Abrahamson, D. (2007). Handling problems: Embodied reasoning in situated mathematics. In T. Lamberg & L. Wiest (Eds.), *Proceedings of the Twenty Ninth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 219 – 226). Stateline (Lake Tahoe), NV: University of Nevada, Reno.

HANDLING PROBLEMS: EMBODIED REASONING IN SITUATED MATHEMATICS

Dor Abrahamson
University of California, Berkeley
dor@berkeley.edu

Fifty 4th-17th-grade students participated in individual interviews oriented toward probabilistic intuition. Participants were given a boxful of equal numbers of green and blue marbles, mixed, and a device for scooping 4 ordered marbles and asked to predict the most common sample. Students replied that the outcome with the highest relative frequency would have 2 green and 2 blue marbles. Their verbal reasoning was accompanied by a deictic–metaphoric gesture to the left then right, as if they were separating the colors in the box. Gesture, I argue, bridges direct intuitive grasps of situations to conscious reflection, thus concretizing the prereflective, possibly grounding it in material form such that it emerges as conducive to further elaboration in mimetic symbolic form. Situated mathematical reasoning transpires largely as embodied negotiation among kinesthetic image schemas afforded by available material resources and epistemic forms.

"The soul never thinks without an image" (Aristotle, *On the Soul*, 350 B.C.)
"To see a world in a grain of sand…" (Blake, *Auguries of Innocence*, ~1800)

Objectives

The objective of this study is to contribute to research on mathematical cognition by illuminating implicit processes of embodied reasoning in situated problem solving. I argue that situated mathematical reasoning transpires as an embodied negotiation between material/perceptual affordances of phenomena and evolved cultural–historical cognitive artifacts that include physical utensils, symbolical forms, and figures of speech. To build this argument, I present empirical evidence of students engaging kinesthetic–imagistic reasoning in solving a situated probability problem. I propose that situated mathematical reasoning transpires neither as direct translation (mapping) from phenomena to symbols nor as chains of signification, as some studies of semiotics or anthropology-of-scientific-practice may suggest, but in embodied modalities drawing on both the material and symbolical and operating in complex dynamical feedback iterations, in which these resources reciprocally constrain each other toward achieving structures evaluated as cognitively coherent, locally effective, and socio–mathematically normative. Specifically, I conjecture that material affordances of situated objects may facilitate a gesture-based bridging from unreflective kinesthesia to representational intentionality, a process that is instrumental to individual reasoning stimulated by and conducted in interpersonal discourse practices. I speculate whether the mathematics-education community currently has theoretical, methodological, and pedagogical wherewithal to successfully interpret the nature and mechanism of such commonplace multimodal reasoning so as to draw implications for the design and facilitation of learning environments. Toward this goal, I articulate the nature of special mathematical learning tools—*reflexive artifacts*—that are conducive to drawing on embodied resources such that the resources are coordinated into emergent solution procedures. These tools may currently be rare, yet articulating their nature could be conducive toward designing new reflexive artifacts that facilitate students' guided mathematical reasoning. As such, this paper is part of an ongoing project to develop a design framework including principled methodology for implementing constructivist pedagogical philosophy in the form of content-targeted artifacts, activities, and facilitation emphases (see Abrahamson & Wilensky, 2007).

Theoretical Background

Following renewed post-Behaviorist interest in the roles of vision in mathematical reasoning (Arnheim, 1969; Davis, 1993; Goldin, 1987), the mechanisms of imagery, in its broad multimodal sense, have become foci of research on mathematics education (Presmeg, 2006; Schwartz & Heiser, 2006). Yet, whereas design-based researchers have been cognizant of the roles of perception in mathematical learning, attention to the microdynamics of multimodal reasoning has not been articulated or consolidated in the form of a methodology with clear directives for practice (but see Case & Okamoto, 1996, for a neo-Piagetian approach to the design of contexts that support students in drawing on multiple resources and integrating these as a 'central conceptual structure,' e.g., the case of counting, in which speech, perception, and gesture are implicated). It thus appears timely to integrate considerations of multimodal reasoning into design-research methodology including emphases for microgenetic data-analysis of learners' moment-to-moment interactions with artifacts designed to support content learning. Toward outlining such a prospective methodology, this paper closely examines a case study as a means for considering several theoretical models and their attendant methodologies, including the following. McNeill's (1992) foundational taxonomy of gesture types facilitates interpretation of students' multimodal actions as thinking-for-speaking with artifacts, with each micro-moment constituting context for the subsequent embodied-cognition act. Grice's (1989) theory of pragmatics helps us interpret students' gestures as including aspects of ostentation and clarification that respond to the interlocutor's conjectured perspective. Pirie and Kieren (1994) offer a methodology for monitoring a student's personal invention, consolidation, and use of images that ground the meaning of a mathematical concept and later serve as resources for further conceptual differentiation and development. Schwartz and Heiser (2006) draw on their empirical studies of students' situated problem solving to discuss the difficulty of coordinating modalities (e.g., motoric and imaged)—work suggesting the importance of learning environments that facilitate such coordination. Fauconnier and Turner's (2002) *conceptual blend* model illuminates cognitive mechanisms underlying the superimposition of images or percepts, offering a viable extension of standard cognitive-science problem-solving models to include attention to metaphor as image- and not proposition-based. Finally, Hutchins and Palen (1998) delineate a distributed-cognition approach to explain the ubiquitous, quotidian, and inextricable roles instruments play in supporting the coordination of multimodal and multi-person resources in routinized, practice-based, problem solving. These learning-sciences resources have informed the development of a constructivist/socio-constructivist approach to design that treats students' mediated with-tools phenomenology as epigenetic of reinvention and conceptual understanding. In particular, I seek to understand the ongoing construction of mathematics as developing webs of coordinated multimodal resources. Data from design-based research studies constitute useful arenas for investigating tool-based epigenesis of mathematical constructs, particularly due to the designer's nuanced understanding of the artifacts students engage in problem solving—the tools' material properties, embedded mechanisms, contexts, and affordances of conceptual emergence.

Data Sources

The data used in this study are drawn from a design-based research project exploring the nature, roles, and mechanisms of intuition in mathematical reasoning and learning. The subject matter content that served as context for this study was probability and basic statistics. The study was conducted in the form of ~75-minute semi-structured clinical interviews in which students engaged individually in problem-solving and construction activities, using a set of innovative learning tools under development. We have interviewed over 50 students, including Grade 4 – 6 students as well as undergraduate and graduate students majoring in mathematics, statistics, or economics programs (for the design rationale, learning tools, and analysis of empirical findings, see Abrahamson, 2007b; Abrahamson & Cendak, 2006). The current study seeks to achieve deep understanding of a particular brief behavior manifested by most of the participant students.

Figure 1. The marble-scooper randomness generator consists of a boxful of hundreds of marbles of two colors and a utensil for scooping out a fixed number of marbles. In the current embodiment, there are equal numbers of green and blue marbles (p = .5), and the scooper accommodates exactly 4 ordered marbles. At the onset of the interview, students are asked, "What will happen when you scoop?" Problem solving and discussion involve several other mixed-media learning tools pertaining to sampling, randomness, distribution, and combinatorics.

Figure 1, above, shows the marble-scooper device developed for our studies. Each concavity in the scooper can hold a single marble. The simultaneous scooping of four marbles out of the hundreds of marbles is arguably commensurate with flipping four coins (or flipping a single coin four times). Thus, the device supports a situated study of the binomial function $(a + b)^4$. There are 16 unique outcomes in operating this stochastic generator (2^4). For equal numbers of green and blue marbles ($p = .5$), the expected outcome distribution in empirical experiments is 1:4:6:4:1, where these five coefficients correspond, respectively, to the cases of selecting exactly 0, 1, 2, 3, or 4 green marbles in any order (the rest would be blue). Thus, the numeral '6' indicates an expected plurality of samples with two green and two blue marbles in any order.

By and large, all our participants predicted the 2-green–2-blue outcome as the most common. When asked to support their prediction, all students initially said either that, "It looks that way" or that they do not know how they know. For example, one applied-mathematics major said, "I don't know the reasoning behind it, but it seems kind of obvious to me." Upon further thought, undergraduate and graduate students articulated this intuition, saying that the most common sample should reflect the green-to-blue ratio in the box. Only upon prompts did these older students apply notions and solution procedures relating to expected value, the law of large numbers, the central limit theory, and the binomial function (the "mathematical reason," as one student called these). This study focuses on students' initial, "non-mathematical," yet accurate judgment. In particular, I examine the microgenesis of seeing and applying symmetry and proportionality in probabilistic reasoning (for a survey of related work, see Jones, Langrall, & Mooney, 2007; in particular, see Tversky & Kahneman, 1974, for classical demonstration of the 'representativeness heuristic'). A broader objective framing my work is to develop content-targeted learning tools, activities, and facilitation supporting students in acknowledging their non-analytic perceptual intuitions and coordinating these intuitions with standard solution procedures—I view mathematics learning as the self study of perception (Abrahamson, 2007a).

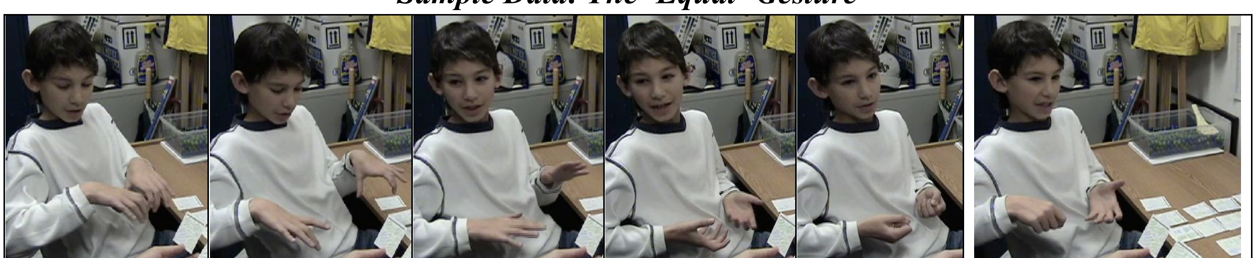
Methods

My research group, Embodied Design Research Laboratory, based in UC Berkeley's Graduate School of Education (http://edrl.berkeley.edu/), operates primarily in design-based research methodology, in which we investigate mathematical cognition through engaging students in activities with objects of our design (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Once we have elicited video/audio data, we study student learning using collaborative microgenetic qualitative-analysis methodology (Schoenfeld, Smith, & Arcavi, 1991). Thus, we select and intensely examine short data episodes in attempt to build as complete as possible an understanding of students' thinking processes. For example—and most pertinent to the current study—, in order to conjecture as to the resources students bring to bear in addressing a situated mathematical problem, we pay attention to students' gestures as they problem solve (see Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999). Through iterated viewings, comparison, and debate, members of the research team become fluent in the entire data corpus, such that analysis of each participant's data is contextualized by the complete interview as well as by all other participants' responses to corresponding items in the interview protocol. Whereas this approach is methodologically incomplete, the potential validity of our insights lies in students' increased facility with learning tools that are improved iteratively, based on these insights, and in subsequent triangulation with the literature and further data analysis. This paper reports on our analysis of students' reasoning processes as they initially brought to bear intuitive resources to respond correctly to the marbles problem. More broadly, we are interested in emergent relations between material substance (marbles, scooper, box) and conceptual constructs (e.g., probability, proportion) and the roles embodiment plays in supporting students' problem-based construction of mathematical content as semiotic coordination of material and epistemic resources. Understanding these complex dynamical processes informs our design of learning environments.

Results, Analysis, and Discussion

While referring verbally to equal proportions of green and blue in the marble box, about 3/4 of the students gestured to one side and then to the other, as though the hundreds of marbles were separated by color to the left and right of the box. In so doing, several students performed a left–right gesture away from the box without any clear referent, some gestured either toward the box itself or to a box they constructed with gestures, and some of the participants indicated—even touched—the middle point of the box immediately prior to gesturing to the "blue half" and the "green half." We will focus on LG, who was typical in manipulating the perceived and imaged.

Sample Data: The 'Equal' Gesture



00:01	00:02	00:03a	00:03b	00:06	00:07
LG (6th grade): If there're equal a*mount*	of *col*ors	on *each*, [hands flap up and down]	*side*,	--	I guess you're less likely [scoops] to get more of one color than the other.

Figure 2. A student engages in multimodal reasoning to support his anticipation of the most frequent outcome in an experiment with the marble-scooper stochastic device (p = .5).

The 4-second pause in LG's utterance suggests that his gestures are integral to the reasoning process, not post hoc. LG's entire sentence is built in the form of an IF–THEN structure: "If…, [then] I guess…" In the IF clause, LG constructs properties of the stochastic generator, whereas in the THEN clause he expresses assumed operative consequences of this construction for the expected empirical outcomes of the projected experiment. Each of these two conjoined clauses is uniquely associated with a proximal seeing of the marble box, yet in the IF clause the marbles are separated, whereas in the THEN clause they are mixed. Thus, in his IF clause, LG considers an event that is complexly related to the distal phenomenon before him. Namely, it is not the case that "there're equal amount of colors *on each side*"—the physical marbles are in fact mixed and not separated. What, then, is the nature of LG's statement? What is he referring to? Why does a casual apprehension of LG's discourse not appear strange to an unreflective interlocutor? Following, I attempt to explain the nature of the gesture—its contexts, mechanisms, and roles.

Data Analysis: The 'Equal' Gesture as a Window Onto Problem-Solving Processes

To begin with, the social situation suggests that there is relevant information to derive from the box so as to respond to the question about the scooper. Hence, the symmetrical shape of the scooper—seen as two concavities on the left and two on the right—foregrounds in the marble box its affordance for bisection. The gesture reveals the body's role in porting these intercontextual constraints back and forth between the material elements of the problem space. In particular, the embodiment of symmetry and balance plays a mediating role between box and scooper. In the process of reasoning through and communicating an idea about the marbles, the student appropriates material properties of available media that include his body and, reflexively, the gesture-constructed marble box. The partitioning gesture toward the constructed marble box is complex deictic–metaphoric. It shows the referents of speech ("amounts of color on each side") in a reconstruction of the box's content, perceived as two en-masse semantic categories. This mental partitioning organizes the marbles such that they better afford mathematization. Finally, note that the marbles are held in a vessel. This particular vessel, due to prosaic reasons of industrial engineering, production, storage, distribution, and marketing, is structurally simple—a translucent rectangular plastic box. A feature of this box, then, is a straight line, the long side of the box, which faces the student. This long side of the box constitutes an ad hoc measurement tool—a primitive number line—upon which LG tacitly offloads the embodied ratio of the green and blue groups, thus inadvertently concretizing the ratio as part-to-part indexes.

By this interpretation, embedded properties of the marbles box (straight frame) constituted a material bridge from a focal artifact (the actual marbles in the rectangular box) reflexively to this artifact's inherent mathematical information in question (green–blue ratio). Thus, pre-verbalized images of symmetry and balance become articulated in bipartite linear form bespeaking proportion. The 'equal' gesture, then, loops from an object and back to it—that is, from the attended categories (color property) of an amorphous object (mixed marbles) and back to the very object (container box), now serendipitously used to ground, elaborate, and communicate embedded aspects of its own properties (half–half). The box of marbles is thus both an originary resource and a vehicle carrying cognitively ergonomic expression of its own properties—it is a *reflexive artifact*, enabling what Noë (2006) describes as using the world to represent itself. Specifically, a reflexive artifact embodies as an affordance an epistemic form for indexing its own properties, thus acting as a cognitive bridge from the phenomenal to the mathematical.

Discussion

Whence did the 'equal' gesture come and what roles did it play? When reasoning and communicating, humans spontaneously leverage their capacity to represent absent objects, or aspects of these objects, within their body space (McNeill, 1992). The epigenesis of a specific gesture is in actual manipulation (Vygotsky, 1978)—the students' 'equal' gesture may be grounded in prior physical actions of sorting and partitioning. Yet it is precisely because the marbles are not readily given to manifesting the 'equal' idea physically, that the participants select an alternative medium, gesture, for conducting and communicating their problem solving. The media-neutral quality of gesture, along with its malleability and portability, make gesture an effective modality for situated problem solving. In particular, gesture marks the body as a buffer for coordinating perceptions of disparate objects that (e)merge as structurally–dynamically homologous. That is, gesture can carry essential structural properties of one element in a situated problem (scooper) to induce it in another element (marbles). Thus, gesture both invests and reveals *epistemic forms* (Collins & Ferguson, 1993), such as *a*:*b*, embedded in situated elements.

In Abrahamson (2004) I suggested that people engaging in situated problem-solving use *embodied spatial articulation*, a type of dynamic visuo–spatial reasoning, to negotiate between body-based mathematical knowledge (kinesthetic schemas) and socially mediated norms of seeing mathematical tools (epistemic forms). That is, people manipulate situations—whether physically or imagistically—so that they can apprehend the situations through familiar schemas that are conducive to determining mathematical properties of the situations (this is the act of modeling, mathematizing). The 'equal' gesture may indicate that the participants were engaging in embodied spatial articulation—they were assimilating the marbles into an embodiment of proportion as a means to anticipate frequency, constructed as expected mode and variance.

Conclusions

Students' mathematical problem solving is not either with or without objects, perceptual or conceptual, situated or symbolic, concrete or abstract. In fact, these pairs of constructs assume an ontology that does not capture the phenomenology of mathematical reasoning. Rather, problem solvers negotiate among embodied schemas afforded by available material and representational resources. In so doing, problem solvers spontaneously conjure and mimetically embody cognitive artifacts that have representational affordances—a symbolic form, a diagram, a figure of speech—as kinesthetic–imagistic substrate supporting the extraction of relevant phenomenal aspects of situated objects. Gesture plays a central role in situated problem solving: gesture bridges from prereflective absorption to reflective attention, from direct intuitive grasps to processes of conscious reasoning and communication. Gesture, a physical action with spatial–dynamic properties, concretizes personal kinesthetic negotiation for inspection, verbalization, and intersubjectivity—gesture grounds embodied quantitative relations for further elaboration.

Reflexive artifacts—e.g., sets of objects affording self-indexing by sorting or tabulation—support embodied mathematical reasoning and therefore merit further design-based research. Of particular interest are cognitive mechanisms governing these perceptual–epistemic negotiations.

Finally, I see a tension between phenomenological and semiotic descriptions of referring-to discourse—a tension that may be hampering productive collaboration within interdisciplinary fields concerned with constructivist pedagogical philosophy, at least with regards to mathematics. This study suggests a need for theoretical perspectives on mathematical cognition that are geared to treat the multimodality of mathematical intuition, reasoning, and practice so as to illuminate issues of design, teaching, and learning. Embodied cognition, informed by phenomenological philosophy, effectively constitutes one such theoretical perspective.

Author Note

Thank you, Katharine G. Young, for tea and a host of figures of speech to embody. Thank you to all members of the Embodied Design Research Laboratory, especially the Undergraduate Research Apprenticeship Program students, for dogged scrutiny and a wealth of insights.

References

Abrahamson, D. (2004). Embodied spatial articulation: A gesture perspective on student negotiation between kinesthetic schemas and epistemic forms in learning mathematics. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the Twenty Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 791 - 797). Toronto, Ontario: Preney.

Abrahamson, D. (2007a). Counter-intuitive reasoning: A design for probability that builds on students' representativeness heuristic for judging outcomes of a randomness generator. *Cognition and Instruction*. (Manuscript submitted for review)

Abrahamson, D. (2007b, April). *The real world as a trick question: Undergraduate statistics majors' construction-based modeling of probability*. Paper presented at the annual meeting of the American Education Research Association, Chicago, IL.

Abrahamson, D., & Cendak, R. M. (2006). The odds of understanding the Law of Large Numbers: A design for grounding intuitive probability in combinatorial analysis. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the Thirtieth Conference of the International Group for the Psychology of Mathematics Education*. (Vol. 2, pp. 1-8). Charles University, Prague: PME.

Abrahamson, D., & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning, 12*(1), 23-55.

Alibali, M. W., Bassok, M., Olseth, K. L., Syc, S. E., & Goldin-Meadow, S. (1999). Illuminating mental representations through speech and gesture. *Psychological Sciences, 10*, 327-333.

Arnheim, R. (1969). *Visual thinking*. Berkeley: UC Press.

Case, R., & Okamoto, Y. (1996). The role of central conceptual structures in the development of children's thought. In *Monographs of the society for research in child development: Serial No. 246* (Vol. 61). Chicago: University of Chicago Press.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Collins, A., & Ferguson, W. (1993). Epistemic forms and epistemic games: Structures and strategies to guide inquiry. *Educational Psychologist, 28*(1), 25-42.

Davis, P. J. (1993). Visual theorems. *Educational Studies in Mathematics, 24*, 333-344.

Goldin, G. A. (1987). Levels of language in mathematical problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 59-65). Hillsdale, NJ: Lawrence Erlbaum Associates.

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909 - 955). Charlotte, NC: Information Age Publishing.

McNeill, D. (1992). *Hand and mind*. Chicago: University of Chicago Press.

Noë, A. (2006). *Action in perception*. Cambridge, MA: The M.I.T. Press.

Schoenfeld, A. H., Smith, J. P., & Arcavi, A. (1991). Learning: The mircrogenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55-175). Hillsdale, NJ: Erlbaum.

Tversky, A., & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases. *Science, 185*(4157), 1124-1131.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge: Harvard University Press.