citation: Abrahamson, D., & Howison, M. (2010, May). *Kinemathics: Exploring kinesthetically induced mathematical learning.* The annual meeting of the American Educational Research Association, April 30 - May 4, Denver, CO.

Kinemathics: Exploring Kinesthetically Induced Mathematical Learning

Abstract: Building on growing evidence that human reasoning simulates multi-modal dynamical imagery drawn from lived experience, we conjectured that some mathematical concepts are challenging because their images are difficult to simulate mentally and that this difficulty, in turn, is directly implicative of the difficulty to enact these concepts physically. For example, two linked equal-rate linear growths are easier to process than two linked different-rate linear growths, because miming the latter is by far more difficult to coordinate ambidextrously. To examine our conjecture, we built a computer-based servo-mechanical device, the Mathematical Imagery Trainer, designed to induce in students the physically challenging concept of proportionality and measured for reflective-learning gains. The full paper reports on a study with thirty 4th-grade students.

1. Objectives

We are in the early stages of conducting research investigating conjectures central to theories of embodied mathematical cognition. Specifically, we are carrying out an empirical study designed to evaluate the plausibility of theoretical models that argue for an embodied and dynamical–enactive basis of mathematical learning and reasoning—models by which situated experiences in the life-world engender durable cognitive simulacrum that, through appropriate instruction, may become objectified in semiotic artifacts as grounded cultural knowledge (Hoffmann, 2007; Radford, 2003). We propose to present a hands-on interactive session in which attendees will have opportunities to work with our innovative instructional materials so as to formulate a first-hand experiential basis for discussing our conjectures and findings.

2. Theoretical Framework

Prior arguments for embodied mathematics either build on theoretical analyses of human reasoning (Barsalou, 1999; Goldin, 1987; Lakoff & Núñez, 2000), empirical studies of human activity in general (Barsalou, 2008; Clark, 1999; Hatano, Miyake, & Binks, 1977), interpretations of mathematics students' behaviors (Fuson & Abrahamson, 2005; Nemirovsky, Tierney, & Wright, 1998), or specifically evidence of gestures accompanying speech utterances produced during the solution of mathematical problems (Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999; Edwards, Radford, & Arzarello, 2009). Whereas these studies furnish strong support for the potential viability of the embodied conjecture, and whereas they have demonstrated the plausibility of an embodied substrate for working memory, they have not established conclusively a *sine qua non* role of multi-modal dynamical imagery in the ontological development of mathematical concepts. Namely, it has yet to be shown compellingly that imagery plays more than a supportive or epiphenomenal role in the instruction of essentially abstract concepts (cf., Schwartz & Black, 1999).

To the extent that the embodied conjecture obtains, a central challenge in evaluating the roles of imagery in mathematical reasoning has been that these psychological constructs are currently inaccessible for measurement. That is, because we cannot see people's "pictures in the head," we instead rely on indirect means of investigating these images. For instance, to examine the nature of procedural fluency that is deeply rooted in object-based visual–kinesthetic experiences, Hatano et al. (1977) had expert abacus users solve arithmetic problems in the absence of an actual abacus yet while performing interfering finger-tapping tasks. Such experimental designs are well geared for studying mathematical imagery, because the researcher can confidently ascribe the experiential basis of the mental operations to interaction with a particular known object. However, cross-sectional studies, such as the above, do not optimally capture individual learning trajectories. In comparison, Abrahamson (2004) introduced new mathematical objects into a middle-school classroom and examined individual students' gestural acts as a window onto the imagery purportedly underlying their mathematical learning. Still, it is difficult to appraise whether these gestures, performed in authentic classroom discussions, were pivotal to the students' reasoning or only epiphenomenal manifestations of normative interpersonal pragmatic discourse. Moreover, students' daily exposure to external influences, during that long intervention, weakens the reliability of experimental instruments allegedly measuring learning exclusively originating in interactions with the new instructional materials within the monitored confines of the research site. Therefore, a study is needed that tracks individual object-based learning through an experimental design that confidently implicates that object as the source of imagery subsequently elicited when the individual engages in content-relevant problem-solving. Thus, rather than seeking out a pre-existing mediating physical artifact, such as the abacus, or waiting for spontaneous gestures to emerge, such as in mathematical argumentation, we take a more proactive approach in this study by directly inducing an unfamiliar, visuo–kinesthetic, initially *a*mathematical image and then guiding participants to reason with it mathematically.

Thus, our research embarks from the premise that mathematical learning is indeed image-based and mathematical reasoning is image-simulated (Case & Okamoto, 1996; Dahaene, 1997; Freudenthal, 1986; Lakoff & Núñez, 2000; Martin, 2008; Nemirovsky & Borba, 2004; Nemirovsky & Ferrara, in press; Nemirovsky et al., 1998; Pirie & Kieren, 1994; Presmeg, 2006). We further assume that people who demonstrate difficulty in understanding a particular mathematical concept nevertheless possess the cognitive wherewithal for leveraging imagery toward mathematical understanding—what these people lack is the initial requisite image for that concept. Implicit in our approach is the view that teaching is a means of creating opportunities for students to develop a family of images that become resources for reasoning about a class of mathematical situations. Our conjecture is that *some mathematical concepts may be difficult to learn precisely because our everyday experience fails to provide adequate opportunities to develop the requisite body-based imagery underlying those specific concepts.* To evaluate this conjecture, we are conducting a study in which we first provide students with a "ready-made" visual–kinesthetic basis for developing imagery pertaining to a difficult mathematical concept and then measure the effects of such provision on their understanding of the targeted mathematical concept. This initially "meaningless" yet embodied experience (aka the "Karate Kid effect") is to play a pedagogical role analogous to concrete artifacts, such as an abacus, pendulum, or dice, in terms of creating opportunities for guided reflection, mathematization, and reinvention of a culturally mediated artifact. Our central research question is:

- What, if any, is the relation between a capacity to perform a physical action and the prospects of constructing the mathematical meaning it may be taken to represent?

3. Modes of Inquiry

3.1 Mathematical content. We have chosen the mathematical content domain of proportionality both because rational numbers are fraught with conceptual challenges (e.g., De Bock, 2002; Davis, 2003; Karplus, Pulos, & Stage, 1983; Lamon, 2007; Noelting, 1980) and because this content lends itself to challenging enactive embodiment. In particular, we focused on the notion of *proportional progression*, that is, a sequence of equivalent ratios or fractions whose ordered pairs respectively increase by fixed amounts at each count (e.g., in the sequence

2/3, 4/6, 6/9, etc., the numerator grows by 2 for every 3 the denominator grows). Our work builds on a pilot study conducted by Anonymous and Author1 (2005), which demonstrated that students' conceptual difficulties in understanding such numerically presented progressions coincided with their physical inability to act out the progressions with their hands (e.g., one hand is to "grow" by 2 units every second *while* the other hand simultaneously grows by 3). Many students maintained a constant vertical distance between the levels of their rising hands, acting out an additive rather than multiplicative relationship—a physical performance error that is reflected in mathematical errors perennially reported in the literature on children's development of rational number concepts.

We conjecture that students' difficulties with proportionality, and in particular their "additive" or "same difference" errors, stem from their lack of a suitable dynamic image in which to ground their understanding of proportionality. Furthermore, it is unlikely that students construct the relevant imagery spontaneously, due to the challenge of ambidextrously executing the *physical* proportional actions themselves. Therefore, we decided to create an opportunity for study participants to undergo this ambidextrous multi-modal embodied experience—initially experiencing the physical concept passively and then gradually assuming representational intentionality, agency, and physical command of the motion—thus allowing us to subsequently trigger and monitor any personal construction of meaning for this physical action. The particular mental model of proportion that we aim to induce is that which Vergnaud (1983) named *isomorphism of measure*, that is, cases in which the x and y in the x:y ratio are functionally related but phenomenologically separate, such that there is no immediate meaning to a mix of these two measures (as compared to *product of measure*, which is akin to Piaget's "intensive quantity"). We have thus elected to construct proportion as a constant *process* (e.g., the same pair of quantities added over and over) rather than as a set of identical sensory perceptions of the *product* (e.g., same sweetness).

3.2 Design. We have developed a *Mathematical Imagery Trainer (M.I.T.)* that "hand-holds" the user through performing arm motions describing proportional growth. Figure 1 shows Becky grasping two weights that are rising at velocities relating as 2:3. The weights then descend at the 2:3 rates. At first, Becky just clenches on and lets her arms rise and fall. Then, she tries to anticipate the motion and flow with it. When an arm moves too slowly, she feels a tug, and when an arm moves too fast, she sees the rope become loose. She thus tunes herself to the "2:3 dance" by synchronizing with the mechanism (see http://www.youtube.com/watch?v=v9XDVYN5zrQ).

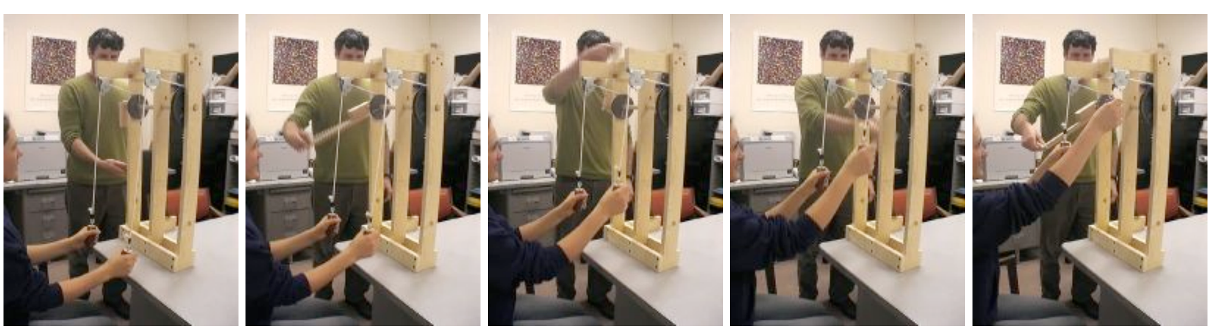

Figure 1. A proportions Mathematical Imagery Trainer (M.I.T.) in action. This is our mechanical prototype for the computer-linked device. The "student," seated, holds lightly onto two weights that are moving up at different rates, maintaining a 2:3 ratio between their respective heights above the table surface. Gradually, the student assumes agency, learning to mime this challenging ambidextrous performance. (The standing person is rotating a long handle, and this work is translated through two pulley wheels of different diameters, here relating as 2:3.)

3.3 Assessment. Before students are introduced to the M.I.T., they will be asked to perform the 2:3 arms motion unassisted several times. Motion capture devices attached to students' hands will provide real-time analysis of their initial competency with this physical performance. Pre-test items borrowed from existing assessments for proportionality knowledge will provide measures of conceptual understanding of the mathematical content as well as procedural fluency. A card-sorting activity (see Figure 2) depicting schematic snapshots of the arms motion in various correct and incorrect states will establish whether or not the student is already fluent in the imagery we are attempting to induce. We then engage participants in a semi-structured clinical interview, in which they interact with the M.I.T. until they are able to perform the arms motion quite accurately without assistance. Next, we guide them through a series of activities aimed at reflecting on the motion as well as quantifying it, using both graphical and numerical instruments and inscriptions. Finally, participants will complete post-test items on proportionality, including the card-sorting task, to gauge any immediate conceptual change or new procedural fluency. Essentially, we will monitor whether the students have built upon the arms motion any meaning for the notion of proportional progression. These behaviors on the post-test will be compared to those of a control group who will not be trained with the M.I.T.

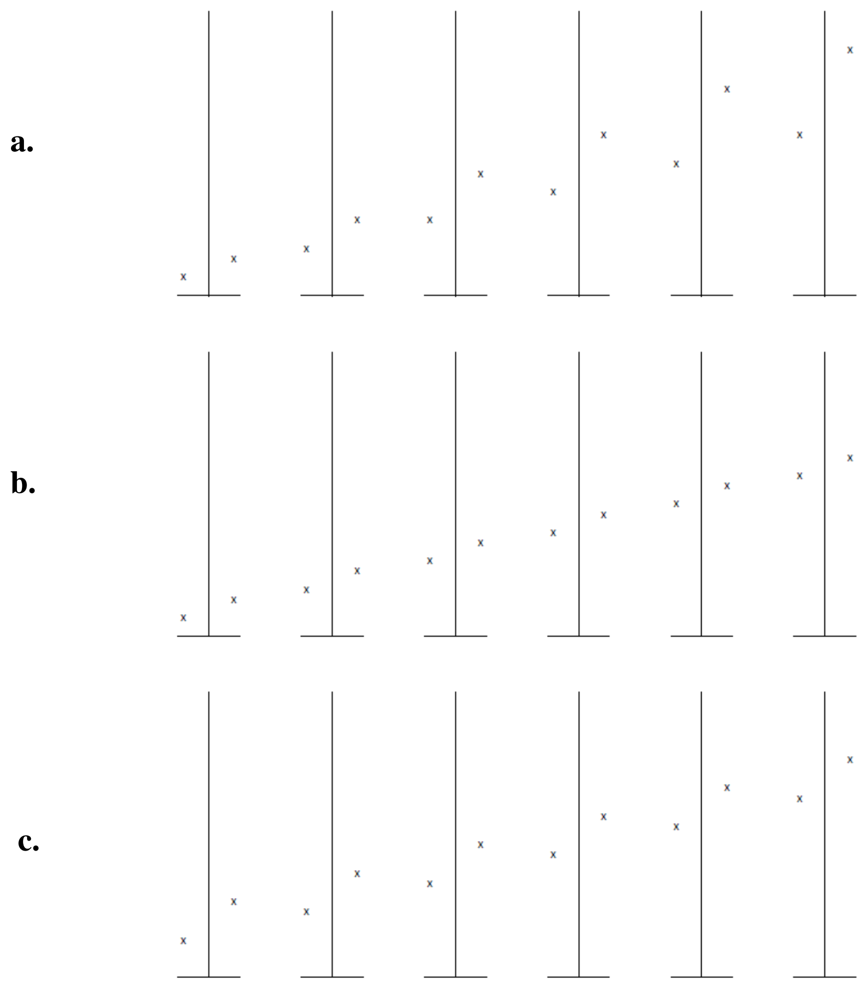

Figure 2. 18 flash cards organized into three sets depict either: (a) proportional progression (different rates of growth); or (b&c) constant-difference equal-rate progressions. The *x*'s mark hand positions. Participants will be given all 18 cards in randomized order and asked to construct sequences that make sense to them.

4. Data Sources

To date, we have gathered pilot data of a dozen adult researchers from a biweekly gesture study group (Undisclosed, *Director*) performing the arms motion in response to seven different instruction prompts (see Appendix A). Although these participants had far more sophisticated knowledge of proportionality than our target age group, these pilot data have established the intricacies of physically performing the arms motion. Adult participants used various coping strategies, such as digitizing the motion, recruiting their expertise on dance (maintaining an aligned body posture) and music (counting beats), and practicing a more intuitive 1:2 ratio before attempting 2:3. Working with the M.I.T., participants discussed the gradual engagement of kinesthetic intentionality, the role of visual feedback as a pedagogically vital sensory modality intersecting with the kinesthetic production, optimal enactment speeds for building an "embodied shape," prospective cognitive and technological mechanisms for mathematizing an "embodied artifact" (a "*gesturoid*") and the affordances and constraints of fragile vs. sturdy machines (the fragile machines immediately demand participant agency, whereas sturdy machines enable the participant to hang the arms as "dead meat" without performing any planning or execution).

5. Results and Conclusions

Clearly, this is work in progress. Yet, we believe there is already sufficient *theoretical* material for sparking useful scholarly discourse. By AERA, we will have collected and analyzed *empirical* data from a study we are conducting this fall with thirty 4^{th}-grade students. Because this is pioneering work, we anticipate intensive development and adjustments in response to initial runs, as we strive to address the current and emerging research questions. A follow-up question will be whether the body, as a medium, is more conducive for constructing multi-modal mathematical images as compared to artifacts embedded in other media, such as video or textbooks, and if so then what is the nature of the advantage of such pre-embodied knowledge.

6. Educational Significance

In tune with the AERA 2010 theme, we are offering an interdisciplinary study aiming at "Pasteur's Quadrant" by understanding the complexity of learning ecologies, wherein students are guided to negotiate embodied and semiotic resources, while at the same time creating viable instruments and activities for fostering successful negotiations. We thus respond directly to the conference call (http://aera.net/Default.aspx?id=7588) by problematizing the question of "how the repertoires that people develop within and across the routine settings of their lives can be recruited to support complex learning" as well as by demonstrating "how educational settings—formal and informal—can be designed to address" this interrelated web of individual experience and cultural heritage.

References

Abrahamson, D. (2004). Embodied spatial articulation: A gesture perspective on student negotiation between kinesthetic schemas and epistemic forms in learning mathematics. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the Twenty Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 791 – 797). Toronto, Ontario: Preney.

Alibali, M. W., Bassok, M., Olseth, K. L., Syc, S. E., & Goldin-Meadow, S. (1999). Illuminating mental representations through speech and gesture. *Psychological Science, 10*, 327-333.

Barsalou, L. W. (1999). Perceptual symbol systems. *Behavioral and Brain Sciences, 22*, 577–660.
Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617-645.
Case, R., & Okamoto, Y. (1996). The role of central conceptual structures in the development of children's thought. In *Monographs of the society for research in child development: Serial No. 246* (Vol. 61). Chicago: University of Chicago Press.
Clark, A. (1999). An embodied cognitive science? *Trends in Cognitive Science, 3*(9), 345-350.
Dahaene, S. (1997). *The number sense*. Oxford: Oxford University Press.
Davis, G. E. (2003). Fractions, ratio, and proportional reasoning. *The Journal of Mathematical Behavior [Special Issue], 22*(2&3).
De Bock, D. (2002). *The illusion of linearity: An empirical analysis of secondary school students' improper proportional reasoning in geometry problems*. Unpublished doctoral dissertation. Leuven, Belgium: University of Leuven.
Edwards, L., Radford, L., & Arzarello, F. (Eds.). (2009). *Gestures and multimodality in the construction of mathematical meaning [Special Issue]*. Educational Studies in Mathematics 70 (2).
Freudenthal, H. (1986). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
Fuson, K. C., & Abrahamson, D. (2005). Understanding ratio and proportion as an example of the Apprehending Zone and Conceptual-Phase problem-solving models. In J. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 213-234). New York: Psychology Press.
Goldin, G. A. (1987). Levels of language in mathematical problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 59-65). Hillsdale, NJ: Lawrence Erlbaum Associates.
Hatano, G., Miyake, Y., & Binks, M. (1977). Performance of expert abacus operators. *Cognition, 5*, 57-71.
Hoffmann, M. H. G. (2007). Learning from people, things, and signs. *Studies in Philosophy and Education, 26*, 185-204.
Karplus, R., Pulos, S., & Stage, E. K. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 45-89). New York: Academic Press.
Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*: Basic Books.
Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629-667). Charlotte, NC: Information Age Publishing.
Martin, L. C. (2008). Folding back and the dynamical growth of mathematical understanding: Elaborating the Pirie-Kieren theory. *The Journal of Mathematical Behavior, 27*(1), 64-85.
Nemirovsky, R., & Borba, M. C. (2004). PME Special Issue: Bodily activity and imagination in mathematics learning. *Educational Studies in Mathematics, 57*, 303-321.
Nemirovsky, R., & Ferrara, F. (in press). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*.
Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119-172.
Noelting, G. (1980). The development of proportional reasoning and the ratio concept Part 1--

Differentiation of stages. *Educational Studies in Mathematics, 11*, 217-253.
Pirie, S. E. B., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*, 165-190.
Presmeg, N. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future* (pp. 205-235). Rotterdam: Sense Publishers.
Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning, 5*(1), 37-70.
Schwartz, D. L., & Black, T. (1999). Inferences through imagined actions: Knowing by simulated doing. *Journal of Experimental Psychology, 25*(1), 116-136.

Appendix A—Pilot Study Instructions

Each of the following activities begins with your two hands lying flat on the table, palms down. Your arms should be comfortable, about shoulder-width apart. In all the following conditions, both hands will be moving up simultaneously. Work on each condition in turn, until you are satisfied you have reached sufficient mastery. In the space provided below each of the following prompts, keep notes of any experiences, challenges, achievements, surprises, thoughts, etc.

So, beginning with two hands flat on the table, palms down, move both hands up and…:

1. *Make sure the vertical difference between the levels of your hands is growing.*

2. *Move one hand at 2 inches per second and the other 3 inches per second.*

3. *Move one hand upwards at a speed of roughly 2 inches per second, making sure the vertical distance up to the level of the other hand is increasing constantly.*

4. *Move one hand upwards at a speed of roughly 3 inches per second, making sure the vertical distance down to the level of the other hand is increasing constantly.*

5. *Move your hands up, each at a constant speed, so that they end with the RH at the level of your forehead and your LH at the level of your chin.*

6. *Move both hands up, making sure that one hand is always at 2/3 the height of the other.*

7. *Move both hands up, making sure that one hand is always at 3/2 the height of the other.*