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1 author:

Dor Abrahamson
University of California, Berkeley

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LEARNING CHANCE: LESSONS FROM A LEARNING-AXES AND BRIDGING-TOOLS PERSPECTIVE

Dor Abrahamson
University of California, Berkeley

The paper builds on design-research studies in the domain of probability and statistics conducted in middle-school classrooms. The design, ProbLab *(*Prob*ability* Lab*oratory), which is part of Wilensky's 'Connected Probability' project, integrates: constructionist projects in traditional media; individual work in the* NetLogo *modeling-and-simulation environment; and networked participatory simulations in* HubNet. *An emergent theoretical model, 'learning axes and bridging tools,' frames both the design and the data analysis. A* learning axis *is the space of potential learning residing between two subconstructs of a domain that the designer identifies as necessary, interdependent, and complementary, e.g., between 'theoretical probability' and 'empirical probability.' The subconstructs are embedded as competing "affordances" of a bridging tool. A* bridging tool *is an artifact, a "mathematical object," designed to foster and sustain students' dwelling in a learning axis and honing the tension between these coupled subconstructs toward coordinating them as a new mental construction. The model is explicated by discussing a sample episode, in which a student reinvents sampling by connecting 'local' and 'global' perceptions of a population.*

INTRODUCTION

This paper is about an experimental unit designed for middle-school students studying probability and statistics as well as about a theoretical model of mathematical cognition. Design-for-learning and theory-of-learning, coupled in a single design-research project, are mutually constructive (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003)—as the project progresses, these efforts incrementally co-coalesce into an integrated coherent framework. Coming into a project, an experimental unit is grounded in and emanates from the designer's initial intuitions as to how a domain might be learned effectively. These intuitions are informed by personal knowledge of the domain, previous contributions to the field, and pilot interactions with students showing difficulty with the target mathematical concepts. Subsequently, as the design-researcher examines students' interactions with the innovative materials and activities under development, the designer's initial intuitions are progressively articulated both into innovative learning supports and into a domain-specific theoretical model of learning and of design-for-learning. This emergent theoretical model, in turn, frames principles of design for mathematical learning tools and contextualizing activities—at the least only for the target domain, and, potentially, for other domains. In sum, although this paper presents as distinct components of a research project both a design rationale and a perspective on students' mathematical cognition, these pieces of the project are only *a posteriori* unraveled, extracted, and parsed, e.g., in academic text; both had been concurrently and reciprocally under development and co-emerging. Finally, the plausibility and utility of a theoretical model could be evaluated in terms of the coherence it lends to the project—framing both the design of learning supports and the analysis of classroom data that are collected during the implementation of the design. This design–theory coherence would not be useful, however, if the design itself did not support classroom activities that the designer–researcher, from the standpoint of particular pedagogical commitments, evaluates as useful learning experiences for the participant students.

Following, I will begin by explaining the learning-axes-and-bridging-tools perspective that grew out of working with students who interacted with the design. Second, I will introduce *ProbLab* (Abrahamson & Wilensky, 2002), an experimental unit in probability and statistics designed for middle-school students—I will explain the design rationale of the unit and several of its key components. Third, I will demonstrate a sample classroom episode that I frame from the learning-axes-and-bridging-tools perspective. Finally, I outline further work underway along the parallel development tracks of design and theoretical modeling and raise questions regarding the nature of mathematical intuition.

LEARNING AXES AND BRIDGING TOOLS (LA&BT)

Learning axes and *bridging tools* (hence, 'LA&BT') are theoretical-cum-pragmatic constructs for design-research in mathematics education. The two constructs are inter-defining, with a learning axis being more about the cognition of mathematics, and a bridging tool being more about the design of learning environments. These two constructs, which have emerged through studies of student learning in diverse mathematical domains, enable articulation of design rationales in terms of understandings of how students learn mathematics. In turn, these constructs enable analysis and description of students' mathematical learning in terms of their interactions with the designs.

X X X
X X X

Figure 1. A bridging tool anchoring student construction of the commutative property of multiplication. Attending alternately to 2 rows of 3 X's or 3 columns of 2 X's—two competing disambiguations of this ambiguous figure—may stimulate student insight.

Figure 1, above, is an example of a picture that could serve as a bridging tool in a mathematics-education design. The picture can be attended to in many ways—it *affords* different perceptions. Two of the possible interpretations of this picture—the picture's meanings—are as 2 rows each made up of 3 X's or as 3 columns each made up of 2 X's. For the learner to build new understanding with these contrasting meanings, s/he would need to be aware of some quality of the picture that is conserved or remains constant across the competing perceptions. In this case the constant is the cardinality of the group (6 X's) or the 'object permanence' of the picture (knowing that the picture does not *really* change).

Between two alternate meanings for a single object extends the *learning axis*, a space of potential learning. The potential learning is evoked when the coexistence of two competing perceptions is foregrounded and problematized, stimulating the learner to construct and possibly articulate a viable reconciliation (a "bridging") of these competing perceptions. In this case, one might construct the commutative property of multiplication ($a*b = b*a$) as a reinvented rule that relaxes the tension inherent in the object's ambiguity.

The LA&BT framework builds upon prior contributions to the Learning Sciences that have responded to a, once prevalent, subject–object dualism by creating models of learning-as-mental-construction and have theorized as to *what* exactly is constructed and *how* (Abrahamson & Wilensky, 2004; Freudenthal, 1973). Mathematical 'knowing' (conceptual understanding) is taken in this framework to be activity based, instantiated in spatial–temporal images that are grounded in learners' embodied interactions with the mathematical instruments and progressively articulated as generative problem-solving skills. Given a constructivist pedagogical commitment and given a target mathematical domain, LA&BT aims to provide specificity for implementing the pedagogy in the form of didactics.

PROBLAB DESIGN RATIONALE AND KEY MATERIALS AND ACIVITIES

ProbLab, created under the umbrella of the Connected Probability project (Wilensky, 1997), is based on a particular reading of previous analyses of student difficulty in understanding the concepts of probability and statistics. This reading highlights pairs of apparently juxtaposed phenomena underlying the domain, such as randomness vs. determinism, dependent vs. independent events, and mean vs. range, and couches students' difficulty with the domain as a struggle to bridge these apparent juxtapositions. This domain analysis, in turn, informs the design principle that students should be given opportunities to focus on the conceptual pairs they have difficulty bridging, i.e. to dwell in the designated learning axes. This design principle is implemented through developing bridging tools and contextualizing activities that support students' experiencing the conceptual tensions of the domain toward unraveling the learning issues as articulated reconciliations of the tensions.

ProbLab is a *mixed-media* unit (Abrahamson, Blikstein, Lamberty, & Wilensky, 2005). It includes activities around bridging tools that are built in traditional media, e.g., crayons and paper, in the *NetLogo* (Wilensky, 1999) modeling-and-simulation computational environment, and in *HubNet* (Wilensky & Stroup, 1999), a technological platform enabling a whole classroom to participate in activity-based inquiry by operating electronic avatars that "live" and "move" in a shared environment. I now briefly overview selected materials, focusing on their embedded learning axes and on the activities that evoke both the construction of these axes and negotiation along them toward reinvention of math constructs.

The basic stochastic variable in ProbLab is a single square that can be either green or blue. In a theoretical, combinatorial-analysis, context, this means that students can choose whether to color in this square with a green crayon or a blue crayon. In an empirical context, such as in a computer-based simulation of a probability experiment, this square would become either green or blue at some chance level. Yet the basic stochastic object in ProbLab—the *"math-thematical"* object—is the *9-block* (see Figure 2, below, on the left), which is a collection of 9 such squares in the form of a 3-by-3 grid. Here again, each of these squares can be either green or blue. There are $2^9=512$ possible arrangements of the green/blue 9-block (see Figure 2, below, second from left, for one possibility of these).

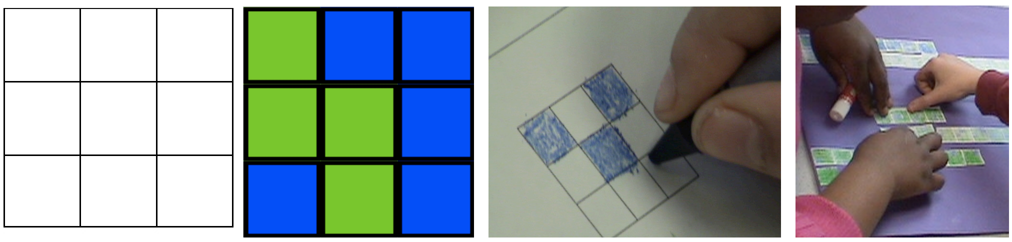

Figure 2. The 9-block, ProbLab's "math-thematical" object.

Working first as individuals and then in a collaborative classroom effort (see Figure 2, above), students create the entire combinatorial space of the 9-block and assemble it in the form of a *combinations tower*. Figure 3, below, on the left, shows a computer image of the entire tower and a detail from its base. In these 10 columns, 9-blocks are arranged according to the number of green squares in them (from 0 to 9). This tower, once displayed in the classroom, extends from the classroom floor up to the ceiling (see Figure 3, below, center).

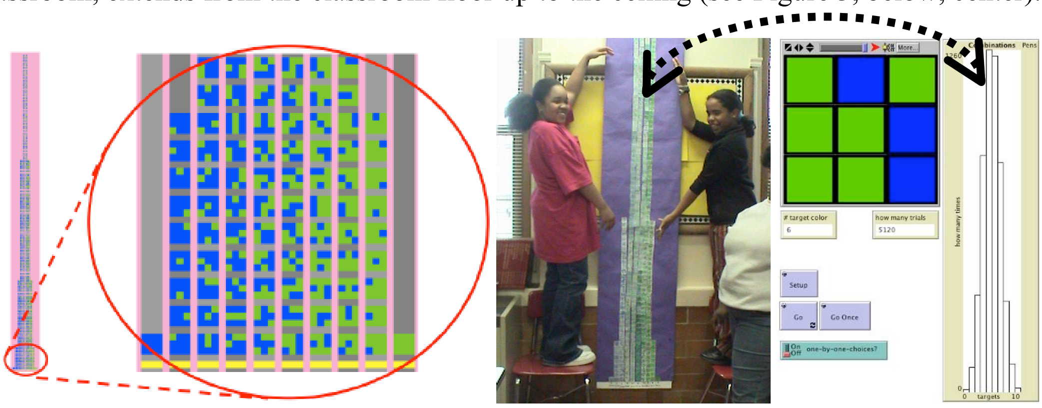

Figure 3. The combinations tower juxtaposed with the outcome distribution.

In a subsequent activity, students interact with a NetLogo simulation, *9-Blocks* (Figure 3, above, on the right). In this model, 9-blocks are generated randomly (each of the 9 squares "chooses" between green and blue). A histogram aggregates the number of occurrences of 9-blocks by the number of green squares in each. For example, in the figure above, there are 6 green squares, and so the "6" column has risen by one unit. Note the figural resemblance between the combinations tower, a structure reflecting theoretical analysis, and the histogram that was produced through random generation of independent outcomes. This shape is a bridging tool—in the context of the theoretical-analysis activity it is a combinatorial space, but in the context of the empirical-experiment activity it is an outcome distribution. How can a single shape be at once both theoretical and empirical?

The insight that reconciles the ambiguity of the tower-cum-distribution shape is that even though the computer is choosing randomly between all possible 9-blocks, it has a greater chance of choosing samples from the central columns of the tower, because there are more samples there to choose from. For example, when taking 5120 samples, each of the 512 unique items appears about 10 times, so the tower "stretches" up discretely by a factor of 10. Even though this scaling up is not uniform—some items chance to appear more than others—the multiplicative relation between columns is by-and-large maintained, since sets of items compensate for each other within columns. This qualitative articulation of randomness captures local uncertainty, i.e., that we can never predict the next sample, whilst maintaining a quasi-determinist sense of the aggregate, emergent distribution, i.e., over a sufficient number of trials, the distribution progressively converges on an anticipated shape.

A SAMPLE CLASSROOM EPISODE FROM THE LA&BT PERSPECTIVE: THE LOCAL–GLOBAL LEARNING AXIS COHERES AS A SAMPLE

Earlier, I explained that the learning-axes theoretical framework is instrumental in the analysis of mathematical domains, and that this analysis, in turn, informs the design of bridging-tools for classroom activities. In this section I demonstrate how the LA&BT perspective also can be used as lenses for analyzing classroom data. Using the same lenses both towards and following classroom implementations helps us evaluate the efficacy of our activity design. In particular, we select episodes in the data that, to our judgment, demonstrate student insight and inventiveness, and we work to articulate this insight in terms of the bridging tools the students were working with, the activity that contextualized the student's work, the underlying learning axis stimulated by this bridging tool, and the statistical construct that the axis potentially coheres as. Following, we will examine a sample classroom episode so as to explain in depth one cluster of the learning axes, bridging tools, and statistical constructs that are enfolded in the design and are enabled through student participation in the classroom activities.

S.A.M.P.L.E.R, Statistics As Multi-Participant Learning-Environment Resource, is a statistics related activity in which students determine the greenness of a "population" of thousands of green/blue squares in a large "mosaic." Without being instructed in sampling, students who engaged in this activity performed quasi-sampling actions. Students: (a) attended to selected spatial locations in the population; (b) used counting actions to inform their sense of the greenness within these selected locations; and (c) coordinated information from these samples so as to determine the population's global value of greenness.

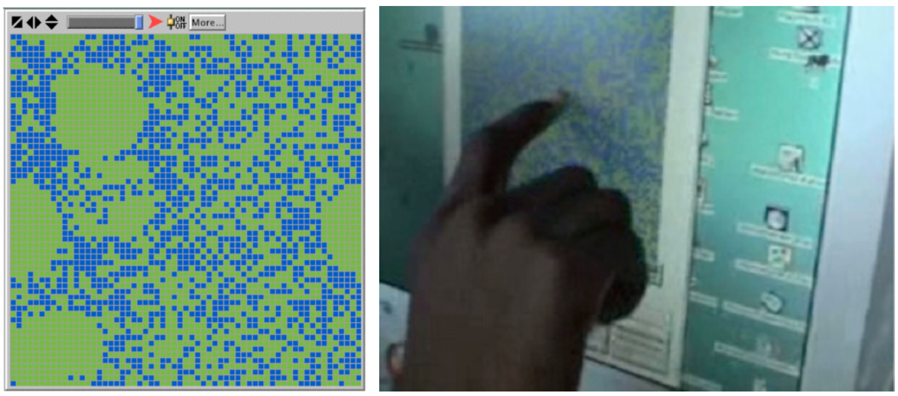

Figure 4. S.A.M.P.L.E.R. population, and a student explaining how he determined a value for the population greenness. The student alternates between 'local' and 'global' modes.

Upon closely attending to students' verbal descriptions and gestures, it appears that their guesses were informed both by counting tiny squares ("local" actions) and by eyeballing the *entire* population and assigning to it a greenness value ("global" actions). So students were using two different methods: local enumeration actions and a global perceptual judgment. Importantly, students did not appear, initially, to be aware that they were using two different methods, nor did they appear to coordinate these methods as complementary. Yet, through discussion with their peers and the facilitator, students had opportunities to connect between these personal resources, as the following transcription demonstrates.

Researcher: [standing Devvy, a low-achieving student, who is working on his individual laptop computer] **What are you doing here?**

Devvy: [gazing at the S.A.M.P.L.E.R. population, index finger hops rapidly along adjacent locations in the population; see Figure 4, above] **Counting the squares.**

Res: **What did you come up with?**

Dev: [hands off screen, gazing at it; mumbles, hesitates] **Around 60 or 59 percent.**

Res: **Sorry… so, show me *exactly* what you're counting here.**

Dev: **Green squares,** [right index on screen, swirls at one location, hops to another, unfurling fingers] **'cause it says, "Find the percentage of the green squares."**

Res: **Uhm'hmm**

Dev: **So if you were to look at it** [left hand, fingers splayed, brushes down the whole population and off the screen] **and sort of average it out,** [touches the 'input' button] **it'd probably equ...** [index on population, rubbing rapidly up and down at center, using little motions and wandering off to the left and then down] **It'd probably go to 59 or 60.**

Res: **And how did you get that number?**

Dev: [index strokes population along diagonal back and forth] **Because it's almost even, but I think there's a little bit more green than blue.**

Devvy's actions are not statistically rigorous—he is not taking equally sized samples, nor is he systematically counting the number of green squares in each sample or methodically averaging values from these counts. But his actions are *proto*-statistical (Resnick, 1992)—without any formal background in statistical analysis, Devvy is going through the motions of statistical analysis, if qualitatively: skimming the population, attending to selected locations, comparing impressions from these locations, and determining a global value. Albeit, Devvy appears to acknowledge the tenuousness of his methods in qualifying his suggested strategy as "*sort of* average it out."

Whereas Devvy's spontaneous local and global methods are as yet disconnected, both methods are grounded in the same object, the S.A.M.P.L.E.R. population. This 'common grounds' constitutes the platform or arena upon which Devvy may negotiate the competing mental resources and bridge them. Devvy may have already begun building a micro-to-macro continuum by attending to clusters of tiny squares, i.e. "samples." Through participating in the S.A.M.P.L.E.R. activities, Devvy's proportional judgments could possibly be connected to his acts of counting. Yet, at this point in the classroom activities, this student's limited fluency in applying proportional constructs does not enable him to quantify his proportional judgment in terms of the local data. Therefore, he begins with a local narrative but, when pressed for an exact answer, he switches to a global approximation.

In summary of this episode, 6^{th}-grade students have personal resources that are relevant to statistical reasoning. The ProbLab activities stimulate these resources and support students' coordination between these resources. Specifically, in the context of determining the greenness of the S.A.M.P.L.E.R. population—the bridging tool in this episode—students invent sampling as an action that reconciles enumeration and perceptual judgment.

CONCLUSION, LIMITATIONS, AND FUTURE WORK

I do not claim that the LA&BT perspective is a design-for-learning panacea for mathematics education. The theoretical model addresses a certain class of mathematical topics structured as pairs of conceptual building blocks that each need to be fit together and that both can, in turn, be embedded as perceptions of a single artifact (object, computer-based simulation, etc.). At this point, I cannot fathom the proportion of mathematical topics that can be fashioned as abiding with these necessary constraints. Nevertheless, mathematical domains can plausibly be characterized in terms of the particular types of difficulties with which they confront learners, and a learning-axes perspective could possibly frame a taxonomy of these difficulties toward developing learning supports and professional-development emphases tailored to particular domains. Questions we could then ask might be: What are the bits of knowledge that need to be conjoined in new ways in order to understand domain *X*?; Why may it be challenging to conjoin these bits of knowledge? For instance, are these conceptual components initially construed as contradictory? Are the axis components by necessity dialectically co-defining? What of learning *systems*, rather than axes—Does learning advance as a gradual *pairing* of stable-enough concepts, or can *several* bits of knowledge come together all at once? Are some learning axes in principle unbridgeable? What is the nature of the relationship between intuitive and formal mathematical knowledge—Do intuitive perceptual judgments ever become *articulated* or are they essentially only *validated* through appropriation of formal procedures? Might "simple" concepts, e.g., multiplication, also be, in fact, opaque composites in need of bridging?

Both the design of ProbLab and its associated research of students' cognition of probability and statistics will continue under the *Seeing Chance* project. Conclusions from the previous implementations of ProbLab have now been applied toward honing the bridging tools, and these improved tools are being researched, beginning with individual students and then scaling up to classrooms, where we will attempt to capture learning gains. The LA&BT perspective will be developed along three tracts: The perspective will be: (a) further refined vis-à-vis Cognitive-Sciences models of learning; (b) couched and packaged in forms that may be useful for mathematics teachers; and (c) applied to other mathematical and, potentially, scientific concepts. Ultimately, the perspective could help practitioners of mathematics education—teachers, designers, and researchers alike—gain insight into students' difficulty with mathematical concepts and respond effectively to this difficulty.

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