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Towards an ecological-dynamics design framework for embodied-interaction conceptual learning: the case of dynamic mathematics environments

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Abstract
Designers of educational modules for conceptual learning often rely on procedural frameworks to chart out interaction mechanics through which users will develop target understandings. To date, however, there has been no systematic comparative evaluation of such frameworks in terms of their consequences for learning. This lack of empirical evaluation, we submit, is due to the intellectual challenge of pinning down in what fundamental sense these various frameworks differ and, therefore, along which parameters to conduct controlled comparative experimentation. Toward an empirical evaluation of educational-technology design frameworks, this conceptual paper considers the case of *dynamic mathematics environments* (DME), interactive modules for learning curricular content through manipulating virtual objects. We consider user activities in two paradigmatic DME genres that utilize similar HCI yet different mechanics. To compare these mechanics, we draw from complex dynamic systems theory a constraint-based model of embodied interaction. Task analyses suggest that whereas in one DME genre (GeoGebra) the interaction constraints are a priori inherent in the *environment*, in another DME genre (Mathematics Imagery Trainer) the constraints are ad hoc emergent in the *task*. We conjecture differential learning effects of these distinct constraint regimes, concluding that ad hoc emergent task constraints may better facilitate the naturalistic development of cognitive structures grounding targeted conceptual learning. We outline a future empirical research design to compare the pedagogical entailments of these two design frameworks.

Keywords Conceptual learning · Constraint · Dynamic mathematics environments · Ecological dynamics · Embodied cognition · Enactivism · GeoGebra · Mathematics Imagery Trainer

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Introduction: digital mathematics environments as a research context to compare vying frameworks for embodied-interaction supporting conceptual learning

Educational designers of instructional activities have recourse to a dazzling array of electronic resources for building multimodal interactions with virtual objects. And yet, how should these resources be selected, and how should they be used? What activities would lead students toward building conceptual understanding and developing new skills? What interaction mechanics would best utilize the technological resources in line with what we know about how people learn? What, then, are effective design principles for engineering technological learning environments? What should students be doing? How might manipulating objects best result in, say, mathematical knowledge?

Left to their own devices, designers of educational modules choose among a variety of existing frameworks and common practices, by which to approach the initial outlining of general patterns for the interaction mechanics that would lead students to conceptual learning (see Fig. 1). What, though, is the merit of these heuristic frameworks, and how do they compare? Does it even matter which framework a designer chooses? Is that choice ultimately consequential for students' learning of the targeted subject matter? To begin evaluating these important and far-reaching questions, this paper takes on a case study of comparing two different design frameworks underlying the activity architecture of interactive mathematics learning modules.

To optimize our investigation, we have identified two genres of embodied-interaction modules (in Fig. 2, see Mod_1 vs Mod_2) that are designed for the same discipline (mathematics), employ a similar HCI (natural user interfaces), and even feature similar types of interactive objects (generic forms, e.g., points and lines), yet differ in the underlying theoretical rationales and activity regimes (DF_1 vs. DF_2). To compare tasks across the two modules, we draw on theories from the movement sciences (viz. ecological dynamics)

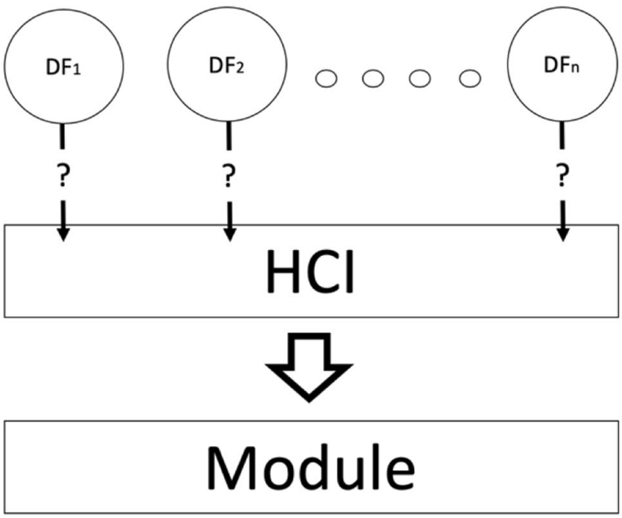

Fig. 1 Educational designers of conceptual learning modules must choose among different design frameworks (DF) in order to select a human–computer interaction (HCI) technology; they then realize the design framework in the form of its interaction mechanics leading to the learning in question

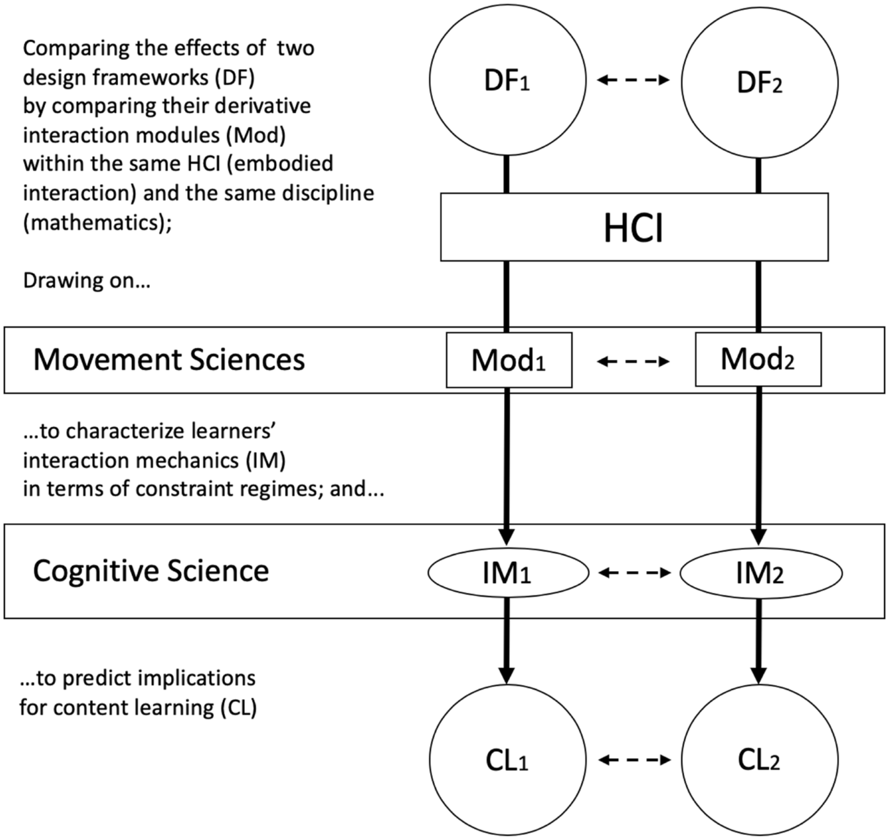

Fig. 2 The rationale of the conceptual study reported in this paper

to apply a criterion (viz. the locus of constraints on action) for the analysis of the module's interaction mechanics (IM_1 vs. IM_2). Then, once we have articulated a critical difference between the modules (viz. task constraint on action vs. environmental constraint on action), we draw on the philosophy of cognitive sciences (viz. Enactivism) to evaluate what this difference in movement regimes across the two activity genres might mean for students' understanding of new concepts (CL_1 vs. CL_2). This evaluation will result in a hypothesis for *how* the two genres differ in their instructional outcomes and *why* this may be the case. We end by offering some guidelines for empirical studies that would follow up on this hypothesis.

The character of this paper is conjectural. We hope to surface for designers of educational technology what we believe is a tacit yet important facet of embodied interaction, namely, students' natural cognitive inclination to develop perceptual structures mediating their goal-oriented manipulation of objects on the screen. These perceptual structures, which we will be calling "attentional anchors," are believed to serve critical epistemic roles in the developmental passage from embodied interaction to content knowledge. That is, when the embodied mind is engaged in managing the dynamical control of objects in the sensory manifold, such as handling an assembly of points and lines on a screen in order to accomplish a task, the mind intuitively comes to form a constellation of these features—a

Gestalt—as its way of maintaining structural invariances, that is, achieving and preserving goal states throughout an interaction. Attentional anchors are like steering wheels that pop up into our engaged enactment, in the sense that they come forth from the background to mediate between our mind and the environment. Attentional anchors are spontaneous psychological forms that may combine actual and imaginary percepts.

Even as they are ad hoc functional structures, in turn attentional anchors can surface to our consciousness as our felt way of doing things. They become the things we refer to as we reflect about and explain how we are doing what we are doing. They become enacted ontologies (or conceptual choreographies, Abrahamson and Shulman 2019). To the extent that it is important for students' conceptual development that they develop attentional anchors through embodied interaction, it would be worthwhile to examine theoretically, and eventually empirically, which activity genres foster the development of attentional anchors. Likely, the two particular types of embodied-interaction learning environments selected for this comparative study will delineate the typology of all such possible environments on a spectrum with respect to their capacity to foster attentional anchors.

More broadly, we believe it is important for the field to develop theories and methodologies by which to characterize what may be critical conditions for students to develop conceptual ontologies through engaging with embodied interaction in educational technology resources. Characterizing conditions for learning through interaction is particularly important as new HCI platforms emerge. Theories of naturalistic sensorimotor learning could help us understand what we could gain as well as what we might lose when digital media supplant pre-digital media.

Pre-digital pedagogical artifacts fabricated from concrete materials, such as a yarn ball, offer naturalistic learning about their qualities, such as their structure, composition, purpose, and function, just as long as their exploration enables embodied interactions, such as manipulation, deformation, de/reconstruction, and application. In the work of Froebel (1885/2005), the yarn ball is animated as a protagonist in kindergarten stories, where it is moved in a variety of dynamic forms, such as hopping or dancing. As pre-digital artifacts are converted into the virtual medium, their multimodal affordances, such as the haptic sensation of a yarn ball's elasticity, the proprioceptive sensation of gripping its spherical structure and rolling it in the palm, or the kinesthetic sensation of swinging it from a string, are all liable to be lost in translation. This is why we are interested in developing a comprehensive understanding, drawing from movement research, on relations between sensorimotor interaction and conceptual learning. Such theoretical understanding, we suppose, would better equip us to create effective digital learning environments.

Motivating a theoretical comparison of two mathematics interaction designs

Digital mathematics environments (henceforth, "DME") are interactive technologies designed for learning content through engaging in inquiry-based activities oriented on manipulating virtual objects. We have selected DME for studying design principles of embodied-interaction conceptual development, because we have discerned two paradigmatic DME genres that differ in ways we that we perceive as consequential for the nature and outcomes of learning. Focusing on one exemplar from each DME genre, we will argue that apparent differences between them are widely important for research and development of educational technology, at least for mathematics. We, thus, aspire to contribute to the field of educational-technology design theoretical knowledge of relations between manipulating and learning (Allen and Bickhard 2015; Ball et al. 2018; Tarasuik et al. 2017;

Martinovic et al. 2013; Sarama and Clements 2009) as well as practical knowledge of heuristic design frameworks (Abrahamson 2014; Chase and Abrahamson 2018; Kali et al. 2009; Pratt and Noss 2010; Yerushalmy 2013).

This is not a first attempt to compare different actionable approaches to designing interactive educational technology (Artigue et al. 2009; Artigue and Mariotti 2014; Drijvers et al. 2013). Yet these comparisons often report back the conclusion that theories of learning cannot speak across epistemological divides. Perhaps a way forward, in comparing embodied-interaction design frameworks, we submit, would be to work with a theory of interaction that is already spelt out in terms that are congenial to the specificities of this particular activity genre, namely, in terms of relations between features of the learning environment, demands of the activity task, and students' entering capacity. Let us briefly elaborate, below, on what DME are, so as to justify our choice of DME as a study context and our choice of ecological movement theory for the comparative analysis of two DME genres.

Context: dynamic mathematics environments (DME)

Imagine a computer screen featuring an interactive quadrilateral. When one drags any of its vertices or edges, it remains quadrilateral. Now further imagine two activities. In Activity A, the quadrilateral is initially a parallelogram, and no matter how one reconfigures it, it always remains a parallelogram; you are tasked to figure out properties of this unfamiliar shape. In Activity B, the quadrilateral is initially not a parallelogram, yet, as one reconfigures it, it turns green whenever it happens to be a parallelogram; you are tasked to keep the shape green as you reconfigure it. Assuming you have never before studied parallelograms, would these activities bear different effects on your conceptual learning? As researchers, how should we approach this comparison? Why might we expect the activities to bear different effects? What epistemological or theoretical grounds frame our prediction?

Both of the above examples are cases of dynamic mathematics environments. DME are inquiry-based, technology-enabled learning environments, in which students develop an understanding of new mathematical concepts through manipulating virtual objects on an interface and reflecting on patterns or principles they discern in so doing (e.g., Abrahamson and Trninic 2011; Leung et al. 2013; Schansker and Bikner-Ahsbahs 2016; Schwartz et al. 1993; Sinclair 2014). Over the years, and across a global range of development efforts, DMEs have come to constitute a broad spectrum of products, and these vary in activity architecture. This variation in activity architecture, in turn, can be attributed to variation in the software developers' underlying design rationales, theory of learning, and pedagogical philosophy.[1] For educational design researchers, this variation in activity architecture creates a serendipitous context to surface, articulate, and compare impacts on cognition, teaching, and learning. Such critical evaluation of design products may, in turn, lead to working hypotheses framing new empirical studies, notably studies that look to juxtapose DME variants with respect to their theoretical warrants, cognitive consequences, and implications for the educational practice of embodied-interaction design.

Our particular interest will be in what might appear as an inconsequential technical feature of DME activity architecture—the question of *Who gets to constrain the student's*

[1] Whereas one may infer designers' frameworks through analyzing their products, these frameworks may or may not be explicit in their public discourse on their work (Barwell 2009; Schön 1983; Vagle 2010).

interaction with the virtual objects: the software or the student?—and the consequences of this design decision for the cognitive process of developing new mathematical concepts. We compare exemplars of each design architecture, at times polarizing them for rhetorical clarity or theoretical acuity. As such, we will accentuate in each environment design those features that are most relevant to the comparison.

Theoretical perspective: ecological dynamics

Ecological dynamics (Araújo et al. 2009) is a conceptual framework for theorizing, designing, and administering the learning and teaching of physical skill. We selected ecological dynamics as a guiding framework for examining DME design, because whereas DME mechanics foreground sensorimotor inquiry, research on DME has not foregrounded sensorimotor behaviors that are involved in learning with DME. In DME, students' digital actions depict, constitute, and transform the mathematical structures they are examining. Learning is experienced through moving objects in the display. Performing these manipulations requires dexterity. Therefore, practicing mathematical inquiry in DME involves the coordination of sensorimotor faculties. As such, research into how people learn in DME appears to require understanding how people learn to move in new ways. We therefore assume that educational designers of embodied interaction could avail of theoretical frameworks for modeling how people learn to move in new ways and how other people might best help them do so by shaping and guiding their experiences (Abrahamson and Sánchez-García 2016; Beilock 2008; Kelton and Ma 2020; Nathan and Walkington 2017). As its name suggests, ecological dynamics is rooted intellectually both in ecological psychology (Gibson 1977; Heft 1989) and in dynamic systems theory (Thelen and Smith 1994, 2006). Ecological dynamics is a systemic framework, in the sense that it models the phenomenon of skill learning not as occurring uniquely within or by an individual organism but as distributed over the natural and cultural environment, which includes material entities but also peers, coaches, artifacts, norms, values, and so on (cf. Anderson et al. 2012; Barab and Plucker 2002; Jacobson et al. 2016). Ecological dynamics approaches the study of learning as an intrinsically systemic phenomenon.

Per ecological dynamics, physical learning depends on the emergence of ecologically coupled perceptual attunements facilitating the enactment of tasks. The organism operates purposefully on the environment to accomplish its goals. Yet in so doing, the organism encounters a range of haphazard constraints. To tackle these constraints, the organism both adapts how it perceives the sensory manifold (i.e., it assembles selected environmental features of apparent relevance to interaction) and reorients its motor actions vis-à-vis these features. This process of sensorimotor adaptation is iterative. From this process emerge ways of perceiving the environment that are conducive to carrying out necessary motor actions. A perception-for-action of the environment is called an *affordance* (see below).

By way of interim summary, perceptual forms—particular ways of construing the environment—emerge through iterative, goal-oriented, explorative action on the environment. Perceptual forms iteratively emerge and stabilize to coordinate the neuromuscular activation of sensorimotor schemes required for successful enactment of movements to accomplish a situated task. As such, researchers working within an ecological-dynamics approach seek to model skill learning as the emergence of human–environment action-oriented relations, namely, affordances.

As epistemological constructs go, affordances constitute irreducibly relational ontologies—they are tacit, pre-reflective, and immersive phenomenological syntheses—situated, enactive, and interactional knowings (Turvey 2019). The idea of an affordance foregrounds the formative role of sensory perception as the human–environment functional bind. Perception is always perception-for-action (Rehrig et al. 2020; Varela et al. 1991). Thus, the interplay of perception and action is co-constructive and mutually serving, with perception guiding action, even as action promotes perceptual vantage (Fiebelkorn and Kastner 2019; Maturana and Varela 1992; Schroeder et al. 2010). Affordances, thus, come forth as the organism iteratively adapts its environmentally coupled motor action to increase access to sensory information conducive to enacting effective movement; which, in turn, reveals opportunities for greater sensory acuity and even new sensory modalities and dimensions for refining motor engagement (Abrahamson 2020). Discovering affordances is, therefore, a recursive process that gradually maximizes an organism's grip on the world (Merleau–Ponty 1962, cited in Dreyfus and Dreyfus 1999, p. 103).

When we detect new affordances in the environment that prove conducive to action, the nature and quality of our engagement may change along multiple parameters. These changes, per ecological dynamics, should be examined systemically. Viewed as elements of a complex, dynamical, adaptive, and self-organizing system, human–environment relations are volatile and susceptible to phase transition. The functional system remains stable in dynamical equilibrium only inasmuch as the organism's neuromuscular capacity can accommodate for changes along various dimensions of the task and/or environment. That is, when an organism–environment dynamical system-in-action is perturbed by changes along dimensions of susceptibility, its structural composition will change. However, the system's reaction to perturbation may vary qualitatively. Transitions in an organism's motor behavior due to systemic perturbation have been characterized either as small changes (shifts) that maintain the system's current organization or large changes (bifurcations) that dramatically reorganize motor coordination (Kelso 1984; and see parallels to Piaget's reflective abstraction, in Abrahamson et al. 2016c).

The theory of ecological dynamics highlights for designers of embodied interaction the ontological difference between a person's observed behaviors and the underlying systemic organization that gives rise to these behaviors. This stark difference, claim theorists of ecological dynamics, implies a pedagogical approach to skill learning that is unlike direct instruction (Chow et al. 2016). Instead, individuals learn to perform particular cultural practices by adaptively reconfiguring their idiosyncratic sensorimotor relations to the environment so as best to achieve task demands inherent to the practice. As such, variation across individuals with respect to skill-learning process is anticipated, embraced, and leveraged. To be sure, ecological-dynamics learning environments are highly designed and rigorously facilitated. And yet ecological-dynamics designs and facilitation seek to optimize for individual adaptivity by deliberately introducing looseness into the human–environment coupling (cf. Newman et al. 1989, p. 62, on looseness). As we will explain in a later section on the work of Karl Newell, a key theoretical construct informing the design and facilitation of motor skill learning, per ecological-dynamics, is that of constraints.

We propose the theory of ecological-dynamics and its corollary *constraints-based pedagogy* (Araújo et al. 2009; Chow et al. 2016) as lenses for examining and comparing two mathematics-education design approaches to building activities for students to learn a new mathematical concept through manipulating objects on an interface. Our comparison will focus on characterizing the phenomenology of manipulation through the lens of interaction constraints. In particular, *we will follow the ecological-dynamics paradigm to implicate the sources of constraints on dynamic manipulation—whether they reside in the student, the environment, or the task—and we will speculate on the consequences of the source of constraint for students' conceptual learning process.*

Ultimately our comparison will dwell on attributes specifically of task constraints that the student constructs through interaction. A self-imposed task constraint, we will argue, may originate in an individual's formulation of a new goal as a personal means of accomplishing the task objective. As compared to other constraints on learned forms of purposeful behavior, we will argue, student-constructed task constraints may bear advantages for the cognitive process of learning a DME's targeted conceptual notions. In particular, we maintain, self-imposed task constraints enable students to learn by accommodating what they are able to do into what they are almost able to do. This accommodation is predicated on gradually assimilating the environment, namely, coming to perceive the environment in a new way. Students construct these action-oriented perceptual forms spontaneously to facilitate the coordinated enactment of movement solutions to the task's dynamic-control problem. Thus, action-oriented perceptual forms constitute products of students' engaged interactive inquiry—the perceptual forms are empirically validated achievements of the student's concerted effort to develop situated know-how. In turn, these perceptual forms become epistemically available and amenable as ontologies for mathematical inquiry (Abrahamson et al. 2016c; Bongers et al. 2018). As such, developing new action-oriented perceptual forms is a pivotal and necessary prerequisite of learning a new concept.

When, in contrast, constraints on action are imposed by the environment, we surmise, the learning trajectory may be different. It may be discontinuous, because some environmental constraints would bar students from engaging the task naturalistically, that is, they would bar the student from: (a) bringing to bear their naïve sensorimotor schemes; followed by (b) recognizing the inadequacy of these schemes; and finally (c) accommodating the schemes. Imagine a toddler drawing a circle under two conditions. In one condition, exemplifying a self-imposed task constraint, she is handed a pencil and asked to trace along a circle that is already printed on a sheet of paper. As she attempts to do so, the pencil keeps trailing off the circumference into linear tangents, and she keeps correcting her actions. Gradually, she learns to guide the pencil along the circular line. In so doing, the toddler has learned to constrain her actions systematically, resulting in the coordination of a new sensorimotor scheme, which she may be able to replicate in the absence of the model circle. In the other condition, exemplifying an externally imposed environmental constraint, the child holds onto a pencil guided by a robotic arm that revolves on paper in perfect circular motion. Here the child need not exercise agency in circumscribing, consequently she may not develop appropriate sensorimotor schemes.

We hasten to acknowledge the rhetorical nature of this comparison between task and environmental constraints, which we have offered to exemplify these theoretical constructs. Our point is not to cast judgment but to pique the field's interest and motivate empirical inquiry into tradeoffs of various pedagogical regimes. At this point, the field does not know enough about relations between embodied interaction mechanics and consequences for conceptual learning, which is why we are calling for empirical work and suggesting a theoretical framework to guide this work. Ecological dynamics will serve us, later in the paper, to hone the comparison of the two DMEs in question. But, first, the section "An overview of the two DMEs in question" will stage the comparison by explaining the two DMEs. Then, the section "Applying the theory of ecological dynamics to investigate learning with DME" will elaborate further on the theory of ecological-dynamics foregrounding its formulation in the work of the kinesiologist Karl Newell, who proposed the ecological view that motor skill emerges within a triadic system of organismic, environmental, and task constraints (Newell 1986, 1996; Newell and Ranganathan 2010). The section "A multidimensional comparative analysis of two DME activities" then applies ecological-dynamics to compare the two DMEs. Following the "General discussion" section, the "Conclusion" section closes the paper with an outline for an empirical project that would strive to hone and resolve the theoretical and pragmatic tensions we hope to raise.

An overview of the two DMEs in question

DMEs emerged in the early 1980s to leverage technological developments in human–computer interaction engineering, such as screen graphing and the computer mouse (Goldenberg et al. 2008). These technological improvements enabled a new vision of children learning through interacting with computers (Papert 1980). In parallel, the computer was endorsed as a potentially powerful environment for implementing earlier ideas that children could learn mathematics through solving carefully constructed problems (e.g., Polya 1945/1988). Several design approaches emerged for leveraging interactive computers in mathematics education, and this article will look specifically at two types of DME.

In the first type of DME, students work on educational problems involving interactions with ontologically stable figural structures, whose consistent properties the student is to identify. We will call this type of DME "xDME"—"x" connotes that the targeted mathematical equivalence class remains closed, that is, intact or immune to violation through direct interaction. This design approach harks back to early reform-oriented pedagogical regimens of offering children prefabricated concrete materials, such as a yarn ball (Froebel 1885/2005). The child is to learn new concepts through inquiry-oriented manipulation, where the object is the focal phenomenon grounding formal reasoning about disciplinary ontologies (Froebel 1885/2005; Montessori 1949/1967). Per this approach, scientific practice is the study of objects, and scientific knowledge is what we learn through analyzing the world. Any physical movement performed in the course of conducting the inquiry, such as reaching, grabbing, and displacing the objects, is elided from analyses of learning, as though motor action is merely a pragmatic, functional, or perfunctory means of obtaining sensory information; as though in and of itself sensorimotor experience or competence carry no conceptual grounding or meaning. To return to an earlier example, xDME might include a parallelogram that can be variably prodded, stretched, and rotated yet will preserve its geometrical structure invariant. You cannot deform it into a trapezoid.

In the second type of DME, students work on educational problems involving interactions with figural structures whose defining essence can be interrupted. We will call this type of DME "oDME"—"o" is for "open," connoting that the targeted mathematical equivalence class can be modified. This latter design approach is inspired by cognitive science theories of embodied cognition, which posit the conceptually constitutive role of purposeful, situated, dynamic, and corporeal phenomenology of perceptually organized action (e.g., Kiverstein and Clark 2009; Newen et al. 2018; Shapiro 2014; Varela et al. 1991). The child is to learn new concepts through inquiry-oriented manipulation, where the focal phenomenon is not the object but the child's *dynamic interaction*, and in particular the emergent sensorimotor coordinations—eyes–hands, in this case—enabling the enactment of task-adapted movement forms. Per this approach, scientific practice is the study of interaction, and scientific knowledge is what we learn through interacting with the world. oDME design is founded on embodied-interaction epistemology, which conceptualizes knowledge not as transmitted information but as situated know-how (Dourish 2001). oDME design thus seeks to occasion opportunities for students to learn curricular content by discovering the affordances of the environment. This paper inquires into DME embodied-interaction design principles that optimize for learning mathematics through discovering affordances.

"Closed" dynamic mathematics environments (xDME): GeoGebra

We use the term xDME in referring to educational technologies for learning mathematical ideas through iterated cycles of building, manipulating, and inquiring into virtual dynamic objects (Bamberger 1999; Hoyles 2018; Hoyles and Noss 2009; Sinclair et al. 2016). In their original conception at the early 1980's, xDMEs' graphical and interaction features were inspired by contemporary advancements in hardware design followed by a boost in software applications, and particularly computer games (Goldenberg et al. 2008). xDME activity design drew its inspiration from pedagogical philosophies, such as constructionism, which champion dedicated environments, where students learn through building, manipulating, and inquiring into structures (Papert 1980; Resnick et al. 1988). xDME was also inspired by mathematicians, whose publications unraveled strategies and heuristics for learning by solving problems (Polya 1945/1988; Schoenfeld 1985).

In its early years, xDME researchers focused mainly on the mathematical content of geometry—hence the commonly used name "dynamic geometry software," or DGS, for geometry-oriented xDME. Three notable environments were created, roughly in parallel: the Geometric Supposer (Schwartz and Yerushalmy 1987), the Geometer's Sketchpad (Jackiw 1995), and Cabri geometry (Laborde and Laborde 1995). Both the Geometer's Sketchpad and Cabri geometry have been developed to encompass other topics in the mathematics curriculum, and their software has been adapted for implementation in various platforms and devices. A more recent and widely used xDME, GeoGebra (Hohenwarter et al. 2009), includes the following features—it is: free and open-source; operable from desktop computers, tablets, and smartphones; translated into 63 languages; integrated into other technological systems, such as the Virtual Math Teams, for remote collaboration (Stahl 2009; Oner 2016); and equipped with automated formative assessment (Olsher et al. 2016).

The last decades have witnessed an increasing research interest in understanding how xDMEs enable users to create learning activities (e.g., Alqahtani and Powell 2017; Leung and Baccaglini-Frank 2016; Soldano et al. 2019) and what content can be learned therein (e.g., Jacinto and Carreira 2017; Sinclair and Yurita 2008). This research interest in xDMEs' potential application and curricular scope is greatly supported by the GeoGebra website, which curates over one million interactive modules designed by users from around the globe.

When xDME users (i.e., teachers and students) construct virtual structures with mathematically defined behaviors, the nature of this construction process depends on the interface, which requires either programming command lines composed of symbolic notation and formal syntax (Hoyles et al. 2002; Kynigos 2004; Papert 1980; Resnick et al. 1988; Roschelle et al. 2000) or applying preset ontological primitives, such as points, lines, and arcs (Arzarello et al. 1998; Chase and Abrahamson 2018; Hohenwarter et al. 2009; Laborde and Laborde 1995; Schwartz and Yerushalmy 1987). The designer of an xDME module may place the mathematical objects they have constructed at selected locations within a virtual Cartesian domain. Once placed, these objects can be dragged about on the screen, via either mouse, touchpad, or direct touch, depending on the particular platform. Typically, the designer will further define mathematical relations *between* two or more objects, for example, one line is defined as perpendicular to another line, or a point is defined as bisecting a segment. These relational properties will remain intact upon dragging the mathematical objects—as one object is dragged, its dependent object will adjust its properties, so as to maintain the encoded relation. As such, the user manipulating an object cannot

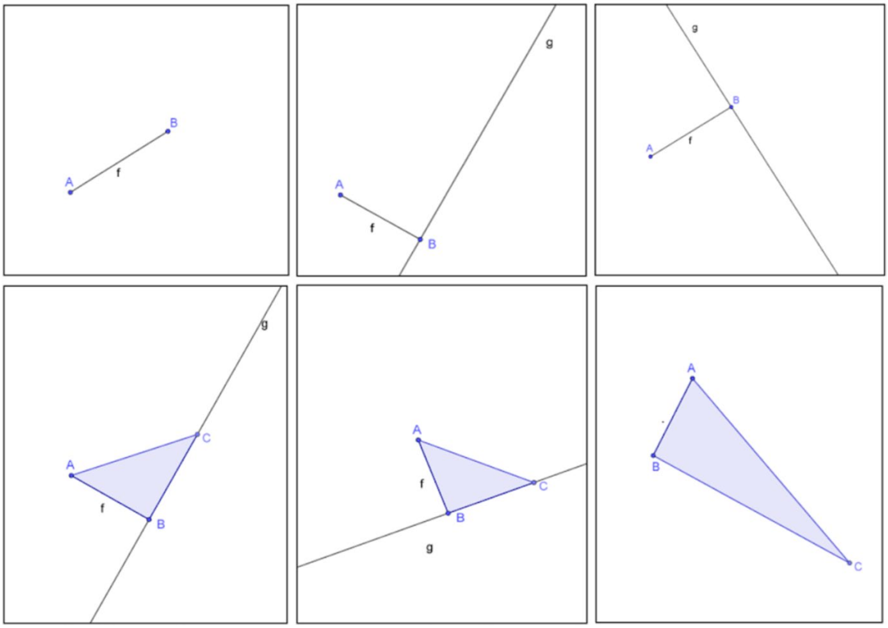

Fig. 3 Constructing and manipulating a right triangle in GeoGebra

violate its encoded interdependencies with other objects, unless the user modifies the module's underlying code.

Figure 3 features phases in the construction of a right triangle as well as some possible subsequent actions for inquiring into its properties. Creating a dynamic right triangle can be done as follows:

1. Set Points A and B on a Cartesian domain to create Segment *f*, defined on these points (top-left). Points A and B are dynamic and independent: dragging Point A keeps Point B static and vice versa, with Line *f* adjusting in length and/or orientation to maintain itself as subtended between the two points;
2. Create Line *g* defined as perpendicular to Segment AB through Point B (top-center). This structure is encoded to maintain intact the defined relationship between the two objects (i.e. the perpendicular relationship between the line and the segment), so dragging either Point A or B would keep Line *g* and Segment *f* perpendicular (see top-right, where Point B was dragged anticlockwise relative to Point A);
3. Next, add Point C on Line *g* (bottom-left) to form Triangle ABC. Point C can be dragged only along Line *g*.
4. Triangle ABC's three vertices are all dynamic, yet dragging any one of them will automatically maintain the triangle right-angled. Dragging Vertices A, B, or C will change the shape of the triangle but will keep it a right triangle (bottom-center), where Segment BC remains perpendicular to Segment AB (compare bottom-left and bottom-center). Vertices A and B remain independent, so moving Vertex A still keeps Vertex B static, and vice versa. Dragging Vertex C is still only possible along Line g. Note that dragging Vertices A and C changes the proportions of the triangle's segments, whereas dragging Vertex B maintains the proportion between the triangle's segments.

5. All the unnecessary objects on the screen (e.g., Line *g*, bottom-right) can be hidden to leave visible only a right triangle composed of three segments.

Whereas the xDME design rationale assumes that students will learn about geometrical forms by building and manipulating them, the software can be used to offer students prefabricated geometrical forms. Research into teachers' instructional practices with xDME has demonstrated that most teachers refrain from asking students first to create interactive mathematical structures from scratch. Instead, the teachers have students interact with prepared objects in available ready-made modules (Abdu and Niv 2019). Based on an extensive survey, Abdu and Niv (2019) attributed teachers' reluctance to use GeoGebra's constructionist functionalities to the teachers' greater comfort with familiar curricula, discomfort with orchestrating new technological environments, beliefs that the software was suitable for advanced but not beginner students, and disengagement with professional-development modules oriented on using the software. Mor and Abdu (2018) further show that teachers may experience discomfort with movement- or transformation-based conceptualizations of mathematical ontologies (a challenge, we add, that would probably hold for both xDME and oDME). Consequently, one popular way instructors use xDME is by creating well-defined dynamic modules, as in Fig. 3. These modules set a problem space, where targeted mathematical concepts, for example, a right triangle, are encoded as the properties of a virtual assembled object. The to-be-discovered properties are invariant—they will not change under transformation (Leung 2003).

A pedagogical practice of having students interact with a prepared object to inquire into its behaviors and properties harks back to the dawn of reform-oriented early childhood education (Dewey 1916/1944; Froebel 1885/2005; Montessori 1949/1967) and resonates with experimental methodologies employed in the empirical work of cognitive-developmental psychologists evaluating the theory of genetic epistemology (Piaget 1968). Moreover, identifying, characterizing, and utilizing invariants are fundamental epistemic practices among mathematicians. Whereas many of the invariants that mathematics educators take for granted are not apparent to children (Vergnaud 1982), children can "reinvent" these invariants through participating in activities that simulate mathematicians' practice (Freudenthal 1991). xDME are designed to provide these opportunities. Lesson designs concerned with the invariant principle can use xDME modules to lead students towards developing proto-concepts and forming conjectures (Hadas et al. 2006; Leung 2011; Leung et al. 2013; see also Meira 1998).

Students' interaction with xDME software consists of exploring the geometrical construction to determine its unknown properties. They generate hypotheses about a class of geometrical objects, such as right triangles, and then attempt to empirically verify or refute the hypotheses. For example, a student may interact with the triangle in the example above: Upon dragging Point A, B, or C, Triangle ABC changes its form yet maintains invariant all its predefined properties (e.g., two edges remain perpendicular, see Fig. 3, bottom-center and bottom-right), while allowing for change along other dimensions, such as orientation or scale.

We have presented a paradigmatic module and explained how it could serve as a resource in a lesson design for investigating mathematical ideas. This general plan for designing, building, and facilitating xDME modules is broadly applicable across mathematical ideas, for example, in determining the defining properties of a parallelogram through manipulating its vertices and edges or in determining relations between a parabola's parameter values and its shape. Working in this context requires learners to explore the range of possible ways in which points, lines, and arcs can be dragged (Leung et al. 2013) to discern invariant properties of mathematical objects (Leung 2003, 2011). Invariance in xDME should emerge for the student as an object's property that does not change as a result of dragging (Sinclair 2018). For example, dragging Vertices A, B, and C (see Fig. 3) and looking for invariants could bring a learner to discern that Segments AB and BC remain perpendicular for all cases.

Having discussed GeoGebra as a case of xDME, we now turn to discuss a case of oDME, the Mathematics Imagery Trainer.

"Open" dynamic mathematics environments (oDME): the Mathematics Imagery Trainer

The Mathematics Imagery Trainer (henceforth, "the Trainer") is a technologically enabled learning environment. The Trainer integrates and applies two cadres of psychology theory: (a) theories that model cognitive capacity as emerging from the organism's adaptive, goal-oriented, and situated multimodal interaction (viz. constructivism, Piaget 1968; radical constructivism, von Glasersfeld 1987; enactivism, Varela et al. 1991); and (b) theories that model human learning as the process of appropriating heritage cognitive routines that enable productive participation in normative social enactment of cultural practices involving material and immaterial artifacts that mediate effective engagement with the environment (viz. cultural–historical psychology, Stetsenko 2017; Vygotsky 1926/1997; Wertsch 1998). Constructivist/enactivist theories and their various derivatives have inspired prodigious efforts in mathematics educational research (e.g., Arnon et al. 2013; Moreno-Armella et al. 2008; Pirie and Kieren 1994; Steffe and Kieren 1994), as have sociocultural theories (e.g., Saxe 2012; Sfard 2008). Some scholars have sought to reconcile these two bodies of work, given their ostensibly disparate epistemic axioms (Cole and Wertsch 1996; diSessa et al. 2015). The Trainer's design rationale likewise nurtures from this theory reconciliation.

The integrated constructivist/enactivist motivation of the Trainer design is lyrically captured by Skemp (1983): "[M]athematics, like music, needs to be expressed in physical actions and human interactions before its symbols can evoke the silent patterns of mathematical ideas" (p. 288). Implementing this pedagogical motto in the form of instructional resources—whether xDME or oDME—would create opportunities for students to develop what Skemp (1976) calls deep *relational* understandings of mathematical content, as compared only to surface *instrumental* understandings. It would be important for the field of educational technology design to understand whether students learn targeted mathematical content better when they enact physical action forms believed to instantiate the relevant cognitive practices.

The sociocultural conceptualization of the Trainer design is modeled on a *field of promoted action* (Reed and Bril 1996), that is, a form of interventional practice, including dedicated space, materials, and activity, that fosters children's development of particular motor capacity (Abrahamson and Trninic 2015). As such, the Trainer is an instrumented field of promoted actions believed to bear *semiotic potential* (Bartolini Bussi and Mariotti 1999, 2008) as instantiating mathematical concepts. By "semiotic potential" Bartolini et al. (1999, 2008) coin a characteristic of embodied cultural activities, specifically manipulation-based classroom instructional tasks. These tasks inhere opportunities for teachers to shift students from informal task-oriented perceptions of, and actions on artifacts to formal redescriptions, significations, and conceptualizations of these orientations in disciplinary forms and nomenclature as instantiating a mathematical notion. Learning is further

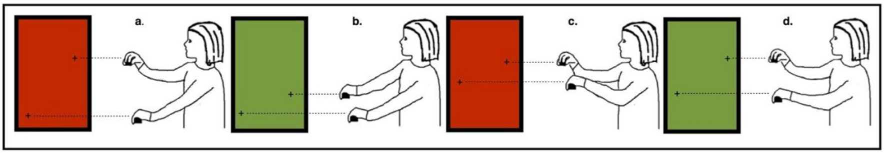

Fig. 4 The Mathematics Imagery Trainer for Proportion: Schematic interaction sequence leading to discovery. Moving one cursor does not affect the location of the other cursor. Art credit: Virginia J. Flood, PhD

actuated through social interaction: When students work with a tutor (Flood 2018; Flood et al., in press; Shvarts and Abrahamson 2019) or with another student (Abrahamson et al. 2011; Abdu et al. under review), mathematical terminology emerges spontaneously as a pragmatic solution to the discursive problem of planning, coordinating, monitoring, and regulating the simultaneous co-enactment of joint action.

Whereas Trainer activity designs cover a gamut of mathematical concepts (e.g., parabolas, Shvarts and Abrahamson 2019; trigonometry, Bongers et al. 2018), we will focus our discussion on a paradigmatic Trainer for proportional reasoning (Abrahamson and Howison 2008; Abrahamson and Trninic 2011).

Trainer activities present students with virtual objects, some or all of which are manipulatable. Figure 4 features a Trainer for proportion, where two cursors can each be independently moved up and down along their respective vertical axes. A green ("success") signal is activated on the screen whenever the two cursors are positioned at locations whose numerical measures, concerning a certain frame of reference, relate to each other by the mathematical function constituting the oDME's pedagogical objective; here, the screen will turn green whenever the cursors' heights, gauged from the bottom of the screen, are related by a 1:2 ratio. Neither the target concept nor the underlying frames of reference or the objects' coordinate values are initially revealed to the students. Students are only asked to "make the screen green" and, once they have succeeded, to move both hands, keeping the screen green. Typically (follow Fig. 4), students will: (a) explore the space; (b) stumble upon a case of the hidden function; (c) reconfigure the assembly of virtual objects in accord with a contextually inappropriate hypothesis for what should remain invariant (here, erroneously keeping invariant the spatial interval between the cursors); and (d) figure out how to move within the intended invariance (here, varying the size of the interval relative to its location up and down along the screen; compare Fig. 3b, d, where the interval could be either the vertical displacement of the two cursors or a diagonal line connecting them). The first phase of Trainer activities ends when the students *demonstrate* they are able to stably move their hands while maintaining the green feedback and are able to *explain* to another person how to perform this feat of motor control.

Once students demonstrate and explain their strategy for "moving in green," the tutor—whether a human or an artificially intelligent interactive virtual pedagogical agent—interpolates onto the screen additional symbolic artifacts, such as a grid, but without specifying their purpose (compare Fig. 5a, b). Abrahamson et al. (2011) found that elementary and middle-school students immediately discern in these supplementary materials utilities for enhancing their action and discourse, namely they detect in these virtual inputs embedded resources for better enacting, explaining, or evaluating their strategy of moving in green. Yet in the course of utilizing these strategic advantages, the students redistribute their sensorimotor routine over the new materials (cf. Kirsh 2010; Martin and Schwartz 2005). The materials thus become frames of reference

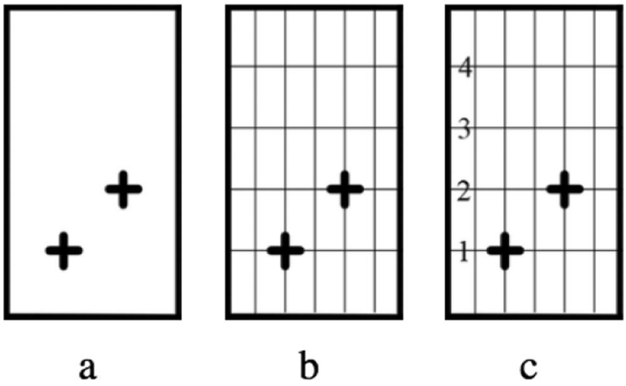

Fig. 5 Symbolic artifacts overlaid onto the Mathematics Imagery Trainer activity space: **a** the two cursors; **b** a grid; **c** numerals

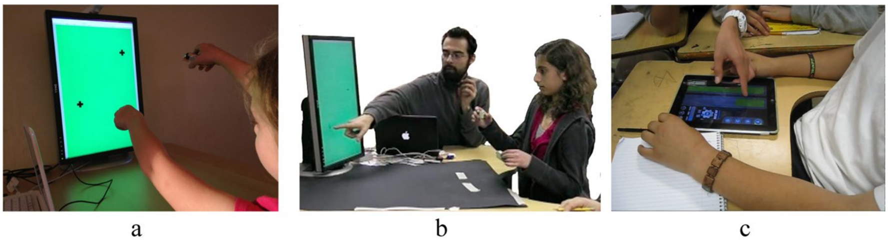

Fig. 6 The Mathematics Imagery Trainer for Proportion in action: **a** a child working on her own; **b** a student working with a tutor; and **c** two students working together on a Trainer iPad app

that reorganize the students' perceptual orientation toward enacting their strategy, consequently transforming the strategy itself, now deployed over and through the supplementary materials. For example, students who had been raising both hands simultaneously within continuous space, explaining their strategy with qualitative language, such as, "The higher my hands go, the bigger the distance between them" (corresponds to Fig. 5a) find themselves raising both hands *sequentially* within *discrete* space, explaining their new strategy with *quantitative* language, such as, "For every 1 unit I go up on the left, I go up 2 units on the right" (corresponds to Fig. 5b). Subsequent introduction of numerical symbols (Fig. 5c) again shifts students' strategy, because they now draw on their arithmetic knowledge to detect and anticipate quantitative patterns: Rather than work iteratively, such as rising 1 unit on the left, then 2 on the right, then over again, the students can now predict all green locations on the screen, such as 5 and 10. They no longer require the green feedback, because they have determined the mathematical function underlying its regimen. Furthermore, students use the frame of reference to coordinate *across* strategies, such as in explaining why raising their hands at different constant units (e.g., 1 on the left, 2 on the right) necessarily increases the spatial interval between the hands (Abrahamson et al. 2014).

Trainer activity design has been integrated in a variety of media, including remote sensors and multi-touch tablets, applied across several mathematical concepts, and deployed in a variety of settings, including classrooms (see Fig. 6). Readers are referred to other publications for further information about these studies (see, e.g., Abdullah et al. 2017; Duijzer et al. 2017; Negrete et al. 2013; Rosen et al. 2018).

Summary and preliminary comparison

This section has overviewed two DME: xDME, represented by GeoGebra; and oDME, represented by the Mathematics Imagery Trainer. Although each DME comes with its own origin story from somewhat distinct intellectual tributaries, clearly these environments affiliate along similar formative dimensions of philosophy, theory, and design. These include: (a) underlying epistemological and ontological frameworks that construe learners' apprehension of new mathematical notions as rooted in discerning invariance across a set of experiences, giving rise to a conceptual class; (b) pedagogical philosophies of discovery-based learning that seek to simulate and facilitate aspects of naturalistic inquiry within dedicated spaces and focused on selected curricular content; (c) activity rationales that include framing a general epistemic practice, providing interactive materials, stating a task oriented on these materials, and supporting student inquiry as they encounter impasses, specifically in discerning, controlling, constructing, and predicting invariant properties of spatial assemblies; and (d) technological implementations in the form of HCI platforms with computationally encoded functional interdependencies among a system of interactive feedback features and behaviors of virtual elements that instantiate the to-be-discovered mathematical principles. In both environments, mathematical notions are thus cast as equivalence classes that students are to detect, determine, and denote through engaging with a malleable multi-component system of virtual objects—a system that gives rise to a new ontology through reflective, task-oriented interaction. In both environments, learning is organized as an activity, in which one tries to accomplish some pragmatic end-goal; where declarative know-that *about* the system issues from, and reflexively serves, ever-refined know-how of operating *on* or *in* the system.

Thus, the two types of DME design genres in question have much in common. From the outside, these environments might be put in the same box, both being computer-based interactive and dynamic mathematics education modules for discovery learning of targeted mathematical concepts (Abrahamson and Kapur 2018; Sarama and Clements 2009). Still, the environments are also dissimilar in ways that, we submit, bear concerted analysis that could be productive both for the scholarship of mathematics education research and the design and facilitation of learning activities. That is, *there may exist a conceptual vantage point, for example, a theory of human learning, from which xDME and oDME could be compared to determine whether or not they are equivalent pedagogical alternatives of compatible cognitive impact, whether they bear complementary advantages, and whether different design objectives should favor one of these*.

Earlier, we outlined the theory of ecological dynamics and foreshadowed that we would be proposing this theory as a conceptual vantage point from which to examine the two DME genres, highlight what may be significant differences in the details of their respective activity design, and suggest how these differences may be formative to designers of educational technology. To bring out those differences, we will first introduce, in the section "Applying the theory of ecological dynamics to investigate learning with DME," a theoretical approach that we then apply, in the section "A multidimensional comparative analysis of two DME activities," as a means of offering what we see as important distinctions between the two paradigmatic DMEs.

Applying the theory of ecological dynamics to investigate learning with DME

The ecological-dynamics framework was developed by sports scientists to investigate how individuals develop athletic skills, such as soccer maneuvers or vault jumping (Araújo et al. 2009; Chow et al. 2016). Conceived more broadly, though, ecological-dynamics could apply to any practice where individuals are developing the capacity to move in new ways, such as in oDME.

Attentional anchors: emergent task-oriented perceptual gestalts for adaptive engagement of the environment

Abrahamson and Sánchez-García (2016) applied ecological dynamics to investigate how students learn to perform a motor-control task in a DME activity. Their choice of ecological dynamics as a theoretical framework for the study was inspired by calls for cognitive scientists to draw on sports science (Beilock 2008) and by enactivist theses on athletic competence (Hutto and Sánchez-García 2015). Abrahamson and Sánchez-García (2016) conducted qualitative analyses of study participants' multimodal action and utterance, as the participants attempted to perform an oDME motor-control task, the Mathematics Imagery Trainer for studying proportionality. Findings suggest that the participants succeeded in performing the situated task by developing *attentional anchors*, action-oriented dynamic perceptual forms that come forth from the background environment to afford the purposeful coordinated activity of sensorimotor faculties. For example, students were able to coordinate bimanual manipulation of two virtual objects moving at different speeds along orthogonal axes by constructing an imaginary diagonal line connecting their right- and left-hand index fingertips and moving *that line* sideways at a constant angle. In turn, these emergent perceptual forms gave rise to cognitive structures that the students could describe (Abrahamson and Sánchez-García 2016), measure (Abrahamson and Trninic 2015), re-enact with concrete materials (Abrahamson et al. 2014), and reconstruct on paper (Bongers et al. 2018). Those findings were later corroborated through eye-tracking studies that evidenced students developing perceptual routines that become objectified as referents for mathematical discourse and symbolic procedures (Abrahamson et al. 2016c).

In applying ecological dynamics to the study of conceptual learning, one necessarily espouses an epistemological position that foregrounds the enactive quality of knowing. That is, one commits to characterizing the sensorimotor actions students perform in DME as conceptually formative. As such, one seeks to model students' sensorimotor action in DME not as background noise between signals but as the signal itself. If to borrow two terms from Kirsh and Maglio (1994), an ecological-dynamics approach interprets dragging not as merely *pragmatic* but as intrinsically *epistemic*—a notion summed in the oft-quoted enactivist dictum that "all doing is knowing, and all knowing is doing" (Maturana and Varela 1992, p. 26).

We have discussed the emergence of attentional anchors as mediating affordances in task-oriented organism–environment relations. The nature and quality of affordances are constrained by three systemic components—organism, environment, and task. The ecological-dynamics view of skill as emerging within a triadic interplay of constraints draws directly from the seminal work of Newell (1986, 1996), who developed an ecological approach to kinesiological education. By way of example, learning to walk is a systemic phenomenon. It may emerge as an individual's new capacity for movement subject to organismic constraints (e.g., muscularity), task constraints (reaching a desirable object across the room), and environmental constraints (e.g., a carpeted floor). An additional cultural agent, such as a parent or coach, may further constrain the learning process by selecting and refining features of all three systemic constraints (e.g., Cole et al. 2012) and by augmenting on the multimodal sensory feedback that the learner receives, such as through highlighting how performance should be improved (Newell and Ranganathan 2010). To clarify, the notion of a constraint is meant here not in its colloquial sense of a negative encumbrance, but in the technical sense of task-enhancing positive reduction in degrees of freedom.

Designers of DME can select, impose, and remove organismic, task, and environmental constraints on students' enactive engagement in activity tasks. To summarize how the theory of ecological dynamics applies to research on mathematics education, we note that: (a) any human–environment relation is perforce constrained; (b) learning is discerning, adapting, and attuning to contextual constraints; (c) designing is the deliberate engineering of constraints; and (d) teaching is the responsive facilitation of constraints.[2]

The ontological status of attentional anchors: from "given that" (environmental constraint) to "such that" (task constraint)

Abrahamson and Sánchez-García (2016) characterized the attentional anchor as evolving, through task-oriented interaction, from an *environmental* constraint, which individuals encounter, to a *task* constraint, which they self-impose on their enactive relation with the environment.[3] For example, a student working on a Trainer task perceives spontaneously an imaginary line connecting two virtual objects. These objects had been presented as utensils for performing the motor-control task, and the imaginary line connecting them comes forth in the student's action-oriented perception as bearing utility in this context, that is, as a prehensible means of controlling the environment, specifically controlling *both* virtual objects at once, as though they were features of a greater object, a Gestalt. One might think of the imaginary line as an auxiliary construction for making global sense of the sensory manifold, collapsing and reducing the sensory attentional routine onto a single thing one needs to engage in the phenomenal world. Initially, this line behaves in an unexpected way, so that the student must explore how to control the environment *given that* the line—an assembly of environmental features—behaves as it does. That is, the Gestalt appears to require some transformation, which the student must exert, to maintain a target feature of the environment invariant, per the task specifications. For example, the imaginary line connecting the two virtual objects needs to extend when it is farther up along the screen. Learning to control the environment given this environmental constraint is the process of assimilating the line into an action routine, thus developing a new way of controlling the

[2] See Greeno (1994), on the ambiguity or complementarity of constraints and affordances; but see Steffe and Thompson (2000), who use "constraint" to denote problematics of students' schemes. Curiously, Steffe and Thompson (2000, p. 267) open their chapter with a quotation of von Glasersfeld (1990), who uses the term "constraint" as follows, in the sense that we are using here rather than in the sense that they use in their own chapter: "The constructivist is fully aware of the fact that an organism's conceptual constructions are not fancy-free. On the contrary, the process of constructing is constantly curbed and held in check by the constraints it runs into" (p. 33). We take these curbs and checks on action to be catalysts, rather than impediments, to learning.

[3] See Abrahamson (2020) and Abrahamson et al. (2016a) for the special case of instructional metaphors as constraints on action. See Abrahamson et al. (2016b) for the case of material artifacts used for entraining novices into physical skills.

environment that is mediated by the line. The student achieves this assimilation by accommodating their sensorimotor scheme *such that* the line moves in some particular fashion that is effective for the task. With practice, the student develops a new way of moving. The line that came forth from the environment through goal-oriented action, hovering between bottom-up sensation and top-down imagination, now equips the student as a perceptual tool for action on the environment: the imaginary line is at once an aspect of the environment as it is an aspect of the task, and these twinned aspects are co-constitutive and mutually defining by virtue of the student's goal-oriented situated agency. An affordance now is in effect.

Attentional anchors are important to theory of learning as much as they are for learning per se. Granted, the emergence of an imaginary line as an attentional anchor, that is, the coming forth of a sensory assembly as a task-constraining perceptual ontology, is quite a modest phenomenological event in an individual's experience. And yet, *observing* this event should be of moment to any research program pursuing the enactive roots of mathematical concepts (Abrahamson and Bakker 2016; Hutto et al. 2015). Varela et al. (1991) assert that "cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (p. 173). We view attentional anchors as cognitive structures, and our empirical work indeed suggests the semiotic potential of these cognitive structures as grounding mathematical concepts.

Comparing two learning regimens: self-organized criticality vs. progressively increasing difficulty

Liu et al. (2012) studied the micro-process of motor learning under different training regimens. They were interested, in particular, in characterizing how learning progresses toward performing challenging motor skills that require tuning one's physical movement patterns to the inner mechanisms of some unfamiliar device one attempts to control dynamically. Previous research on analogous skills had demonstrated a non-linear learning progression, where considerable practice led to a critical point, when new coordination became manifest. The study was motivated by a broader inquiry into the pedagogical implications of non-linear profiles of motor-skill learning: If individuals' learning experiences are non-linear and idiosyncratic processes, rather than straightforward and predictable cumulative progressions, what could be best practices for training programs?

The researchers found considerable advantages to an adaptive training regimen that enables participants to exercise agency in determining the pacing and difficulty level of their training (*self-organized criticality*, or SOC), as compared to an immutable regimen that pre-determines the pacing (*progressively increasing difficulty*, or PID). They concluded that learners do best when they are empowered to tune task parameters that affect its difficulty level vis-à-vis their current proficiency, such as by deciding how fast they move. The findings also supported a prevalent notion in the research field of motor-control learning regarding a tradeoff concerning training regimens for developing skills that demand phase transition—a tradeoff with implications for instructional design. Whereas "self-discovery learning situations afford a wider range of learner strategies than under directed learning environments,…some of these self-selected strategies may not be effective" (p. 53). That said, the SOC participants evidenced greater retention of their new skill as compared to the PID participants.

Liu et al.'s (2012) line of research is highly relevant to the study of DME, at least as viewed through the lens of embodied cognition. Students appear to do better, in learning a new sensorimotor coordination, when they are: (a) delegated agency to self-impose task constraints on their actions as compared to coping with externally imposed constraints outside of their control; and (b) enabled to govern their search strategies and thus exercise self-control and self-organization. At the same time, designers should monitor and tune student interactions so as to promote occasions for effective self-organization.

Having outlined the theory of ecological dynamics, we are now prepared to apply elements of this theory in comparing the design rationales of the two technologically enabled educational activity genres for mathematics learning. *We will examine the sources of constraints imposed on students' explorative actions as they attempt to accomplish assigned tasks in "closed" DME (xDME,* viz. *GeoGebra) versus "open" DME (oDME,* viz. *the Mathematics Imagery Trainer).* The results of this comparison will then serve as a basis for conjecturing about the cognitive effects of engaging with these activities, namely, how different types of constraints impact students' conceptual learning.

A multidimensional comparative analysis of two DME activities

Table 1 compares xDME and oDME along multiple dimensions of their design rationale, including their respective interaction mechanics, assigned task, learning process, and the psychological role of movement in task performance.

Table 1 implies that the design rationales of xDME vs. oDME activities differ in terms of their constraint loci and, consequently, in terms of the user's agency, movement, and deliberations in maintaining invariant the relational structures on display. Whereas we hope through this comparison of two exemplar environments to refine our typological juxtaposition of embodied interactions with respect to their capacity to foster learning, we wish to underscore the conjectural and still tenuous reach of any inferences drawn from this table. In particular, we do not have the empirical data necessary to adjudicate whether attentional anchors emerge from manipulating objects in xDME. As such, we will advocate a conservative view that casts xDME and oDME along a design continuum, between antipodes of environments that do not and those that do occasion opportunities for students to develop these conceptually important perceptual structures.

xDME designs bear what we might call *hard constraints*: a priori environmental constraints on movement that the designer encodes and the user is to discern. That is, the designer implements constraints prospectively on the user's scope of permissible display configurations. Consequently, the user need not exercise agency in keeping the mathematical ontology intact: no matter how the user moves (drags) elements of the structure, its relational organization consistently manifests the mathematical category. Users cannot work around these constraints, unless they access the source software code. In xDME, the designer constructs constraints on possible movement.

oDME designs bear what we might call, by way of contrast, *soft constraints*: ad hoc task constraints on movement that the user discovers and self-imposes. Moving under these self-imposed constraints, the user is better adapted to perform the assigned motor-control task of transitioning between permissible configurations of the display. The student thus assimilates a dynamical instantiation of a mathematical concept. In oDME, the student constructs constraints on possible movement.

To be sure, oDME designs, just like xDME designs, are by definition heavily constrained, in the sense that educational designers make deliberate choices that create conditions for a particular form of student behavior to arise through interaction. However,

Table 1 Comparing the design rationale of a "closed" and an "open" DME

Dimension	DME Type	
	Closed DME (xDME)	Open DME (oDME)
Example Module	*GeoGebra*	*Mathematics Imagery Trainer*
Design Rationale	Students learn curricular content by manipulating an interactive display. They notice and articulate invariant properties of an emergent conceptual class	
Input–Output Regimen Mechanics	The designer hard-codes a mathematical ontology as an inherent and immutable relation structure among elements of the display. The software automatically maintains these structural properties under any reconfiguration. For example, the ontology "right triangle" remains consistent across permissible modification to the properties of local elements, such as line lengths or orientations. The user's task does not include maintaining structure, and so the system does not explicitly signal "success" or "failure"	The designer hard-codes a mathematical ontology as a feedback regimen privileging a subset of possible display configurations receiving differentiated feedback. The display, thus, exhibits properties of the conceptual class dependent on the users' actions. For example, the ontology "parallelogram" may be instantiated in the display contingent on the user's reconfiguration actions. The user receives positive feedback (e.g., green background) on generating category exemplars and negative feedback (e.g., red background) on all other configurations
Functional Relation Between Manipulable Elements	Manipulable elements of a virtual display may or may not be interactionally dependent: moving one element may or may not cause automatic movement of the other one	Manipulable elements of the virtual assembly are interactionally independent: moving one element never causes automatic movement of another one
Task	Users are (t)asked to manipulate figural elements of the display in an attempt to discern and articulate what properties of the display remain invariant across manipulation (ontic, i.e., what there *is*)	Users are (t)asked to manipulate figural elements of the display in an attempt to receive "success" feedback, maintain this feedback over continuous manipulation, and then articulate their manipulation strategy (epistemic, i.e., what I *do*)

Table 1 (continued)

Dimension	DME Type	
	Closed DME (xDME)	Open DME (oDME)
User Task Experience	Users drag figural elements of the interactive display, hypothesize invariant properties of an emerging structure, verify or refute the hypotheses, and compose conjectures about the structure's properties. Dragging can become heuristics-based, for example, by searching for the structure's extreme cases at the screen edges or by overlapping two points. The to-be-learned ontology materializes through reflecting on the varied states of the figural elements	Users drag figural elements of the interactive display, experiment with manipulation strategies to satisfy the task objective, and use mathematical frames of reference to articulate effective strategies quantitatively. Initially, the user *can* and probably *will* violate the encoded mathematical ontology, because they have not yet constructed it and the technology does not keep it intact. The user responds to the feedback regimen by searching for, and gradually coming to anticipate, the set of privileged configurations. The user develops strategies for transitioning dynamically between these configurations. These strategies emerge in the form of new sensorimotor schemes oriented on new perceptual structures (attentional anchors). The to-be-learned ontology materializes through reflecting on the movement strategies
Attentional Anchors	It is not clear whether, or to what extent attentional anchors will emerge in this context, because there is no explicit movement task and no need for coordination of actions, since dragging does not violate ontological identity	Attentional anchors are expected to emerge as the user's means of facilitating the coordinated sensorimotor enactment of movements that satisfy the task specification
The Psychological Role of Movement	Movement serves a pragmatic function by mobilizing the search for information common across a succession of static images. The user moves objects on the screen to generate multiple configurations of the intact form in an attempt to discern across them invariant properties of the emergent ontology	Movement serves a conceptually formative epistemic function in searching for a dynamic form that would enable performing the assigned task effectively. The user attempts to discover and describe a task-effective strategy for moving objects on a screen. The sense of invariance, and thus the new ontology, sprouts from discovering, executing, and then articulating consistency in this dynamically stable, task-effective form

oDME, more so than xDME, implicitly delegates to students their complement in the systemic achievement of configuring their own targeted behaviors. oDME modules mete out to students a requisite participatory share in the agentive construction and maintenance of the totality of constraints necessary for imposing an envelope of constraints that together mold dynamical embodied interaction into particular movement forms.

Our analysis suggests that a designed activity's locus of constraints on movement may shape students' attention more so to objects (xDME) than actions (oDME). Objects and actions are different ontologies, and this difference in the locus of attention likely bears differential implications for processes of conceptual development. A reading into theories of genetic epistemology (Piaget 1968) and enactivism (Maturana and Varela 1992; Varela et al. 1991) would suggest that not objects per se but actions on objects—and more specifically action-oriented perceptual routines—constitute naturalistic foci for the subjective emergence of cognitive structures mobilizing mathematical reasoning.

Working with an xDME, students begin with a pre-existing object presented on the screen. It is a compelling perceptual gestalt composed of a coherent assembly of structurally conjoined virtual elements, such as points, lines, and arcs. Students are invited to reconfigure this object. However, given a set of computationally encoded propositions hidden from the students, they will necessarily be operating within this object's constrained morphological degrees of freedom, whereby certain types of reconfiguration are enabled, while others are not. The task is to describe the object's invariant properties, that is, to characterize what it is about this object that remains constant (consistent, conserved, "the same") across all these possible states that one is enacting. In a sense, one is asked to compose an idea of what this object is, where "is" denotes its haecceity, that is, its unique distinguishing properties, in contradistinction with other familiar (geometrical) objects.

Working with an oDME, students begin not from a prefabricated inspectable structure on the screen, such as a geometrical form, but from virtual utensils (handles, appliances) for operating on a particular property of the background world (e.g., its color). The students are tasked to handle these utensils such that a particular state of this background world is achieved and instated, namely, the state of a particular value of the color parameter (viz. green). In the course of this work, a subjectively emergent structure, the attentional anchor, may coalesce and foreground surreptitiously on the students' sensorimotor operatory interface with the technological environment. The attentional anchor comes forth as an imaginary figural effigy bearing phenomenological facticity. The students, who are further invited to describe their strategy, may refer directly to this new structure they are perceiving and explain how they are handling it. Here, the process of articulating the strategy unfolds as enunciating what one is keeping constant as one operates the apparatus. That is, the stable mathematical attributes of the system come forth as ontologies (what is) through the discursive delineation of one's operatory regimen (what I am doing). Indeed, "cognition does not concern objects, for cognition is effective action; and as we know how we know, we bring forth ourselves" (Maturana and Varela 1992, p. 244). There is no a priori object. Objects do not exist as phenomenal ontologies in immersed motor intentionality. Rather, an object comes forth when we "stop to think," or, perhaps, rather, "start to think," that is, when we stop doing and, instead, think about what we are doing. And when we think about what we are doing, the things that come to mind are the perceptions guiding our actions, that is, the attentional anchors. "Cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (Varela et al. 1991, p. 173). A thing is born.

In operating an xDME, we hypothesize, one is less likely to develop attentional anchors than in oDME, because basic xDME tasks do not require developing a new movement forms. As one manipulates the cursor on an xDME screen, the cursor may resist relocating to your desired destination, because the environment constrains that degree of freedom. This brief breakdown in the flow of manipulation, granted, may surface the object's properties. Moreover, one might initiate for oneself the challenge of performing a fluent manipulation form, a task that would require anticipating where the cursor will consequently arrive. Yet xDME do not foreground action as the essential task objective. Ultimately, only empirical work could resolve these questions.

General discussion

Designing technological resources for educational practice affects the nature, quality, and outcome of students' learning. It is therefore important that designers understand the effects of their design decisions. Modeling the relation between design decisions and learning outcomes may require approaching the examination of activities from a new perspective. We have been considering the phenomenon of DME activity from the theoretical perspective of ecological dynamics. The premise of this analytic approach is that operating DME solicits from students the enactment of new movement forms. To effectively manipulate objects in a DME is to enact environmentally coupled movement forms within a set of stable constraints that come forth through interaction (Abrahamson and Sánchez–García 2016). Enacting these movement forms is thus contingent on having developed appropriate sensorimotor schemes that hinge on constructing invariant perceptual structures one brings forth in the DME. The thrust of this paper is that the nature of these perceptual structures, and therefore the sensorimotor schemes that one develops through working in DME, is contingent on the source of the constraints shaping the interaction—whether these constraints are: (1) a priori embedded in the *environment* as mechanically interlinked inviolate properties of the virtual objects being manipulated, and which students are to notice; or (2) emergent through interaction as properties of the *task* that the student learns to perform, that is, self-imposed by the student on how they are handing the objects to enable the achievement of task goals. This section will propose what the cognitive implications may be for students' encounters with either environmental or task constraints. We are thus asking, should it matter who constructs constraints?

Design heuristics for the relation between instructional objectives and the location of constraints

Both xDME and oDME are, by design, highly constrained learning environments. One defining difference between xDME and oDME, however, is that oDME users must imagine covert perceptual structures that in xDME are overt. xDME users imagine these structures so as to facilitate task-effective interaction. As such, oDME users, possibly more so than xDME users, must develop and sustain task constraints to regulate their interaction per the specified task objectives.

Yet why might it matter whether constraints on the enactment of task-effective movement reside in the environment or the acting organism? To answer this question, we must perform a task analysis to determine the role of movement in attaining the task goal—whether the student's enactment of movement serves the designer as an epistemic function critical to the conceptual learning in question or is merely serving a pragmatic function of rendering environmental information accessible to relevant sensory modalities. The following juxtaposition of two hypothetical pedagogical objectives and their recommended design entailments is not meant to map directly onto xDME vs. oDME as much as to clarify a tension through pairing two antipodal generic situations.

If an educational designer's objective is for the student to detect figural invariance by comparing and contrasting among various sensory displays generated by the manual operation of virtual objects—as in the case of xDME— then enacting movement has a narrowly circumscribed role: It mobilizes and facilitates a line of reasoning, but it does not, and perhaps need not, constitute or bear in-and-of-itself conceptual potential. Here movement is analogous to the action of paging through a book to witness multiple examples of some notion: the sensorimotor composition of the paging action per se—establishing index-finger traction at the page corner, grabbing the corner by index and thumb, and flipping the page, perhaps using the remaining fingers to complete the trajectory—is not in any significant way deeply consequential to learning the targeted notion. It is only making available to visual perception a succession of images, each on a different page. For this content-neutral interaction function, turning pages could be equally or even preferably achieved by pressing a button. For this task goal, constraints on movement should reside in the environment, to minimize cognitive resources allocated to generating the array of sensory displays for the student's scrutiny.

If, however, movement itself is intended by the designer to serve a key objective in the learning process—if figuring out how to enact a specific movement constitutes the student's formative struggle bearing the desired learning outcomes—then a priori environmental constraints on movement may not occasion opportunities for students to realize this epistemic function. Here the designer would rather position movement as a phenomenon of inquiry in its own right, and so the designer would want for students to attend to movement.

The design problem of focalizing student attention on movement was solved in the Trainer activities by creating a motor-control task that initially is framed not in terms of the sensory assembly being manipulated. Instead, a task goal was created with respect to an ancillary feature of the environment—generating a particular sensory event (making the screen green) that is not a property of the objects being manipulated (see Abrahamson and Bakker 2016, on a distinction between proximal actions and their distal effects). As students engage in the activity, its task goal comes to constitute for them online feedback on the efficacy of their solution strategy. Once the students have solved the motor-control problem of enacting the solution movement, the ongoing feedback still serves to monitor the enactment quality, and yet the students can increasingly anticipate the feedback regimen. As they mathematize their movement with the new instruments interpolated into the learning environment (see Fig. 5 in the section "“Open” dynamic mathematics environments: the Mathematics Imagery Trainer"), the feedback functions as a conceptual placeholder for a yet-unnamed invariance class, a placeholder that will ultimately be replaced by a model and nomenclature for the mathematical notion in question (Trninic and Abrahamson 2011). That is, the real-time feedback signal is rendered redundant by the emergent anticipatory mathematical model.

A theoretical framing of the relation between constraint locus and cognitive learning

Still, what is the relationship between learning to move in a new way and learning a new mathematical concept? In other words, if designers have instantiated a mathematical concept in the form of a manipulation scheme, why should we expect that students will learn this concept through engaging in the activity? What does manipulating objects have to do with learning concepts?

The enactivist philosophy of cognitive science offers an account for modeling the process by which manipulating virtual objects on a computer screen can give rise to mathematical understanding (Abrahamson and Bakker 2016; Abrahamson and Trninic 2015; Hutto et al. 2015). Varela et al. (1991) maintain that cognitive structures emerge from recurrent patterns of perceptually guided action. Broadly, the enactivist account resonates both with: (a) the ecological-psychology account, by which humans develop, through interacting with the environment, tacit action-oriented perceptual constructions of the environment, namely affordances (Gibson 1977; Heft 1989)[4]; and (b) the genetic-epistemology account of cognitive structures coalescing, through reflective abstracting, when sensorimotor operatory schemes must accommodate drastically (Piaget 1968). All these accounts point to the pivotal role of emergent goal-oriented perception in organizing sensorimotor capacity (Mechsner 2003; Mechsner et al. 2001).

Our studies support the above implications of developing sensorimotor perception as pivotal to conceptual growth. As we have explained earlier, we have documented students' formulation of new perceptual Gestalts, as they attempt to solve the Trainer's motor-control problem. These Gestalts, in turn, enable the students to assimilate the embedded functional invariance of the technological systems they are learning to control, and then to articulate this invariance multimodally as an action strategy that is ripe for mathematical modeling. In the absence of a movement task, however, students need not self-impose task constraints on their sensorimotor actions, and therefore they need not develop perceptual structures facilitating these actions. As such, students are likened to a person driving their car according to a sequence of piecemeal instructions from a GPS. The car travels along an ideal path toward its destination, and yet the person driving the car may be completely oblivious to the shape of this path and, upon returning home, have no clue as to where in the world they have just been relative to any frame of reference, such as the cardinal directions. It might as well have been a self-driving car.

On apples and oranges: considering perceived limitations of the comparison

In comparing the two DMEs in question, it could be argued, we are comparing two very different things. The GeoGebra case dealt with an activity for geometry, whereas the Mathematics Imagery Trainer dealt with proportions. One could argue that geometrical shapes and proportional relations are different ontological entities—a geometrical shape is a quasi-real object, whereas a proportion is an equivalence between dimensionless quantitative relations. We agree that the concepts of a right triangle and a 1:2 ratio belong in different mathematical branches and are formally of disparate ontology. However, from a developmental perspective, we regard these concepts both as psychological notions, related to the mathematical discipline, that arise through reflection on situated sensorimotor interaction in a DME. From a mathematics-education perspective, we therefore conceptualize our task as determining how these notions arise through sensorimotor interaction (e.g., Kim et al. 2011) and, as such, how best to design for this process. Whereas both xDME and oDME offer

[4] Whereas enactivism and ecological psychology resonate in broad strokes, the compatibility of their epistemological and ontological groundings has been questioned (Di Paolo et al. 2020).

students experiences that appear beneficial for developing conceptual understanding, we are interested in theorizing the specific role of movement across these activity genres.

One might also argue that the students' assigned tasks in the two examples were different. The student's assigned task in the particular GeoGebra activity showcased in this paper was the task of noticing invariant properties of the manipulated objects—it was never about the motor enactment of some goal movement form, as in the case of the Trainer, so that comparing the DMEs with respect to aspects of movement is illogical.[5] We agree that "on paper" these two assignments are distinct. However, we wish to point out that the motor actions performed in GeoGebra are informed by the student's inquiry goals, which iteratively emerge in response to the shapes the student thus generates and as contingent on the environmental constraints over permissible reconfiguration. As such, the student's motor actions correspond tightly to the task. One might hypothesize that over additional tasks with different shapes, possibly under different environmental constraints, the student would develop, generalize, and apply heuristic movement meta-forms for efficient geometrical inquiry in this DME, such as dragging points along cardinal directions, dragging points parallel to lines, or rotating them. As such, regardless of a task's didactical definition, for example as mathematical reasoning about properties of objects, its interactive manipulatory architecture tacitly constitutes a form of sensorimotor learning.[6]

Finally, one could remonstrate that for this comparison we elected to offer a strategically partial view of the learning potential inherent to dragging. Indeed, both GeoGebra and Trainer activities can present students with the task of determining and tracing a path that keeps constant some property of the environment (e.g., see in Leung et al. 2013). However, there are several differences between these similar tasks. First, in GeoGebra, the property to be maintained constant, as specified by the task, is inherent to the objects being manipulated (figural elements of a parallelogram), whereas in the Trainer activity the property is not inherent to the objects being manipulated (a peripheral signal). Second, we focused on the baseline GeoGebra task that students first encounter and, arguably, spend most of their time on. It is the definitive task of this environment, which draws much interest in the literature (Mor and Abdu 2018). Third, the GeoGebra path-tracing task might presumably be self-assigned by students engaged in inquiry, whereas the Trainer assignment is defined by the designers. Fourth, note that our intention was never to evaluate any specific software package. Rather, we were hoping to compare two design architectures, and so we needed suitable contexts of implementation. We might, therefore, have compared two different tasks within GeoGebra. However, we sought to juxtapose packages whose design was based on different theories.

[5] As hinted in Sect. 1.2, each design pattern could, in principle, be applied to the other. oDME could be designed to support learning of a geometric concept, such as the properties of a parallelogram: the four vertices can be designed as independent, where green feedback is given to the user whenever the vertices constitute a parallelogram. xDME could be designed to support the learning of the concept of proportions, if the lengths of the manipulated bars are co-dependent ("yoked") and only one bar can be dragged at a time.

[6] Smith et al. (1999) draw on their dynamic systems theory to critique of Piaget's "cognitive" A-not-B behavior as implicitly shaped by tacit sensorimotor qualities (see also Bateson 1972, on deutero-learning).

Conclusion

This paper has tackled a critical challenge educational technologists face in selecting design frameworks by which to plan the mechanics of interactive modules for conceptual learning. Drawing on the theories of ecological dynamics and enactivism—that position sensorimotor activity at the core of all learning—we have foregrounded the construct of a constraint as a lens for reasoning about the mechanics of embodied-interaction modules. More specifically, we have been concerned with the relation between the sources of an interaction constraint and the learning that it engenders.

Our constraints-based conceptual analysis of interactive educational technologies was contextualized in the mathematics domain. We characterized two types of design architectures for digital mathematics environments (DME), "closed" DME (oDME) that constrain possible manipulations and "open" DME (oDME) that do not constrain possible manipulations. These design architectures use essentially the same hardware, have similar HCI, and broadly share instructional objectives, but they differ in their interaction mechanics. We selected exemplars for each of these two DMEs architectures, GeoGebra and the Mathematics Imagery Trainer, and analyzed their interaction-mechanics regimen.

From ecological dynamics, we foregrounded the constraints-based account of movement learning (Araújo et al. 2009; Chow et al. 2016; Newell 1986, 1996) so as to implicate the locus of constraints on students' interactions, within each DME, as either embedded a priori in the *environment* (GeoGebra) or constructed by the students ad hoc to constitute *task* constraints (Trainer). Building on enactivist accounts of learning, we submitted that the locus of constraint, environment vs. task, may bear differential results for learning, because the cognitive process of constructing task constraints grounds mathematical concepts in sensorimotor activity. Specifically, we discussed Trainer students' spontaneous perceptual constructions as constituting self-imposed constraints on action. We suggested that this solution activity of assembling perceptual structures for coordinating motor action is conducive to forging cognitive continuities from informal sensorimotor operations through to formal mathematical actions. These principles may obtain outside of mathematics to other STEM domains and perhaps beyond (q.v. Glenberg et al. 2004, for the relations between moving and reading).

This has been a conceptual paper, and so our conclusions are tentative at best, based on a chain of theoretical deductions and inferences. We hope that essential issues discussed herein will give rise to empirical research that rigorously compares apples and apples to evaluate for the impact of constraint source on learning outcome. Such a research design should optimally hold constant the technological environment, content, and interactive objects, varying only structural constraints on the objects as either hard and closed (x) or soft and open (o). Where the research design might require much creativity is in building assessment tasks that enable participants in both study conditions to demonstrate their new skills and understandings in a different context. Empirical comparisons of "x" and "o" environments would create investigative contexts to evaluate for tradeoffs, complementarities, and unique affordances of these and related genres as practiced across diverse educational settings with their varied epistemic and cognitive objectives.

In examining the future of educational technology, how radical shall we get? As new human–computer interaction platforms enter the bastions of mathematics-education research, they act as Trojan horses, by transforming the *practice* of learning to a point where the common *theory* of learning is stretched and eventually disrupted (Dyson 1996). Scholars of embodied-interaction conceptual learning, such as Leung et al. (2013), have offered profound contributions to the field by stretching essentially cognitivist theories of learning so as to accommodate new forms of exploration enabled by manipulation functionalities. We concur that these HCI platforms are changing the playing field, creating opportunities to bring about what Wilensky and Papert (2010) call "restructuration" of conceptual domains. At the same time, we wonder whether new HCI could create opportunities for researchers to transform the theory of learning more fundamentally. In particular, we wonder whether interactive learning environments should be approached by a theory of learning that foregrounds dynamism as the essential feature of cognition (Abrahamson 2018; Abrahamson et al., in press). Such a theory would elevate the epistemic status of movement from a mere actuator of higher-order reasoning to the very embodiment of reasoning. Modules designed with movement in mind would bring the history of manipulation-based learning full circle, to the point where HCI enables us to offer students learning experiences that simulate and enhance naturalistic sensorimotor exploration (Abrahamson and Bakker 2016). Within these quasi-naturalistic environments, which return movement centrally into educational activities, our theories of learning should attend to movement as the core epigenetic phenomenology of cognitive development (Abrahamson and Sánchez-García 2016; Sheets-Johnstone 2015). Learning, we submit, is moving in new ways.

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