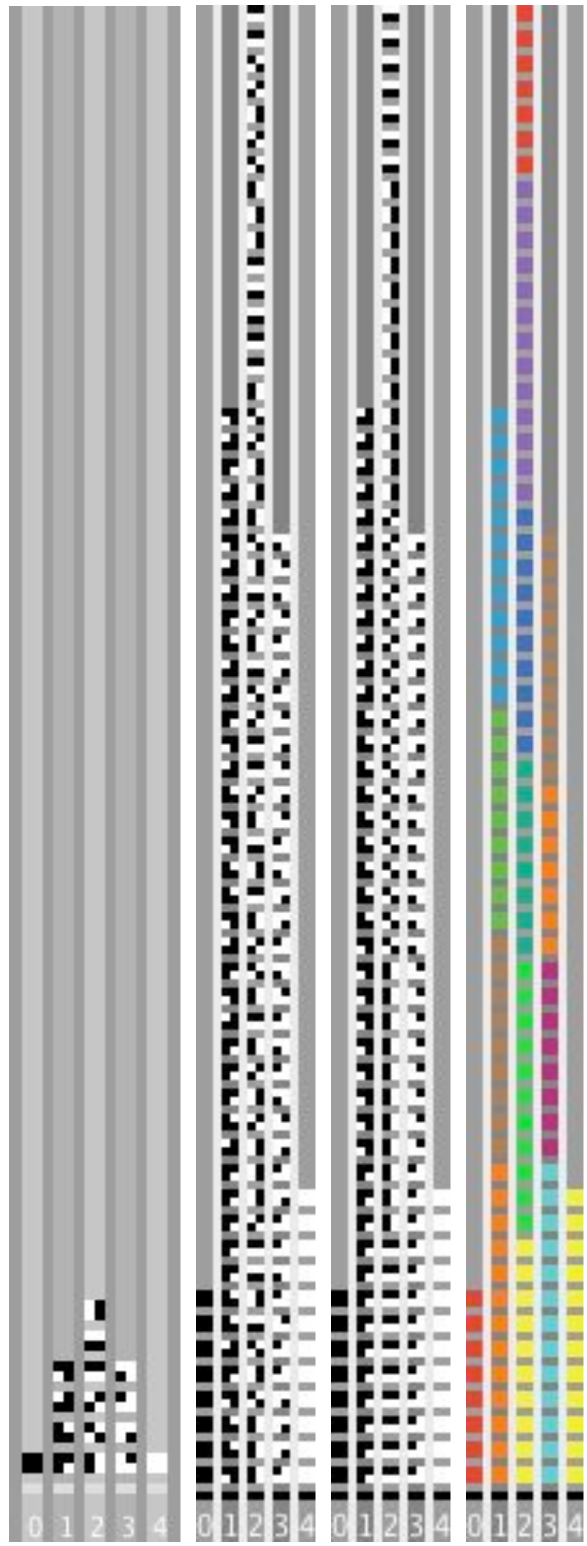


Figure 5. *4-Block Stalagmite*, a computer-based simulation, supports a seeing of empirical distribution as emerging from the sample space. From left: 16 unique possible outcomes in a combinations tower; empirical results from a simulated experiment where random outcomes "drop" into respective columns; a re-ordering of these outcomes by type; and color-coding the 16 groups.

Summary and Conclusion

Mathematics, at least K – 12 constructivist curriculum, is an inherently empirical discipline: As in the discipline of physics, mathematical knowledge develops as the methodical articulation of perceptually apprehended regularities governing relations among quantities situated in phenomena under inquiry. As in physics, direct apprehension of phenomenal regularity is limited by available resources: perceptual and para-perceptual mechanisms, memory, computational algorithms, and representational forms. Some regularity is perceptually privileged, such that we apprehend it synoptically with little or no mathematical training. Such seeing lends an affect of knowing, yet the work then becomes to articulate quantitatively this empirical apprehension—to synthesize intuition and mathematics (Schön, 1981). This paper examined four case studies of constructivist design to investigate conditions supporting such tacit–explicit synthesizing. Learners' initial apprehension of the perceptual information was tacit proportional. Guided attention to discrete properties of the situated quantities enabled students to construct an enumerative–additive rule, such that tacit proportional judgment was reformulated as explicit multiplicative, with the additive procedure acting as a buffer from the tacit to the explicit.

Note that the explicit does not directly articulate or translate the tacit (Abrahamson & Cendak, 2007), because these faculties are epistemologically incompatible. Rather, the explicit voices, concretizes, and ultimately enhances the tacit, rendering the somatic semiotic, i.e., in the form of a socio–mathematical artifact bridging the personal prereflective into the disciplinary domain. Curiously, whereas the mathematical disciplinary continuity is from the addition operation to the multiplication operation, the phenomenological trajectory supported by constructivist discovery-based design may traverse from multiplicative intuitive apprehension to additive analytic procedures. Thus, counter to common premises of curricular design, some additive procedures may be grounded in multiplicative intuition.

Broadly, designed embodiment of mathematical concepts can play critical roles in facilitating learners' tacit–explicit negotiation: Learning tools—their inherent perceptual information and measurement tools—should alternately afford analytic handles on embedded quantitative dimensions or tacit faculties for evaluating the veracity of emergent explicit assertions. These designed embodiments act as 'bridging tools' (Abrahamson & Wilensky, 2007)—they mediate tacit–explicit reciprocal negotiation fostering epistemic synthesis. Specifically, learning activities that afford quantitative intuition grounded in proportional judgment should provide tools for articulating this intuition through pertinent additive processes. Otherwise, students' intuitions remain encapsulated, inarticulate, uncoordinated with robust solution procedures—the students sense they understand a concept, but such understanding is likely no more than tenuous.

In sum, students used additive models to ground explicit multiplicative reasoning in tacit proportional apprehension. I therefore implicate the proportional-vs.-enumerative perceptual tension as a contributing factor to pedagogical challenges of rendering design for multiplicative constructs effective. Moreover, it is perhaps the phenomenological incompatibility of holistic and analytic cognitive resources that raises concerns among some mathematics-education researchers of an overly additive articulation of multiplicative constructs (e.g., Confrey, 1995). Yet, I submit, it is precisely in painstaking synthesis of the embodied and calculated, the synoptic and sequential, the privileged and earned, that core conceptual learning transpires. Thus, designers and teachers should create opportunities for students to coordinate holistic multiplicative action models for situated mathematical problems with distributed-addition solution procedures (Fuson, Kalchman, Abrahamson, & Izsák, 2002); to fit all-at-once synoptic sense with step-by-step sensibility; to express how much as synthesis of rhyme and reason.

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