Building Algebra One Giant Step at a Time: Toward a Reverse-Scaffolding Pedagogical Approach for Fostering Subjective Transparency Through Engineering Levels of Interaction With a Technological Learning Environment

by

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A dissertation submitted in partial fulfillment
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Professor Dor Abrahamson, Chair
Professor Susan Courey
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Abstract

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The project brings together and studies the intersection of three ‘big’ ideas from educational research and practice. Firstly, this project is built on the constructivist assumption that meaningful learning occurs when students interact with educational materials that occasion problem solving and reflection. Secondly, this project elaborates on the theoretical construct of transparency by tracking its subjective development. Finally, the project reexamines the notion of scaffolding – the socio-cultural idea that novices receive expert intervention in the form of supports until mastery is achieved – specifically scaffolding that enables an authentic discovery-based learning experience. This work occurs within the context of early algebra. The story of learning algebra in schools is often told as the challenge of progressing from arithmetic to algebra. A main character in this story is the “=” sign or, rather, students’ evolving meanings for this sign (Herscovics & Linchevski, 1996). The *balance metaphor* is undoubtedly the most common visualization of algebraic propositions. Still, students’ persistent difficulty in *transitioning* from arithmetic to algebra suggests that the balance metaphor may not be the ideal method for building transparency for a relational understanding of equations (Jones, Inglis, Gilmore, & Evans, 2013). Therefore, this study investigates an alternative approach. Using a technological-enabled constructivist learning activity, Giant Steps for Algebra, students construct models of realistic narratives. As they build a virtual model of a problem situation, students discover technical principles for assuring the model’s fidelity to the situation. These construction heuristics, are precisely the conceptual foundations of algebra, and the activity’s *situated intermediary learning objectives (SILOs)*. To enable to gradual development of transparency for Giant Steps, at each interaction level, the student discovers a SILO, and then the technology takes over by automatizing that SILO, thus freeing the student for further discovery. This activity architecture is called *reverse scaffolding*, because the tools relieve learners from performing what they *know* to do, as opposed to what they *do not know* to do. In a quasi-experimental evaluation study (Grades 4 & 9; *n*=40), reverse-scaffolding students outperformed baseline students, for whom the technical features were pre-automatized upfront, on measures of transparency for the SILOs. I thus conclude that discovery-based learning activities are advantageous, and that reverse-scaffolding technological activities can level the gradual development of subjective transparency.

Keywords: Discovery-based learning, Transparency, Scaffolding, Reverse-Scaffolding, Algebra.

Dedication

This work is dedicated to my students, past, present and future.

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CHAPTER 1: Introduction

Over a century ago Dewey wrote *Democracy and Education* (1916), in which he advocated for instructional practices that focus on providing meaningful experiences. He wrote:

> careful inspection of methods which are permanently successful in formal education, whether in arithmetic or learning to read, or studying geography, or learning physics or a foreign language, will reveal that they depend for their efficiency upon the fact that they go back to the type of the situation which causes reflection out of school in ordinary life. They give the pupils something to do, not something to learn; and the doing is of such a nature as to demand thinking, or the intentional noting of connections; learning naturally results. (Dewey, 1916/1944, p. 154)

Fast-forward to the work of Piaget who wrote “knowledge proceeds neither solely from the experience of objects nor from an innate programming performed in the subject but from successive constructions" (Piaget, 1977, p. v). Knowledge is thus built through recursive interaction with the world. Views inspired by Piaget’s genetic epistemology, such as constructivist pedagogy (Kamii & DeClark, 1985; Warrington & Kamii, 1998) or radical-constructivism (von Glasersfeld, 1983), as well as its adaptations to technological environments (Harel & Papert, 1991; Papert, 1980) harken back to aspects of Dewey’s vision, most prominently the notion that the learner should not be a passive recipient of new knowledge but an active participant in constructing his or her own new understanding. As expressed by Ackermann (2004), “*ideas get formed and transformed when expressed through different media, when actualized in particular contexts, when worked out by individual minds*” (original italics, pp. 20-21). Several mathematics education researchers have taken up this approach. Consider the Dutch school of Realistic Mathematics Education (Freudenthal, 1968, 1973, 1983; Gravemeijer, 1994; van den Heuvel-Panhuizen, 2003). One could further mention the monumental work of Zoltan Dienes (1916-2014), Ephraim Fischbein (1920-1998), Caleb Gattegno (1911-1988), and Richard Skemp (1919-1995) (for a review, see Reid, 2014).

For those of us who design and study technology-based artifacts for teaching and learning, the most notable extension came from the work of Seymour Papert. His oeuvre contributed a theory of learning called constructionism (Papert, 1991b). Papert builds on the constructivist foundation adding the notion that new knowledge is constructed while the learner actively builds meaningful artifacts. Consequently, the resources that are strewn across the makers ‘work-bench,’ and the contexts in which ideas are actualized, influence the constructor’s experience and their learning. And designers hope and pray that their designs are a positive influence and not peer-pressuring the learner into a narrow understanding.

Thirty-five years after Seymour Papert’s emphatic assertion that students learn by constructing (Papert, 1980) we are perhaps making some gains towards making this a reality in Science Technology Engineering and Mathematics (STEM) classrooms. These approaches reject the standard instructional sequence wherein the teacher demonstrates a particular set of procedures, and then students practice these procedures. Schwartz, Chase, Oppezzo, and Chin (2011) call this practice “Tell-and-Practice.” Catrambone (1998) succinctly characterizes the concerns associated with this approach, “Students tend to memorize the details of how equations are filled out rather than learning the deeper, conceptual knowledge” (p. 356). The alternatives to the “Tell-and-Practice” approach essentially flip the script and are designed to enable the students to solve problems in whatever ways they can – inventing procedures (Roll, Aleven, & Koedinger, 2011), discovering critical features of the problem, analyzing their peers’ work (Kapur, 2010) – and only then receiving explicit instruction. These problem-, project-, inquiry, and discovery-based approaches each uniquely attempts to foster learning experiences like those described in the introductory quote, experiences where the learner actively constructs his or her understanding through interactions with specific resources.

Over the last several decades there has been a significant amount of research to establish the validity and effectiveness of these problem-, project-, inquiry and discovery-based approaches. Findings suggest some specific cognitive, theoretical, and pedagogical implications. Firstly, having an opportunity to try something out before receiving explicit instruction, while placing a higher cognitive load on the learner’s working memory, results in better understanding, multiple solution strategies, and improved transfer of knowledge to new situations. Secondly, simply watching a peer attempt to problem-solve, or examining a worked example, is not as effective as engaging in the problem oneself (Kapur, 2014). Thirdly, where an artifact’s structure instantiates content-critical functions the user may not *see* these contributions and therefore be unable to cognitively incorporate these into the emerging schema (Chase & Abrahamson, 2013). Therefore resources and materials designed to support a constructionist learning experience must be transparent to the learner (Hancock, 1995; Meira, 1998).

An artifact is transparent when its content-relevant features that measure, quantify, calculate, convert, or otherwise manipulate information as intermediary steps toward solving a given problem are salient to the user. The theoretical construct of *transparency* captures relations between, on the one hand, artifacts inherent to a cultural practice, and, on the other hand, a social agent’s understanding of how features of these artifacts mediate the accomplishment of particular practices (Meira, 1998). In general, cognitive artifacts bear information structures, logical relations, and activity constraints that offload intentionality onto external features (Kirsh, 2010; Martin, 2009). However, artifacts that are used to foster content learning should be transparent, because tinkering and figuring out how they work is tantamount to understanding the content (Pratt & Noss, 2010; Wilensky & Reisman, 2006). Therefore, whereas industrial designers often wish to obscure the workings of artifacts in the service of efficient use, pedagogical designers might choose to encumber use in the service of learning.

Encumbered interaction and the associated cognitive load is one of the major criticisms of the constructionist agenda and inquiry-, discovery-based instructional approaches. Increasing the cognitive demand with minimal guidance can result in students generating faulty ideas that are hard to override. Additionally, the whole process is time consuming, students do not have opportunities to practice and elaborate on solution strategies, and transferring knowledge to new situations can be impacted (Kirschner, Sweller, & Clark, 2006; Klahr & Nigam, 2004). So as we turn towards a future in which technology will play a central role in instructional settings, designers are challenged to conceive technology-based discovery-learning interfaces that enable the authentic construction of new ideas, opportunities to practice and elaborate discoveries in new problem contexts, and most importantly provide appropriate support and guidance.

So how do we do this? Roll et al. (2011) write, “Students who invent gain the knowledge of key requirements of formalisms, of reasons for these requirements, and of mathematical tools that satisfy (and fail to satisfy) these requirements” (Roll et al., 2011, p. 2829). How, then, do designers proceed to create inquiry-based interfaces that enable students to invent these mathematical tools? Additionally, discovery-based activities are conceptualized as providing opportunities for students to invent and articulate their solutions, as well as build knowledge that is consistent with a particular domain (Hammer, 1997). This creates tension between "respecting children as mathematical thinkers" and helping them to "acquire particular tools, concepts, and understanding" (Ball, 1993, p. 384). Therefore, how can technology-based resources help to bridge this gap? Lastly, what heuristic principles support the design of technology-based interfaces that enable the learner to discover underlying domain specific structures and procedures? What does scaffolding this discovery look like?

This dissertation project seeks to pursue the answers to the above questions by addressing the general design problem of scaffolding discovery-based content learning. The project is centered on a technologically implemented early-algebra modeling activity, Giant Steps for Algebra (GS4A). The central goals of this research project are to answer the following questions:

1. What are effective design heuristics for creating discovery learning activities? How might such activities avail of technological functionalities?
2. In particular, is it possible to create a microworld based on constructionist principles, wherein students learn mathematics content through building artifacts? Can a learning design balance constructionist principles with specific curricular goals? What particular activity architecture would achieve this balance?
3. To the extent that the activity is effective, how exactly does building artifacts lead to content learning?

The project brings together and studies the intersection of three 'big' ideas from educational research and practice. The first, as described above, is the constructivist approach to learning. More specifically, this project is built on the assumption that meaningful learning occurs when the student interacts with educational materials that occasion problem solving and reflection. Secondly, the project elaborates on the theoretical construct of transparency by tracking its subjective development. Finally, the project reexamines the notion of scaffolding -- the socio-cultural idea that novices receive expert intervention in the form of supports until mastery is achieved. Arguably, constructivism, transparency, and scaffolding make for strange bedfellows, given that it is not at all clear how an expert might offer supports that do not give away solutions. In any case, such scaffolding might be different from what the word has come to imply. As such, the crux of this project is to design and investigate a new approach to scaffolding that enables an authentic discovery-based learning experience.

Below, I briefly introduce the theoretical positioning for this study. I discuss the historical emergence of scaffolding as a pedagogical approach and elaborate on how scaffolding might be reinvented so as to serve a constructivist approach to learning. I then elaborate on the domain knowledge that this design addresses, early algebra.

1.1 Theoretical Positioning

1.1.1 Scaffolding.

Regardless of their epistemological commitments, scholars of pedagogy who speak of scaffolding appear to agree that novices to a disciplinary domain of practice need some form of support to make progress (Wood, Bruner, & Ross, 1976). Experts help novices learn by creating inclusive conditions that enable the novices gradually to develop skills and knowledge relevant to the practice (Lave & Wenger, 1991). Such common cultural activity of deliberately fostering learning is often perceived metaphorically as “a kind of ‘scaffolding’ process that enables a child or novice to solve a problem, carry out a task or achieve a goal, which would be beyond his unassisted efforts” (Wood, Bruner, & Ross, 1976, p. 90). Its ecological authenticity notwithstanding, the participatory view of education has historically jarred with constructivist philosophy of knowledge, by which meaningful learning is the process of subjectively reinventing cognitive structures for effectively enacting cultural practice (von Glasersfeld, 1983). Are these seminal perspectives on education terminally incompatible? How can an instructor scaffold an individual student’s personal construction of cultural-historical knowledge? Specifically, does not simplifying a task for the child or co-enacting the task with a child rob that child of critical opportunity to reinvent those very operations and structures they were thus relieved of (Freudenthal, 1971)?

Not necessarily. For example, Abrahamson (2012a) offered a sociocultural interpretation of guided reinvention. The interpretation relied on a theoretical distinction between a solution *process*, such as enacting the combinatorial analysis of a novel random generator, and the *product* of that process, such as the resulting visual display of the random generator’s probability sample space. Grade 4 – 6 participants in Abrahamson’s study, who had argued that the solution process was arbitrary and counterintuitive, nevertheless identified its product as meaningful. They were then willing retroactively to accept the process by which this product was created. What students reinvented in this activity was the normative mental construction of the completed product—they figured out how to visualize the mathematical product as affirming their perceptual judgment for the situation that this product was said to model. Abrahamson therefore argued that students are able to reinvent received cultural forms, but only if the learning environment enables them to experience parity between their naïve and mediated inferences or actions. Abrahamson’s project demonstrated his “product-before-process” design principle and as such offered one way of reconciling constructivist and sociocultural implications for pedagogy. Yet we remain with the question of whether the notion of scaffolding per se is still useful or even tenable for envisioning, building, and conducting mathematics education.

The didactical metaphor of scaffolding has become so ubiquitous in the rhetoric of education researchers and practitioners, that its meaning has become diffuse, its theoretical rationale unquestioned, and its pedagogical operationalization vague (Pea, 2004). One of the objectives of this project, therefore, is to offer a constructivist critique of scaffolding—a critique that maintains the formative role of cultural agents in shaping learning experiences yet underscores the essential role of learners in reinventing cultural practice. Stemming from the critique is a proposal to completely rethink standard or “direct scaffolding” and practice instead what could be called “reverse scaffolding.” As I explain and demonstrate, to direct-scaffold instruction is to support learners by enacting for them what they are not yet able to do, whereas to reverse-scaffold instruction is to support learners by enacting for them only what they have already figured out for themselves. The study presents and discusses findings from an empirical study that suggest the potential of reverse scaffolding as a pedagogical design framework for meaningful mathematical learning. In particular, results from comparing the study participants’ learning gains across instructional conditions of reverse-scaffolding and a control condition of the same experimental unit indicate that reverse scaffolding is more effective.

As such, this study presents and evaluates a radical variation of scaffolding called *reverse scaffolding*. Reverse scaffolding adheres to a traditional definition of scaffolding as cultural agents both structuring for learners critical aspects of disciplinary practice and participating in the enactment of core activities. And yet, the reverse-scaffolding instructor introduces these co-enactment supports into the students' activity only once the students have struggled to construct these critical aspects of enactment for themselves. Reverse-scaffolding activities thus provide supports for the enactment only of what students already *know* to do, rather than what they *do not yet know* to do, thus essentially reversing the scaffolding didactical paradigm.

The reverse-scaffolding thesis emerged in the context of conducting and reflecting on an educational design-based research project that investigated an innovative activity for early algebra (Abrahamson, Chase, Kumar, & Jain, 2014). This study reports on a quasi-experimental research design subsequently conducted to compare learning outcomes gained respectively under reverse-scaffolding and control instructional conditions. As I explain in the next section, the pedagogical rationale rests on the theoretical construct of transparency.

1.1.2 Transparency.

Transparency captures novices' understanding of how features of artifacts they are using function in the accomplishment of situated goals (Meira 1998). Often cited in the case of mechanical artifacts, transparency of cultural artifacts obtains equally in the case of procedures involving the handling of non-substantive objects, such as protocols for generating and manipulating symbolic notation in the solution of algebraic problems. Each representational element of an algebraic proposition has particular properties and serves particular functions within the algebraic system. This system of meanings -- latent structures, relations, and functions of algebraic propositions -- must become transparent to the learner.

Transparency is the key hypothetical construct of this dissertation. The design of educational materials and activities as well as the design architecture of reverse scaffolding were all oriented on student development of subjective transparency for the algebra artifact. And the empirical study reported herein evaluated the Giant Steps for Algebra design by operationalizing, coding, and measuring students' development of transparency for algebraic solution procedures.

In order to assess learning as the development of subjective transparency, I first need to operationalize what transparency would mean in considering the target concept of early algebra. I will discuss how the operationalization of transparency led to the development a new epistemological construct, "situated intermediary learning objectives" (SILO). These SILOs spell out the knowledge elements of subjective transparency for algebra solution procedures. SILOs are what students understand when one says that they understand early algebra. And yet by articulating students' emerging knowledge as a delineated set of psychological constructions, I gained greater traction also on the rich empirical phenomena of multimodal mathematical discourse that includes student and tutor speech, gesture, and the production of concrete and virtual objects. As such, operationalizing transparency in the form of the hypothetical construct of SILOs enabled me to visualize transparency—I could point out indications of each SILO in the video data of tutorial interactions and evaluations as well as in the artifacts that students generated as they engaged in the activities, and I would thus also reflect on the apparent absence of particular SILOs for particular students.

1.2 A Design-Based Research Study of Early Algebra

1.2.1 Design problem.

Within the scope and sequence of mathematics curricula, the transition from arithmetic to algebra presents cognitive challenges to many students (Herscovics & Linchevski, 1994; Sfard, 1994), so that numerous students struggle to pass algebra. Failing algebra, in turn, impedes students' progress within the educational system, so much so that Algebra has been referred to as the 'gate-keeper' course for college admittance and completion (Moses & Cobb, 2001) Thus, much is at stake for US students who are unable to transition toward algebraic notions, notations, and reasoning (Jitendra, Star, Dupuis, & Rodriguez, 2013).

1.2.2 Design rationale.

Perhaps the issue with algebra, I conjecture, is not so much with curricula and instruction as much as with a more core element of this subject matter content, namely the most fundamental imagery and notions by which the content is first introduced to students, even prior to the introduction of the its symbol system. This conjecture led to an investigation of these imagery and notions, to ask whether they implicitly help or hinder a deep understanding of the content. Followed by the search for alternative imagery and notions and building an empirical evaluation program around an experimental introductory algebra unit that I designed and implemented.

A convergent body of research on algebra learning suggests that one challenge in the passage from arithmetic to algebra is the evolving meanings of the equal (=) sign (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Radford, 2003; van Amerom, 2003). In arithmetic thinking the equal sign is most often conceptualized operationally such that its syntactic function is understood as an imperative to arrive at a solution, for example, the student interprets "2 + 3 = ___" as an imperative to determine the sum, "5" (Carpenter, Franke, & Levi, 2003; Sáenz-Ludlow & Walgamuth, 1998). And yet this conceptualization of the "=" sign is absent of a relational sense, by which the two sides of the equation are somehow similar to each other (Jones, Inglis, Gilmore, & Dowens, 2012)—a sense that is pivotal for conventional treatment of algebraic equations (Knuth, Stephens, McNeil, & Alibali, 2006). The implicit carry-over of meanings for "=" from traditional arithmetic exercises to the conceptualization of algebraic expressions should thus be regarded as one source of the cognitive challenge algebra presents for many learners.

1.2.3 Design process.

The design work began with a conjecture associating students' poor understanding of algebra content with the pervasive metaphor underlying their conceptualization of algebraic equations. However, before designing an alternative model I had first to analyze the current, twin-pan model. Namely, the conventional metaphor for algebraic equivalence is that of balance, often elicited through invoking prior interactions with relevant cultural artifacts such as a balance scale, teeter-totter (seesaw), etc. (see Figure 1).

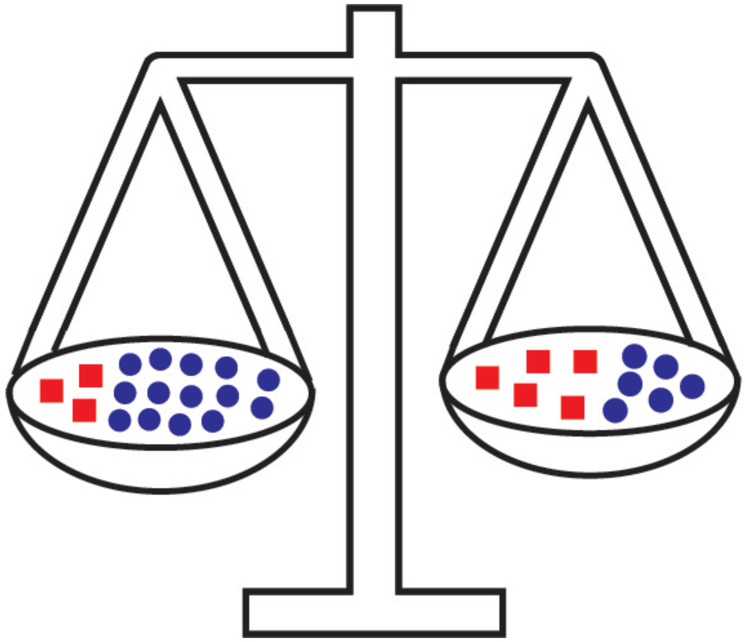


Figure 1. “3x + 14 = 5x + 6” on a balance scale.

A strength of the equivalence-as-balance conceptual metaphor (see Lakoff & Núñez, 2000) is in grounding elements of formal algorithms with symbol manipulation, such as, “We remove 3*x* from both sides of the equation,” in informal actions upon figural or material elements of familiar artifacts, such as, “We keep it balanced by taking off 3 red squares from each side.” The balance metaphor grounds procedural operations isomorphically, that is, whatever operation you perform on one side of the balance scale you must perform also perform on the other side.

Vlassis (2002) explored the implications of using the twin-pan scale model on students’ development of algebraic problem-solving skills. The results suggested that algebraic problem-solving could be categorized in two ways, either arithmetic or non-arithmetic. Arithmetic problems are those that can be solved using guess-and-check strategies and contain a variable on only one side of the equation. Non-arithmetic problems, on the other hand, are problems that contain the variable on both sides of the equals sign. Non-arithmetic problems can be further subdivided into two subcategories, either attached to the model (in this case the balance scale) or detached from the model. In the case of a non-arithmetic problem, Vlassis (2002) demonstrated that problems that are attached to the model could be characterized as pre-algebraic in the sense that the problem-solvers use the model to approximate formal transformations. Problems characterized as detached from the model are problems that contain negative integers, negative solutions, or more than one variable. The study’s findings suggest that formal algebraic manipulations were the only ways that problem-solvers could solve these problems. Vlassis concluded that students might be said to understand algebra only when they “[understood] 1) the principle of transformations in equivalent equations (performing the same operation on both sides), 2) having extended their numerical range with negative integers and 3) understanding the letter as an unknown” (p. 355). The balance model is only a useful representation for accomplishing the first of these, but not the second and third. Based on this finding it seems that there is a need for alternative models to represent algebraic equations, and specifically models that instantiate negative integers and the manipulation of variables.

1.2.4 Design precedence.

Dickinson and Eade (2004) used the number line as diagrammatic form for plotting linear equations. Their conjecture was that by engaging the learner in a more familiar mathematical representation educators could take advantage of the learner' prior knowledge of number systems and previously established formal algorithms. In Figure 2 we see the illustration of the number-line instantiation for "$3x + 14 = 5x + 6$." Note that the left-hand expression of the symbolic equation is mapped above the line, whereas the right-hand expression is mapped below *the same line*, with one-to-one correspondences between linear extents of respective variable quantities (the mirrored *x*'s index the same intervals subtending between marks). Further note how this diagram "discloses" that $2x + 6 = 14$ and, down the line, that $2x = 8$, so that $x = 4$. It appears that this construction of the algebraic equation is more conducive to diagrammatic reasoning (C. S. Peirce - see Bakker & Hoffmann, 2005, p. 334) than the balance model, possibly because the concretization of corresponding symbols has been pre-aligned for visual inspection, thus supporting quantitative reasoning and inference.

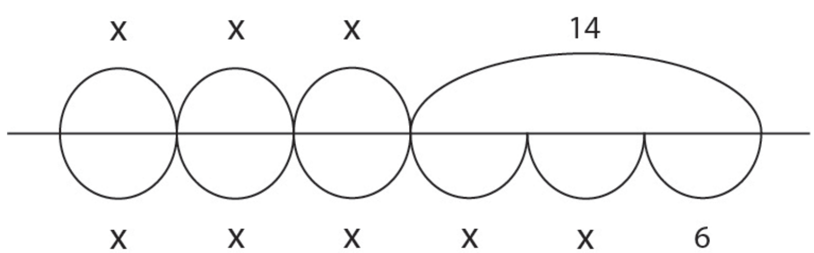


Figure 2. Number-line representation of "3x + 14 = 5x + 6" (image taken from Dickenson and Eade [2004]).

The cognitive operations demanded by this form of algebraic reasoning are different from moving symbols across the equal sign (Goldstone, Landy, & Son, 2010; Wittmann, Flood, & Black, 2013) and therefore bear different pedagogical affordances. More specifically, Dickinson and Eade (2004) found that when participants matched the variables across the top and bottom of a number line, they were able to quickly see relevant similarities and differences both within- and between the equivalent expressions. It thus appeared that displaying an algebraic system of quantitative relations in this particular diagrammatic system was a means of enabling students to visualize the relationships between variable and integers. The cognitive task of comparing and contrasting that students used seems inherently different to performing two separate tasks (such as removing $3x$ from both sides of the equation). The former relies on affordances of the enactive landscape (Kirsh, 2013) and visual thinking (Arnheim, 1969). The latter relies on algorithmic, decontextualized knowledge of how the algorithm is meant to work. Finally, Dickinson and Eade (2004) emphasized that the strength of their method is in creating greater accessibility for early algebra learners—the method should be regarded as transitional toward meaningful application of standard symbol-based algorithm for more complex equations.

1.2.5 Design solution.

Using the number-line model for algebraic propositions, I developed a computer-based discovery-learning activity for algebra, called Giant Steps for Algebra (GS4A). The GS4A project seeks to investigate the potential of a new pedagogical approach to constructing algebraic transparency. The number-line visualization of algebraic equivalence appears to facilitate an offloading of source information onto the diagram's inherent figural constraints. Consider the algebraic proposition "$3x + 14 = 5x + 6$," in this model, we can arrive at the solution, $x = 4$, by a sequence of visual deductions. We conjectured that the number-line model therefore bears greater potential, as compared to the balance-scale model, for students to develop subjective transparency of algebra situations. In particular, the spatial features of the number-line model render highly salient the logical relations between variable and integers, both within- and between expressions. We used the number-line model in designing our learning activity. GS4A is a situation-based model (Walkington, C., Petrosino, A., & Sherman, M., 2013). Per the embodied-design framework (Abrahamson, 2009, 2012b) GS4A seeks to engage and leverage students' tacit knowledge about simple ambulatory motion and spatial relations. In the GS4A activity, students are tasked with finding buried treasure, see Figure 3.

A Giant walked 3 steps and then another 2 meters. She buried the treasure. On the next day, she wanted to bury more treasure in exactly the same place, but she was not sure where that place was. She walked 4 steps and then, feeling she'd gone too far, she walked back one meter. Yes! She found the treasure!

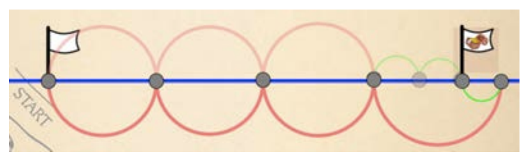


Figure 3. A sample GS4A narrative (on the left) and its model (on the right). On Day 1, above the line, and Day 2, below the line, the giant travels from the flag. Red loops represent giant steps, green loops represent meters.

It was during the development of GS4A that the idea of reverse scaffolding emerged as a design architecture (Chase & Abrahamson, 2015). In GS4A, users read texts about problematic situations they are to solve, and they are given a set of generic tools to model these situations. In the course modeling the situations, students are encouraged to reflect on their models and note these embedded structural and functional patterns: (1) consistent measures; (2) equivalent expressions; and (3) shared frame of reference. These goals are conceptualized as the activity's "situated intermediary learning objectives" (SILOs)

As students progress through the activity, they realize the shortcomings of the available generic tools for generating and maintaining the SILOs. Through this reflection, the students formulate simple understandings of powerful algebraic ideas. With each articulation of what the mathematical tool should do, the technological learning environment takes on the enactment of that particular feature, thus relieving the students from enacting it themselves. As such, the GS4A technological system co-constructs the model only once students understand the necessity and functionality of each specific property of the model. Thus the pedagogical system relieves users from executing what they know to do rather than what they do not know to do.

I suggest that reverse-scaffolding a students' emergent conceptual understanding hinges on students building subjective transparency for features of the number-line model. In the following chapter, I will elaborate on the classical idea of transparency and how it acts as an organizing principle in GS4A, where students build subjective transparency as they advance through the activity. In order to reverse scaffold, I have created a series of levels that scaffold the subjective development of transparency. The conjecture is as follows: Learners will optimally achieve subjective transparency of mathematical concepts when they themselves have wished for the productive interaction constraints that support their modeling activity. In chapter 3, I embellish on the Giant Steps for Algebra design, including more details about how this system reverse scaffolds. In chapter 4 I outline the empirical design of this study. In chapter 5 I discuss the results. In chapter 6 I share the implications of the results and possible future work.

CHAPTER 2: Theoretical Perspectives

This chapter elaborates on the construct of transparency by discussing prior research on the ontological and epistemological nature of learning tools. The construct of transparency is particularly relevant with respect to learning tools designed to scaffold students' emerging understanding. In order to understand the relationship between scaffolding and knowledge acquisition, this chapter will also explore of the notion of scaffolding. Finally, this chapter will offer connections between developing transparency and scaffolding.

2.1 Transparency: A Past and Present Lens on Educational Design

Transparency is defined here as a psychological construct related to objects and procedures inherent in cultural practice—it is the social agent's understanding of how purposeful artifact-mediated actions, such as cutting paper with a pair of scissors, changing a bicycle gear, or solving an algebra problem by manipulating symbols, accomplishes their objective. As such, transparency is, or should be, of central interest to media scholars as well as educational theoreticians.

Within sociocultural literature, learning tools are conceptualized as instrumental to the mediation of disciplinary content, because they organize and embody routine practices. Thus action taken with learning tools instantiates conceptual schemas by constituting and enabling the performance of problem-solving. Unsurprisingly, scholars of learning have forever been fascinated by the process through which manipulating objects transmogrifies into conceptual knowledge.

As I explain, the construct of transparency can be instrumental in organizing investigations into tool-mediated conceptual learning. I begin my discussion of transparency by stepping back to introduce the perspective of distributed cognition. The theory of distributed cognition shares with the theory of transparency particular attention to relationships between the latent properties of artifacts and the conceptual schema that these same properties may render concrete. However, whereas the theory of distributed cognition pertains to the interplay between human activity and the artifacts they use, the theory of transparency emphasizes the explicit pedagogical implications of these interactions. Transparency engages central ideas of distributed cognition yet, taking a pedagogical view, shuffles the elements so as to offer implications for design and instruction.

2.1.1 Distributed cognition.

Distributed cognition is a theory of human practice that focuses on the structured relationships among human participants to a practice and the media that implicitly mediate the activity. Clark (2003) writes

> What is special about the human brains, and that best explains the distinctive features of humans intelligence, is precisely their ability to enter into deep and complex relationships with non-biological constructs, props, and aids. (p. 5)

In particular, Clark coins the phrase '*information-processing merger*' to describe how humans so naturally create tools that become central to their problem-solving processes.

Consider the array of tools, both new and old, that support our thinking: pen and paper, abacus, calculator, computer. We extend our thinking process by developing routines for engaging the world via cognitive artifacts. For Clark (2003), the work of cognitive scientists is in understanding how human thought and reason is born out of looping interactions between material brains, material bodies and complex cultural and technological environments (p. 11).

From a distributed-cognition perspective, the pedagogical utility of learning media can be theorized as offloading aspects of the cognitive process. In so doing, "physical actions [upon media] enable people to query the environment to test their ideas" (Martin & Schwartz, 2005). In turn, the incorporation of technology in practice lead[s] to the creation of extended computational and mental organizations: reasoning and thinking systems distributed across brain, body, and world. (Clark, 2003, pp. 32-33) It therefore stands to reason that we should understand what types of information-processing mergers are most beneficial to learning, what tools offer these mergers, and what are the cognitive affordances of different external thinking media.

Building on Gibsonian ecological psychology as well as phenomenological philosophy Kirsh (2013) develops the notion of the *enactive landscape*, "the structure that an agent co-creates with the world when he or she acts in a goal-oriented manner" (p. 10). More specifically, agents operate on objects, and "[m]oving the object and attending to what that movement reveals pushes us to a new mental state that might be hard to reach without outside help" (p. 2). When these objects are tools that we learn to use in engaging the world, our enactive landscapes become object-mediated. As such, when we appropriate cognitive artifacts, that is, when we create new information-processing mergers, we extend our cognitive enactive landscapes; we instrumentalize our reasoning.

Learners' capacity to adopt a new tool thus depends on a panoply of historical, technological, and experiential factors. At the same time, how effectively we instrumentalize our reasoning with new cognitive artifacts is contingent on properties of our learning experience, which may vary. For example, perceptually rich artifacts may detract novices from their cognitive function (DeLoache, 2000; Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009). Also, contexts of engagement play a role in the quality of adoption: "meaning is given to the manipulation of objects through the siting of the objects within familiar contexts" (Williams, Linchevski, & Kutscher, 2007, p. 154).

Given a particular content-related goal, what objects should educational designers create? A single mathematical idea might be instantiated in many different forms that may bear different learning affordances. Martin (2009) offers a fresh perspective on the role of manipulation in conceptual development. In their view, manual actions on the environment make manifest vital aspects of the conceptual modeling process: the selective organization of phenomenal features into working models highlights latent quantitative properties and relations and brings these closer to familiar, more manageable goal structures and thus renders them better suited for visual reasoning. As such, modeling activities bear the potential of fostering an enduring learning. What we learn about content by modeling is to solve prospective problems of the same conceptual class, because those future problems, too, will necessitate reconfiguring resources. This perspective resonates with the notion that artifacts offload cognitive content in the service of mundane practice (Hutchins, 1995a) and that we think by adapting the environment—manipulation can be epistemic activity (Kirsh, 1996).

To conclude, the effect of tools on our thinking is dialectical: even as we learn to act and think in new ways as framed and facilitated by these tools, the tools bear the potential of reifying for our reflection what and how we are acting and thinking. While distributed cognition is a helpful theory for explaining the cultural practice of thinking with objects, this theory does not pay specific attention to the properties and structures of the tools themselves and the pedagogical implications of these properties and structures.

2.1.2 Tools for learning.

A study of transparency as it relates to learning is situated intellectually in research on concrete and virtual objects in the service of STEM education, namely manipulatable objects or just "manipulatives." It is, therefore, important to step back and contextualize our discussion of transparency in the literature on artifacts and tools for learning.

Artifacts are essential to reform-oriented mathematics education. When designers create situated phenomena as instantiations of mathematical concepts, these situations create opportunities for teachers to guide students toward building meaning for symbolic expressions (Herscovics & Linchevski, 1994; Mariotti, 2009; Pratt & Noss, 2010; Radford, 2003; Sfard, 1994).

A wealth of previous research has portrayed the complexity of circumstances under which the use of manipulatives as pedagogical tools for mathematics learning are more or less efficient (e.g., Edelson, 2002; Greeno, 1998; Martin & Schwartz, 2005; Nemirovsky, 1994). In support of using manipulatable instantiations of mathematical concepts, Meira (1998) considers the "intrinsic qualities of material displays and…. how those qualities might promote individual cognitive efficiency by enabling users to see underlying principles and relations through them" (pp. 123-124). Highlighted here is the *potential* of seeing these principles and relationships and thus grounding otherwise abstract principles and relationships. Meira (1998) goes on to write that 'seeing' is contingent not so much on the properties of the tool itself as much as the prior sociocultural experiences that the participant brings to the table. In like vein, Ball (1993) argues that learning with manipulatives requires certain entry content knowledge. The abacus, for instance may assist in strengthening concepts of place value by providing imagery and bodily engagement. However, the place value relationships are not inherent in the physical material and can only be seen by someone already familiar with these ideas.

As such, pedagogical instantiation of abstract concepts as manipulatable materials bears the potential for dual-representation (Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009), whereby certain "peripheral" properties of a manipulative may hinder the learning process it should be supporting. Specifically, a manipulative's contextually irrelevant affordances--such as color or symbolism--may evoke the user's previous experiences or interests in ways that ultimately derail the learning process and do not support the teacher's objectives. Consequently, it is important that educational designers who develop manipulatives understand theories of tool-mediated learning; and that these theories influence and inform their designs.

2.1.3 Epistemic fidelity.

In an effort to better understand the qualities of learning tools that either help or hinder cognitive activity, researchers began focusing on what is cognitively visible or invisible to the learner. An artifact whose structure and mechanism were objectively accessible for agents to explore and interpret was thought of as transparent, for example, a wristwatch that revealed its inner workings was considered transparent. These tools were also thought to be on a continuum of epistemic fidelity (Roschelle, 1990). The term *epistemic fidelity* was introduced as a way of gauging the degree to which a cognitive artifact indeed represents its ostensible target ideas.

Initially, the terms “transparency” and “epistemic fidelity” were taken as two sides of the same coin, with “transparency” being the psychological counterpart of the “epistemic fidelity” engineering facet. That is, an artifact with high epistemic fidelity was considered as per force transparent. Yet Meira (1998) found that higher epistemic fidelity did not always result in improved learning outcomes. He split an 8th grade classroom into three groups, and each group used different tools with which to explore a particular subject matter. All tools were designed as instantiations of linear functions, with two physical instantiations and one technological instantiation. The first group used a winch with two weighted ropes that dropped down from two spools that moved on one axis (see Figure 4). The spools were different in size and therefore, when the axis turned the ropes rose at different rates.

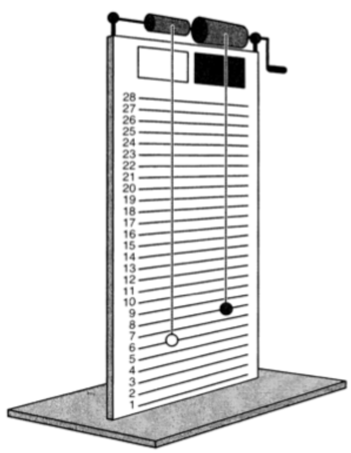


Figure 4. The Winch Mechanism [Image taken from Meira (1998)].

The second group used a similar mechanism, yet the spools were replaced with springs that differed in elasticity. There were hooks on the end of each spring so that the participants could attach small weights to each spring (see Figure 5)

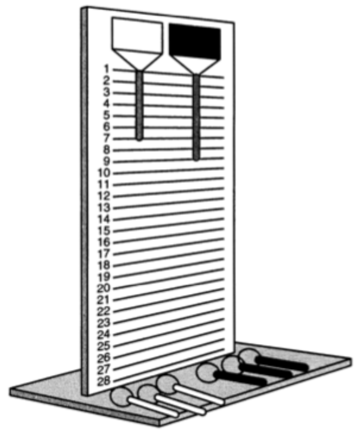


Figure 5. The Spring Mechanism [Image taken from Meira (1998)].

The third group worked on a computer program. The program screen displayed two tables showing different input–output relationship. In each condition the participants were tasked with determining the relationship between two sets of numbers, i.e. the linear function. The researcher hypothesized that the tangible systems would ignite students' embodied understanding of how things work and that this would be applied to this new context in support of building mathematical understanding. For example, the participants would be able to articulate how the spool's different diameters would impact the rates at which the ropes are retracted. Meira considered the epistemic fidelity of each of these conditions and determined that the first, the winch mechanism, was the most transparent, with the highest epistemic fidelity, because "all parts of the mechanism (as well as their connections) can be directly manipulated, observed, and measured" (Meira, 1998, p. 130). The springs were considered less transparent, because the elasticity of the springs is an invisible characteristic. Lastly, the computer program was thought of as the least transparent, with the lowest epistemic fidelity, because all characteristics of the device were internal, invisible, and non-manipulatable.

The findings indicated that participants quickly identified that the relationship between the physical devices and the changes, either the length of the rope or the length of the stretched springs, resulted from an internal characteristic of the devices. Yet, those participants who worked with the computer program could not initially determine the relationship. However, as the task progressed the students working with the computer program were able to decipher the rate of change and quantify this numerically, whereas those participants working with the physical devices struggled with this same calculation. Meira (1998) writes, of those participants working with the springs, "they were able to identify stiffness as a salient feature of the device, but it took them some time to integrate this factor into their problem solving" (p. 135).

In summary, whereas a learning tool's epistemic fidelity is of concern for designers and researchers, this analysis must not neglect the context in which these tools are used and understood. A device's transparency does not simply relate only to its physical or virtual design, but "the transparency of a device emerges anew in every specific context and is created during activity" (p. 138). This finding prompted refinement of the term transparency such that it includes contextual parameters as well as the learner's subjective factors. Transparency is therefore a *psychological construct*—it captures the relation between a person and an artifact (see Hancock, 1995).

2.1.4 Designing for transparency.

> Constructionism boils down to demanding that everything be understood by being constructed. (Papert, 1991a, p. 2)

The notion of transparency suggests that designers should create learning tools that enable students to build conceptual transparency within a given activity. Constructing transparency via psychological actions takes on a new facet when the instructional activity incorporates physical actions of construction. Educational designers then ask: Does the learning tool enable the user to conceptualize and construct, both physically and psychologically, the intended learning objective? What is the relation between physical and psychological construction? Consider the case of designing for subjective transparency of mathematics.

When we say that a mathematical artifact is transparent, we refer to the subjective salience of its content-relevant features that measure, quantify, calculate, convert, or otherwise manipulate information as intermediary steps toward solving a given problem. Therefore, where industrial designers often wish to obscure the workings of artifacts in the service of efficient use, pedagogical designers might choose to encumber use in the service of learning. For example, in a study of physically distributed problem solving, Martin and Schwartz (2005) found that participants generated more salient and transferable conceptualizations of fractions when using a set of "obdurate" square tiles as opposed to classical pie-shaped manipulatives. Whereas both sets of manipulatives—the square and the circular—can technically serve in modeling part-whole relations, pie pieces were found to obscure the notion of "whole" precisely because students did not need to assume agency in distributing onto those media their emerging sense of whole. The visually compelling circle-whole did the work for them, and so they did not exercise the psychological process that would foster a new schema. On the other hand, students working with square tiles had to create the representation of the whole for themselves, and that experience endured. More generally, the learner's agency in building a representation has been shown to contribute to learning (Gravemeijer, 1999).

Intriguingly, though, under certain conditions making practice transparent might impede rather than support learning, for example by deflecting classroom discussions from goal content. Adler and Lerman (2003) name this phenomenon "the dilemma of transparency." The teacher's dilemma is also the designer's dilemma. How does a designer decide which elements of a pedagogical tool should be engineered as transparent and which should remain covert? Which tasks might render opaque features of an artifact transparent?

In summary, a theoretical position that learning mathematical content is tantamount to developing subjective transparency for mathematical procedures is important to consider when designing instructional artifacts. The enactive landscape that is constructed through iterative manipulations should reveal critical characteristics of the instantiated conceptual content. If the artifacts' function is transparent to the users, these users stand to develop the target content.

2.2 Scaffolding: A Pedagogical Approach to Learning

When a small child is first learning how to walk there are several ways that the adults can respond. The adult can either let the child figure out how to walk, or the adult can enact aspects of the walking activity for the child (see Figure 6). As children learn to walk, speak, ride a bike, and read and write, there are many more instances in which the surrounding adults are faced with this same dilemma.

Figure 6. “Two women teaching a child to walk.” Sketch by Rembrandt van Rijn. [1]

Learning theorists may differ in their proposed models of learning, yet by-and-large they all attend to the roles of adults, tools, and peers in the learning process. Most educators, educational theorists, curriculum writers, and even learners are familiar with the idea that the learner can benefit from some assistance to be able to achieve a learning objective. In particular, the term “scaffolding” is often used to characterize the various contributions made by agents and artifacts to an individual’s learning. Once the learner is proficient in executing some target skill, the litany goes, this scaffolding can be removed.

This section seeks to thoroughly explore the notion of scaffolding. I do this to clarify the rationale of a proposed design framework, which I will elaborate on in later sections. And I begin by asking the question “just because designers are building a learning environment where people are progressing conceptually along a learning trajectory that they have designed, does that necessarily imply that designers are scaffolding?” Moreover, how do designers conceive of the concept of scaffolding, an ostensibly sociocultural term, if they call themselves constructivists or even radical constructivists?

This section first explores the origins of the pedagogical concept known as scaffolding. Following this, I reintroduce my alternative genre of scaffolding and examine its implications for the design of GS4A. I conclude this section with an argument for a new approach to the design of constructivist learning activities, one I call *reverse scaffolding*.

2.3 Theoretical Underpinnings of ‘Scaffolding’—a Brief History

Where did the idea of scaffolding come from? Vygotsky (1978) developed a socio-cultural perspective on learning that explicitly included an outsider who participates in the learning process by augmenting the child’s “zone of proximal development” (ZPD). Just as an adult can perform for a child aspects of her physical actions, such as walking, the adult can

[1] Figure used in Levin and Cole (2007) to illustrate the complexity of teaching.

perform for a child aspects of her conceptual actions. Within the Vygotskian worldview, this co-enactment is necessary, because some elements of the activity are not performable by the novice alone. Understandably, some scholars attribute the notion of scaffolding to Vygotsky (Estany & Martínez, 2013). And yet Vygotsky never quite coined the metaphor of scaffolding, let alone can he be said to have formally operationalized scaffolding, classified its variants, or specified its desired attributes.

2.3.1 What does this mean for learning?

For Vygotsky, performance precedes consciousness. Novices participating in social activities achieve rudimentary competence prior to their being able to articulate their understanding. In a sense, it is *through* participating in the enactment of social activity and its discursive envelope that learners appropriate cultural forms, develop their own conceptual understanding, and refine their ideas for subsequent expression.

Let me elaborate. The ZPD is the range between what the child can achieve independently and what the child can achieve with intervention. Or as Vygotsky (1978) puts it

> [ZPD] is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with a more capable peers. (p. 86)

From this I am then led to consider how "the notion of zone of proximal development enables us to propound a new formula, namely that the only 'good learning' is that which is in advance of development" (Vygotsky, 1978, p. 89). I take this to mean that the enabled performance of the learning objective allows the learner to eventually advance developmentally, and that this is classified as a positive learning experience.

The Vygotskian axiom of production prior to conception caught the attention of a Jerome Bruner. In his research on the nature of skills acquisition, Bruner was interested in developing a sociological analysis of learning. The description of this perspective is that:

> It involves a kind of "scaffolding" process that enables a child or novice to solve a problem, carry out a task or achieve a goal, which would be beyond his unassisted efforts. This scaffolding consists essentially of the adult "controlling" those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon and complete only those elements that are within his range of competence. (Wood, Bruner, & Ross, 1976, p. 90)

This characterization clearly builds on Vygotsky's perspective, and indeed Bruner attributes this work to his discovery of the Soviet psychologist's writing (Bruner, 1986).

However, while Woods, Bruner and Ross's investigations into the nature of scaffolding within learning environments shares intellectual territory with Vygotsky, they disagree on one tenet. While Vygotsky asserts that production precedes understanding, Woods, Bruner and Ross (1976) write:

> It is quite obvious why comprehension *must* precede production and why in most instances it *does*. It must because without it there can be no effective feedback. One must recognize the relation between means and ends in order to benefit from knowledge of results. (p. 90 original italics)

This notion of comprehension prior to production is grounded in studies of language acquisition, wherein there is empirical evidence that children comprehend language before they can produce it.

If I apply this understanding to a scaffolded learning situation, it seems clear that comprehension is contingent on the learner being able to understand the co-enacted solution. Furthermore, in order for the outcome of a learning activity to be recognizable as a solution the learner must begin the learning activity with some understanding of what a successful solution will look like as well as an idea about how to reach this solution.

It would now appear that these two perspectives are antithetical. The distinction between co-enacting as a way of coming to know something, and having some knowledge against which production can be measured seem like two different things. And indeed they are. In this distinction between "do before know" and "know before do" I begin to recognize a fundamental fissure that I elaborate on in the remainder of this section—a fissure that I view as critical to educational theory and design. Perhaps, I submit, designers should reserve the notion of scaffolding to the "do before know" conceptualization of learning.

This short history described the perspectives of the two theoretical titans who have been credited with introducing the term scaffolding. While their perspectives are united in many ways, this section also highlights where they differ, as this difference is one that plays a fundamental role in how the concept of scaffolding has been applied in subsequent research. The following sections will explore how theorists and researchers have taken up defining scaffolding in different learning environments, and scaffolding and constructivist epistemology. Lastly, I will present an argument against the use of scaffolding within constructivist or discovery-based activities and instead propose a new term – reverse scaffolding.

2.4 Scaffolding Learning Environments

The term "scaffolding" has been used so frequently and in so many distinctly different contexts that it is difficult to pinpoint a coherent and unified understanding of its meaning. In its colloquial use, scaffolding is a structure that is erected on the façade of a building while it undergoes renovation. In education parlance, the colloquial sense of scaffolding is used metaphorically to describe a situation in which a tutor structures elements of an educational task while the learner apprentices into this same task. And yet educational theorists and designers also speak of scaffolding as the contributions made by specific tools, not a human agent, in the accomplishment of a certain task. In some cases the structure guides the learner through a predetermined series of steps that eventually enable their independent performance of the goal skill. Yet in other cases the learner is called to tinker with materials until they achieve the desired learning objective. Whereas this is by no means an exhaustive list of examples, it serves to demonstrate how widespread and pervasive the term of scaffolding has become. "Scaffolding" is now almost synonymous to "supporting" or just "teaching." This is cause for concern for the field of educational research. As Pea (2004) so aptly noted, "the concept of scaffolding has become so broad in its meanings in the field of educational research and the learning sciences that it has become unclear in its significance” (p. 272). Pea (2004) goes on to discuss how this dilution in the meaning of scaffolding has created so much variance that establishing empirical boundaries has become difficult. In an attempt to provide a container he proposes that there are two main axes that can describe how scaffolding provides support to the learning process. The first is social and thus pertains to the agents in the learning environment, and the second is technological and relates to how artifacts are designed.

In the following sections I survey some of the uses of the term ‘scaffolding’ in relation to learning. This exercise will enable me to clarify what educational researchers consider when they utilize this term. In so doing, I will demonstrate the need for, and lay a foundation for, my nascent theoretical contribution, the concept of *reverse scaffolding*.

2.4.1 Scaffolding in classroom contexts.

Cazden (1979) was one of the first researchers to apply Vygotsky’s description of interpersonal learning, through scaffolding, to the teacher–student relationship. Because the classroom is the primary context for teaching and learning, it stands to reason that this context is also rich with examples of how scaffolding arises between teacher and student. As characterized by Stone (1993) “the student is not a passive participant in teacher–student interaction but scaffolding is seen as a fluid, interpersonal process in which both participants are active participants” (cited in Pol, Volman, & Beishuizen, 2010, p. 272). Consequently, the process of scaffolding learning is highly personalized, there are many different operational definitions, and therefore empirical findings are all over the map.

In a meta-analysis of papers published between 1997-2007, Pol et al. (2010) sought to synthesize empirical and theoretical work on scaffolding in the classroom in hopes of building a united framework upon which they based their proposed research program. Their framework is described in the following ways. There are three characteristics that are part of all instances of teacher–student scaffolding: (1) contingency; (2) fading; and (3) transfer of responsibility. Contingency describes the degree to which the scaffolder assesses the learner and adjusts their approach contingent on the results of the assessments. Fading describes the process by which a particular level of scaffolding is gradually removed as the learner gains competence. Transfer of responsibility works in consort with fading to describe the process by which the learner slowly takes up agency over aspects of the task that were previously performed for them. Since then, numerous researchers have investigated scaffolding within an educational context. Cazden (1981) examined teacher pupil scaffolding interactions, focusing on the use of interrogation to guide learning. Examining whole-classroom scaffolding interactions, Smit, A. A. van Eerde, and Bakker (2013) found additional scaffolding tactics, specific to those contexts, that instructors use in their didactical efforts to hand over agency to the learner. These additional tactics are:

- *layered*—teachers attend to multiple types of emerging information, both during and after the lesson;
- *distributed*—a teacher’s responsiveness to student performance occurs across multiple lessons; and
- *cumulative*—students’ increasing independence over an instructional unit cannot be attributed to any particular point of contact with a teacher.

Thus the notion of scaffolding has evolved. In particular, scaffolding is not exclusively inherent to the actions of a co-present teacher per se. Rather, aspects of the instructional rationale and interaction may be cumulatively layered and distributed onto other classroom instructional resources. In particular, scaffolding practices are distributed over educational infrastructure—informative, functional, and interactive tools or media that complement, emulate, and possibly enhance the variety of customized supports that co-present human agents provide (Meira 1998).

2.4.2 Scaffolding in technology-enhanced environments.

As such, when educators invest and distribute their pedagogical efforts into a variety of interactive elements within a learning environment, we might still conceptualize these concrete or virtual features as bona fide scaffolds—scaffolds that are embodied, embedded, and latent to the artifacts until students engage, mobilize, and leverage them (Barab et al. 2007). In these environments, it may not be the direct "live" actions of an adult that scaffold the child's assigned task by co-enacting it but rather elements of the artifacts that co-enact the task "remotely" (Quintana et al. 2004; Reiser 2004).

In the last several years there has been a wealth of studies interested in understanding technology-enhanced learning. Within the birth of this new learning environment researchers are interested in questions of design, questions of effectiveness, and questions of epistemology, just to name a few. One interesting line of inquiry tackles questions related to the relationship between scaffolding and the technology tools. Sherin, Reiser, and Edelson (2004) developed a framework for analyzing scaffolding within technology-enhanced learning environments. The premise of their framework was to devise technology-enhanced learning environments that have a scaffolding feature, and a matching environment without this same feature, so that researchers can measure the impact that the scaffold has on learning. A simple example that they use is a word problem about two trains leaving a station, where one train is traveling 30 miles an hour and the other train is traveling 55 miles an hour in the opposite direction (see Figure 7). The students are to determine the total distance between the two trains after a given period of time, in this case 3 hours.

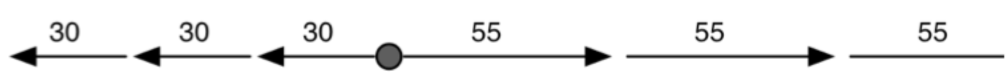


Figure 7. Example train problem (Sherin et al., 2004, p. 390).

In the scaffolded environment the students have a calculator, and in the unscaffolded environment there is no calculator. In this example a calculator provides computational support so that the students can focus their cognitive attention on designing the process for solving this problem, and monitoring their execution. The calculator, in this instance is performing aspects of the task that, hypothetically, students could do on their own without it. I ask the question, should a calculator be considered as a scaffold? If the learning objective of this environment is to calculate, then the calculator is indeed performing the calculations with the student and the solution is co-enacted. Yet, presumably a learning objective is for students to understand the solution process. Therefore, the calculator, in this instance, is not scaffolding the learning process, but rather providing support.

This distinction is interesting because technology-enhanced learning environments can be programmed to perform computational tasks quite easily. therefore, designers should determine what features of the environment actually serve to scaffold the essential cognitive shifts in the learning progression, and what features simply support auxiliary interactions with the interface, if any.

Looking closer at the variety of artifact elements bearing the potential to scaffold the learning process, I will now focus on symbolic elements, because these are important to processes of mathematization. In their analyses of educational technology, Quintana et al. (2004) located scaffolding affordances in representations, specifically in features of the learning environment that highlight for learners how certain interactive features demonstrate content related structures. For example, selectable hints would appear on a computer monitor to suggest the meanings of symbolic notations and animated procedures. Reiser (2004) argues that "scaffolding" is also taken to mean that the software both structures and problematizes the situation, stewarding the students toward discovering the instructional unit's target content (Reiser 2004). That is, the software *scaffolds discovery*. As such, constructivist parlance appears to have usurped the sociocultural term without adhering to its ideological underpinnings.

Whereas by no means have I exhausted the different contexts in which notions of "scaffolding" have been applied, I hope to have demonstrated how widespread, varied, and pervasive the term of scaffolding has become (for a broader review see Smit et al. 2013; Pea 2004). Pea discusses how this dilution in the meaning of scaffolding has created so much variance that establishing empirical boundaries has become difficult. He summarizes

> The goals of scaffolding research going forward should be to study how scaffolding processes—whether achieved in part by the use of software features, human assistance, or other material supports—are best conceived in ways that illuminate the nature of learning as it is spontaneously structured outside formal education and as it can most richly inform instructional design and educational practices. (p. 446)

Indeed, views of learning outside of the mathematics classroom suggest types of authentic instructional methodologies that flout the very rationale of scaffolding. For example, Reed and Bril (1996), cultural anthropologists of skill acquisition, have documented a pervasive parenting practice in which mothers create for their infants opportunities to develop new motor-action coordinations. Rather than model, explain, or directly help the infants achieve the target skill, the mothers instead create for the infants what the researchers call a "field of promoted action"' in which infants discover for themselves how to negotiate the challenging situation into which they have been thrust. As such, the infants develop effective motor-action responses customized to their own musculoskeletal complex. From a distant yet complementary perspective, sports scientists informed by Nikolai Bernstein's theories of kinesiology and biomechanics have put forth the hypothesis that athletes learn better when they must each invent for themselves their personal solutions to motor-action problems (Chow et al. 2007; Vereijken & Whiting 1990). It would make little sense to name these various cultural practices as "scaffolding"' because they are founded on the principle of not-helping rather than helping—in each of these practices the learner is set in a dedicated micro-ecology geared to promote the personal discovery of new affordances for action.

As such, given the numerous operationalizations and pedagogical commitments of "scaffolding" as well as the conflicting precedents of non-scaffolding cultural practices, it becomes questionable whether researchers and designers should persist in using the same word to describe any dyadic, whole-classroom, and technological types of tutorial tactics. This issue becomes acute when technology designers engineer stand-alone learning environments wherein pedagogical methodology is "hard-coded". If educational researchers are to keep using the term "scaffolding" in their practice and discourse, what conceptualization of scaffolding, could we possibly use? Specifically, what form of scaffolding should we implement in educational software programs? Our concern is that where a tutor or tool perform aspects of the learning activity for the learner, there is no guarantee that these scaffolding actions be transparent.

2.5 Scaffolding in Constructivist Epistemology: an Oxymoron?

Genetic epistemology (Piaget, 1968), often dubbed "constructivism," is a philosophical perspective on how humans come to make meaning of their environment. Researchers committed to constructivist philosophy are interested in the relation between individuals' interactions in the environment and their construction of understanding "in the head" (Cobb, 1994). This notion of building upon experiences began with Piaget (1968), who argued, that "children are active thinkers, constantly trying to construct more advanced understandings of the world" (p. 45). Because the learner's process of knowing is deeply embedded in his or her own experience, "truths are replaced by viable models—and viability is always relative to a chosen goal" (von Glasersfeld, 1992, p. 382). Knowledge production or knowledge growth is the process in which "perturbations that the cognizing subject generates relative to a purpose or goal are posited as the driving force of development" (Cobb, 1994, p. 14). The presence of these inconsistencies instigates the process of restructuring a model of the world or a model of how that world works, and this is what is called learning.

2.5.1 Constructivist theories of learning.

> Et, quoi qu'on en dise, dans la vie scientifique, les problèmes ne se posent pas d'eux-mêmes. C'est précisément ce sens du problème qui donne la marque du véritable esprit scientifique. Pour un esprit scientifique, toute connaissance est une réponse à une question. S'il n'y a pas eu de question, il ne peut y avoir de connaissance scientifique. Rien ne va de soi. Rien n'est donné. Tout est construit. (Bachelard, 1934, p. 17)
>
> And, whatever one might say about this, in scientific life problems do not pose themselves. It is precisely this sense of a problem that gives the mark of the true scientific spirit. For a scientific spirit, all knowing is a response to a question. If there were no question, there would be no scientific knowledge. Nothing comes from itself. Nothing is given. All is constructed. (translated by the author)

Constructivist epistemology has given rise to several theories of learning. Specifically within the domain of mathematics education, constructivists believe that students learn by re-inventing mathematical procedures. In fact teaching young children arithmetic algorithms is harmful, because it fails to enable the children to ground the content in their earlier understandings while, worse, training the children to ignore their own thinking (Warrington & Kamii, 1998).

To ameliorate these alleged ills of mathematics instruction, Freudenthal (1983), father of the Realistic Mathematics Education (RME), developed the *didactical phenomenology of mathematical structures*. This pedagogical methodology is based on the principle that children create their own models of problematic realistic situations. Gravemeijer (1999) explains the function of modeling activities in mathematics learning, emphasizing the imperative of letting students' models emerge.

> The premise here is that students who work with these models will be encouraged to (re)invent the more formal mathematics. … [F]ormal mathematics is not something "out there" with which the student has to connect. Instead, formal mathematics is seen as something that grows out of the students' activity. The students are expected to develop formal mathematics by way of mathematizing their own informal mathematical activities. (pp. 159-160; original italics)

Gravemeijer (1999) proposes to view conceptual learning as a structured developmental sequence of iterated semiotic emergences, wherein each cognitive structure can be viewed as referring to the last one in an extending chain of significations. The designer analyzes a target concept by investigating its phylogenetic and ontogenetic origins so as to hypothesize how the historical process may be simulated for individual learners under controlled settings. The designer then creates problems and materials to instigate and mobilize this reinvention.

How does this vision of knowledge production compare to socio-cultural perspectives? Socio-cultural theorists contend that knowledge is negotiated and acquired through social interactions whereby the less knowledgeable individual becomes acculturated into an existing cultural system or practice. Constructivists, however, maintain that knowledge production is an internal and personal negotiation in which the learner must necessarily reinvent the formal systems underlying those would-be cultural practices. In so doing, subjective models of phenomena are reorganized so that personal experiences form a foundation for further interpretation and knowledge building.

And yet, it is too simplistic to pit these two camps against each other. Past decades have seen work to reconcile these seemingly opposing theories (Cole & Wertsch, 1996). One vein of analysis that unites these two theories is very relevant to this discussion, namely the use of tools or artifacts in the learning process. From a socio-cultural perspective the learner becomes indoctrinated with societal norms and practices when he or she engages in activities involving tools, where for Vygotsky almost anything is a tool, including language. As Cole and Wertsch (1996) point out, cognitive activity is shaped by prior cultural practices, for the tools with which people think have been shaped by thinkers who have come before them. Consequently, as humans come to know their tools they reinvent aspects of this historical thought process.

In sum, the concept of scaffolding is broadly associated with the socio-cultural camp and as such is taken to describe deeply embedded social practices as shaping the learning process. Indeed, for Vygotsky production precedes conception. In contrast, for Constructivists production is contingent on assimilating the available means of production to pre-formed goals. Is it appropriate to transport the term scaffolding from its natural habitat in socio-cultural theory and adopt it for use as a constructivist pedagogical practice? I think not. Moreover, since socio-cultural and constructivist theories maintain unique perspectives on how learning happens, I submit that these theories should inform unique perspectives on how teaching should be organized.

2.5.2 Constructivist design heuristics for learning activities.

When constructivist designers refer to the notion of scaffolding within their pedagogical designs, what exactly are they referring to? Reiser (2004) contributed some ideas about how technology can scaffold inquiry-driven learning by identifying two specific ways that software systems support learners. These mechanisms are *structuring* and *problematizing*. To address the former,

> [I]f reasoning is difficult due to complexity or the open-ended nature of the task, then one way to help learners is to use the tool to reduce complexity and choice by providing additional structure to the task." (p. 283)

For example, a software tool designed to scaffold writing-up scientific investigations can display a list of the student's research questions, inferences, and observations, so that the student can easily refer back to these. This description is well aligned with notions of distributed cognition (Hutchins, 1995b; Zhang & Norman, 1994), as the tool does the remembering and categorizing, allowing the learner to focus their cognitive attention on writing their composition.

Problematizing, however, is an entirely different way of providing support for the learning process. This design heuristic is employed when the software system highlights a discrepancy in the student's observations, thus suggesting the need for further problem-solving. This strategy may not simplify the task; in fact it will most likely render the task more difficult initially. Let me look more closely at an example of problematizing to better understand how this particular interface scaffolds. Reiser (2004) describes how the software system ExploreConstructor uses the problematizing scaffolding strategy by "highlighting epistemic features of scientific practices and products" (Strategy 7d in Quintana et al., 2004) when it presents students with dialogue boxes that problematize the difference between forming observations about a situation and making interpretations about a situation. This problematizes an aspect of the scientific domain and thus highlights it. The technology interface is described as setting the stage, through supportive structure and problematizing, for users to explore the distinction between observation and interpretation, one that is central to achieving a sound scientific argument.

Reiser (2004) urges designers to find the balance between providing adequate structure, so that students can be successful, and problematizing, so that students are engaged with and capitalize on their intuitive understandings for constructing new knowledge. One way of achieving this in software systems is through the use of representations that build conceptual bridges between intuitions or prior knowledge and domain-specific ways of understanding. For Quintana et al. (2004) a "representation" is something that shows or explains how the functionalities of the software system work. Representations, in this context, highlight why particular features function in the ways they do, for example, graphs, computer-generated hints, or tools that suggest the domain's formalism so that users' interactions can better approximate expert practices. A representation allows the learner "to think about the deeper concepts and structure of disciplinary relations and not get caught up in surface details" (p. 347). These kinds of representations are described as scaffolds for "grounding learner understanding by helping learners access familiar ideas on which more formal concepts can be built" (Quintana et al., 2004, p. 347).

In conclusion, technology-based learning environments designed using constructivist principles can offer a wide variety of mechanisms that are characterized as scaffolds for performance, and yet these differ dramatically. Much the same way that a human can scaffold a learning process by reducing or chunking elements of the process, a technology-based environment can provide structure, as defined by Reiser, and representations, as defined by Quintana et. al., problematizing serves to situate a leaner in the task and activate prior knowledge and intuitions, a mechanism that is comparable to Woods, Bruner, and Ross's first scaffolding strategy of recruitment. Problematizing could also serve to situate the learner, analogous to Woods, Bruner and Ross's second strategy (reduce the degrees of freedom) as well as the fourth strategy (marking critical features), but this is contingent on the problematizing itself. When the nature of the task, presented in a technology-based learning environment, is problematized to scaffold learning, the problem must be authentic to the learner, and allow the learner to begin thinking. They must have the opportunity to construct a hypothesis about the problem, they must develop some intuitions about the problem that can then emerge as a foundation upon which to build new understanding.

What is missing from the above descriptions about the potential for scaffolding within technology-based constructivist learning environments is an analysis of whether these tools allow the users to construct subjective transparency for the aspects of their productions that are being scaffolded. For example, if an interface contains a structure that performs a critical function, such as the ExploreConstructor described earlier, at what point in the learning progression are the users called to construct its criticalness for themselves, or do they simply use the functionality to perform a specific task?

For Reiser a tool is transparent if the interaction with the tool is analogous to the type of thinking that the designer is eliciting. By this definition, I see ExploreConstructor as transparent with respect to observation and inference because, users must distinguish between them. However, as Pea (2004) writes:

> The general issue with which I am concerned is whether students using a constrained set of forms for producing the work artifacts of scaffolded scientific inquiries are "parroting" back disciplinary forms of thinking rather than performing with understanding of what they have created. (p. 436)

As a designer I wish to challenge Reiser's particular notion of transparency. While this software system does allow users to generate content within these distinct categories, it robs the users of opportunities to construct the scientific importance for this distinction in the first place. I would argue that the ExploreConstructor interface does not scaffold *why* this epistemic difference exists. The interface simply supports the user to perform the distinction, with the assumption that this *why* will become transparent.

2.6 Conclusion

When educational designers and researchers design artifacts that will be manipulated by learners, what is the intended outcome? Likely, the designer is hoping that the learner will learn.

Likely, the learner is hoping to figure out how the artifact works and using it to solve a particular problem. Therefore, how the artifact functions, in relationship to the structures of the problematic situations must become transparent to the learner. The notion of scaffolding describes a variety of pedagogical moves, taken by either a teacher, tutor, or artifact, that con-enacts the problem solution and procedures with the learner. This dissertation seeks to build a framework for describing how these two theoretical constructs interact.

Traditional scaffolding connotes that a more knowledgeable party, a teacher, tutor or artifact, is performing aspects of a solution procedure that the learner *does not yet* know how to do. This robs the learner of the opportunity to construct subjective transparency for this particular performance. I propose that we 'flip the script.' In a Reverse Scaffolding pedagogical model, a teacher, tutor or artifact only performs aspects of a solution procedure that the learner *already knows* how to do. Reverse Scaffolding enables the learner to develop subjective transparency for aspects of a solution procedure and only then offload these performances onto features of the learning environment.

CHAPTER 3: The Design of Giant Steps For Algebra

3.1 Why This Design? Some Background Knowledge

The story of algebra learning in schools is the story of progressing from arithmetic to algebra. A main character in this story is the "=" sign or, rather, students' evolving meanings for this sign (van Amerom, 2003; Filloy, & Rojano, 1989; Herscovics, & Linchevski, 1996; Molina, & Ambrose, 2008). Students' implicit framing of this symbol is *operational*, because the framing is fashioned by a history of solving arithmetic problems such as "3 + 14 = __", where you operate on the left-hand expression and then fill in your solution on the right (Carpenter, Franke, & Levi, 2003; Sáenz-Ludlow & Walgamuth, 1998). Ultimately, algebraic conceptualization of the "=" sign should be *relational*—the proposition means that the expressions on the two sides of the "=" are equivalent (Jones, Inglis, Gilmore, & Dowens, 2012). Given that the arithmetic visualization of "=" is impeding students' transition to algebra, how might this visualization be countervailed? One way that educators approach the work of building students understanding is through the use of metaphor.

The *balance metaphor* is undoubtedly the most common visualization of algebraic propositions. This metaphor is typically introduced to students by invoking interactions with relevant cultural artifacts such as the twin-pan balance scale (see Figure 8).

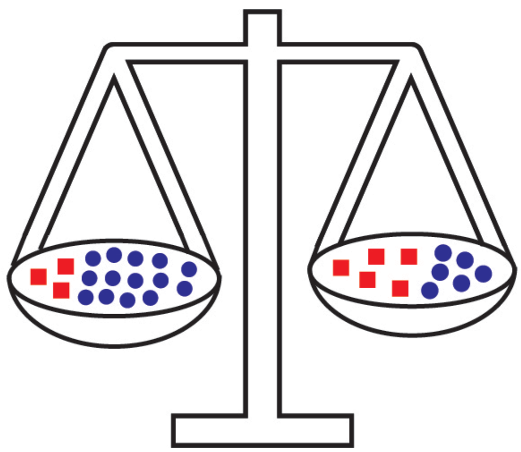


Figure 8. Balance scale showing "3x + 14 = 5x + 6."

However, although the balance metaphor is well suited for grounding the fundamental algebra *algorithm*, students' persistent difficulty in *transitioning* from arithmetic to algebra suggests that this metaphor nevertheless may not be the ideal method for building a relational understanding of equations (Jones, Inglis, Gilmore, & Evans, 2013). Moreover, the historical substitution of twin-pan scales with electronic scales may have rendered the twin-pan scale unfamiliar to many students. I therefore asked, "What alternative metaphor might facilitate students' passage from arithmetic to algebra?" A search revealed that Dickinson and Eade (2004) had tackled a similar design problem. They used the number line as a diagrammatic form for modeling linear equations (see Figure 9).

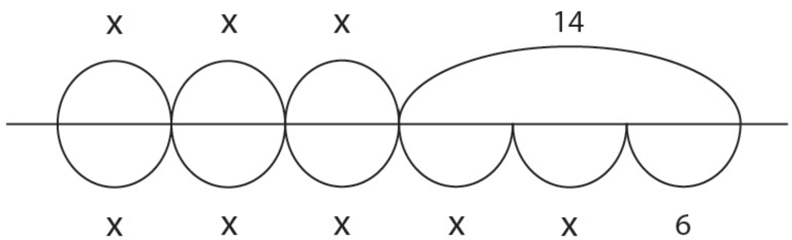


Figure 9. Number-line instatiation of “3x + 14 = 5x + 6” (Dickinson & Eade, 2004).

Note in Figure 9 how the left-hand (“3*x* + 14”) and right-hand (“5*x* + 6”) expressions of the algebraic proposition are mapped, respectively, above and below the same number line. In particular, pairs of *x* symbols above and below the number line each denote an arc subtending *one and the same linear interval*. Looking again at the number-line diagram (Figure 9), I want to highlight that the combination of above-the-line and under-the-line symbolic indices of one and the same line segment offers two perceptually contrasting yet conceptually complementary visualizations of a single perceptual stimulus. Note further, looking on the right of Figure 9, how this number-line diagram “discloses” that $2x + 6 = 14$, so that $2x = 8$, and therefore $x = 4$. Students model not two sets of voluminous quantities of equal mass (twin-pan model, Figure 8) but a single linear quantity bearing two alternative indices of its extent (number-line model, Figure 9). In the number-line model but not in the twin-pan model, students are able to construct logical relations between variable and integers directly by attending to spatial properties such as adjacency and containment. As the students scan its components, the number-line model appears to do the work for them, so that they can do the math less so “in the head” and more so “on paper.”

3.2 Giant Steps for Algebra

The Giant Steps for Algebra design, described in the following section, is based on this “double-measuring-stick” model. The Giant Steps problem narrative depicts a quasi-realistic situation, in which Egbert the giant performs two consecutive journeys along a path. These two journeys—Day 1 journey and Day 2 journey—begin at the same point of departure and end at the same destination. However, the journeys differ in terms of the agent’s process in traversing from the start point to the end point. The two journeys correspond to two equivalent algebraic expressions. For example the algebraic proposition “$3x + 2 = 4x – 1$” is rendered into the progressions “$3x + 2$” (Day 1) and “$4x – 1$” (Day 2). The narrative of this particular example reads as follows:

> Egbert the Giant has stolen the elves’ treasure. He escaped their land and voyaged to a desert island. After mooring, Egbert set off walking along a path. You are Eöl the Elf. You are positioned on this island, and you are spying on the giant to find out what he does with the treasure.
>
> Starting from the port and walking along the only path there, Egbert the Giant walked 3 giant steps and then another 2 meters. He buried some of the treasure, covered it up really well, and then went back to the ship, covering up his tracks.
>
> On the next day, Egbert wanted to bury more treasure in exactly the same place, but he was not sure where that place was. Setting off along the same path, he walked 4 steps and then, feeling he’d gone too far, he walked back one meter. Yes! He’d found the treasure.

> He buried the rest of the treasure in exactly the same spot. Egbert then covered up the treasure as well as all his tracks, so that nobody will know where the treasure is. He returned to the ship and sailed off.
> Your job, Eöl, is to tell your fellow elves exactly where the treasure is. You must tell them how many meters they need to walk from the docks to the hidden treasure.

As the design of Giant Steps unfolded and coalesced, I realized that it would be an environment wherein students could develop the notion of a variable as a specific quantity: a numerical value that is consistent within a local situation. The specific value of the variable would initially be unknown to the student but could eventually be determined through the student's triangulation of available information about the two situated expressions, that is, the Day 1 and Day 2 journeys. Yet triangulating depictive information—as I learned by tinkering with the design myself and observing children attempt to solve the problem—carries certain implicit demands of structural precision and coordination. These smaller structural productions are what facilitate a logical progression whereby triangulation can be realized. I call these smaller productions Situated Intermediary Learning Objectives (SILOs).

Qualitative data analyses suggested three SILOs for the Giant Steps design.

1. Consistent measures. All variable units (giant steps) and all fixed units (meters) are respectively uniform in size both within and between expressions (days);
2. Equivalent expressions. The two expressions (Day 1 and Day 2) are of identical magnitude—they share the "start" and the "end" points, so that they subtend precisely the same linear extent (even though the total distance traveled may differ between days, such as when a giant oversteps and then goes back);
3. Shared frame of reference. The variable quantity (giant steps) can be described in terms of the unit quantity (meters).

Users can articulate SILOs mathematically before they attempt to solve the problem. Rather, the users may know the SILOs informally. For example, they may think about their own steps and perhaps of having buried an object in their backyard and using an informal measurement system to later determine its location. In the next section, I explain how the emergence of Giant Step's SILOs allowed the design team to consider a new design architecture based on the emergence of these critical intuitions within the discovery-based learning process.

3.2.1 The design architecture of a discovery-based learning sequence.

The following section reflects the work of my design team. We used the SILOs to imagine a unique learning progression within Giant Steps. We decided to use the design's SILOs to plan a technological version of the Giant Steps activity. The SILOs would form a blueprint for an activity architecture, wherein transitioning from each interaction phase to the next would be linked to demonstrating mastery over one of the SILOs.

The idea was thus to step learners through the design, all the while enabling them to build and sustain subjective transparency of the emerging model. Each SILO is one aspect of the model that the learner would be required to build manually (virtually) before that property was instantiated and monitored automatically. Borrowing the notion of "levels" from popular computer games—that is, the gradual rewarding of manifest competency with increased power that is linked to increased task demand—in Giant Steps I *level transparency.* As users master each SILO, they receive new control over the environment in the form of enhanced affordances that instantiate that specific SILO: the system then interprets the user's intention to act and facilitates its execution by offloading the technical details of enactment.

In Giant Steps, leveling transparency is engineered as follows. The user encounters a problem narrative and is encouraged to solve it on the screen. On the left there is a small flag (the "start" location). Below the line there is a standard drawing toolbox with buttons for either selecting a color (giant steps are red, meters are green), toggling between journey days (Days 1 or 2), or editing (by either removing or clearing model elements). A floating "treasure box" (see in Figure 10, top-right corner) can be placed at any location. If a user selects the "Giant Step" button and then clicks on the screen, a red arch appears that connects the giant's last location along the path (a grey node) to the clicked location (a new grey node). Similarly, "Meters" are green arcs. Using these interface utilities, the user is to solve the problem.

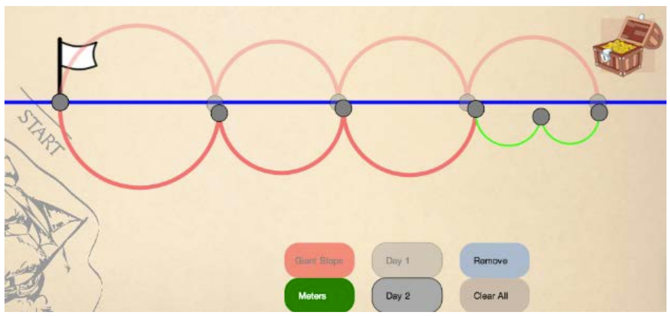
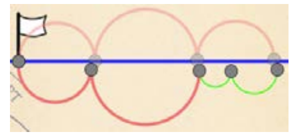
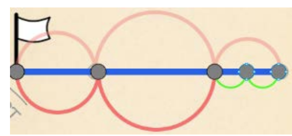
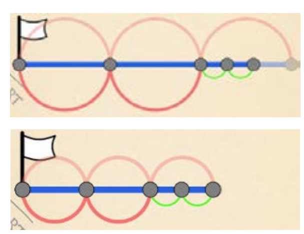


Figure 10. In Level 1, Free Form, users create all parts of the model manually. Note that the giant steps (red arches) are not quite uniform size; neither are the meters (green arches).

3.3 Sequencing Without Scaffolding: The Notion of Reveres Scaffolding

SILOs are psychological constructs—they are about what a child knows (or, at least, about the design-based researcher's best understanding of what the child knows). Levels, on the other hand, are technical constructs—they are about an activity's technological affordances, that is, what a pedagogical system demands of, and performs for the user. And yet SILOs and levels are closely related: Each SILO articulates a knowledge criterion for entering a new level, and then each level, in turn, orients the child to achieve some next SILO. Table 1 delineates this relation between SILOs and levels in Giant Steps. The SILO sequence in this table should be interpreted as paradigmatic, not dogmatic. It is the typical "story" of a child going through what may or may not become a canonical learning sequence in this design.

Table 1. *Leveling Transparency: Matched SILOs and Levels in the Giant Steps Technological Design*

SILO	Level	System Constraints, User Activity, and Behavior Criterion	Interface
1. Consistent Measures	1. Free Form	System offers no support in coordinating units or expressions.	
	Activity	User builds all parts of the model manually; is perturbed by units' unequal lengths within and between days; tries to equalize units via small adjustments, but witnesses that increasing one unit decreases an adjacent unit sharing a node.	
	Criterion	User expresses frustration in equalizing units.	
2. Equivalent Expressions	2. Fixed Meters	System generates meter units in predetermined size and maintains uniform size automatically.	
	Activity	User builds variables manually; is perturbed by variable units' unequal lengths within/between days; tries to equalize variable units but witnesses that increasing one unit decreases an adjacent unit sharing a node.	
	Criterion	User expresses frustration with managing uniform variable units particularly in an attempt to equalize the two propositions (the lengths of Days 1 & 2).	
3. Shared Frame of Reference	3. Stretchy	System monitors for manual adjustment to the size of *any* of the variable units and accordingly adjusts the size of *all* variable units.	
	Activity	User adjusts the variable size to equalize the two propositions	
	Criterion	User reads off the value of a variable unit in terms of the number of known units (meters) it subtends, e.g., one giant step is 2 meters long.	

I shall now elaborate on this table, referring to its screenshot images. In Level 1, "Free Form," users construct all elements of their model in freehand, analogous to drawing with pencil and paper. Some production imprecision naturally ensues, such as steps that are not quite the same size (see also Figure 10). The importance of precision (SILO 1) will arise only once the learner attempts to coordinate measures across two journeys, marked above and below the path, and encounters misfits that impede progress in building the model. Once users have articulated the imperative of consistent unit size and labored over implementing this aspect in their models, they are evaluated as having graduated SILO 1, “Consistent Measures.” As a first concession, the program enters Level 2, “Fixed Meter,” in which the system relieves the learner of producing uniform meter units (see also Figure 11).

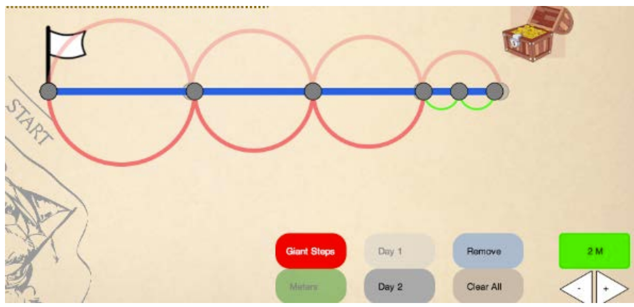
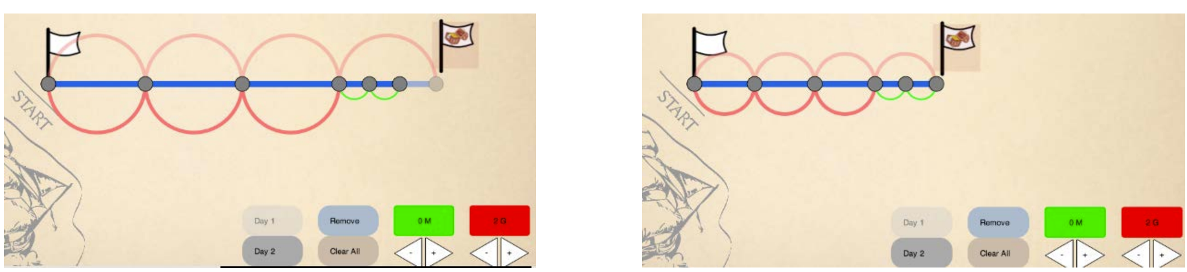


Figure 11. In Level 2, Fixed Meter, the meters are uniform in size while the giant steps remain variable. Users interact with a symbolic control (bottom-right corner) to generate meters.

Unburdened by the task of maintaining uniform meters, the user now attempts to equalize the two journeys (Day 1 & Day 2) by adjusting the variable size. Note that one and the same variable, a giant step size, applies both within *each* journey day and across *both* days. Once the user articulates that the variable is consistent across the entire model, the interface enters Level 3.

In Level 3, “Stretchy,” not only is the meter unit size maintained automatically, as in Level 2, but here also the variable size, which is consistent across the model and changes uniformly. So when the user drags any of the nodes to the right or left along the path line, both the variable and fixed units react. The variables change in size accordingly (compare Figure 12a and Figure 12b), and the fixed units move left or right along the path line yet remain fixed in size. This supplementary affordance enables more felicitously to match the end points of Day 1 and Day 2, as follows.



a. b.

Figure 12. In Level 3, Stretchy, green arches (meters) are invariable, while red arches (giant steps) are variable via uniform scaling. A new control (bottom-right corner) now enables the user to generate a specified number of giant steps, not only meters.

Note, in Figure 12a, that all the variable units are uniform, both above the line path (Day 1 journey) and below it (Day 2 journey), and yet the two journeys do not end at the same location—the top trip ends farther from the start than the bottom trip. Recall that the green meter arcs cannot be adjusted ("Fixed Meter"), so that the only way of aligning the two trips would be by changing the uniform size of the red arcs. That is precisely what our hypothetical student did, so that the two trips ended in the same location (see Figure 12b).

A new hypothesis arises from the "leveling transparency" technological design architecture—a hypothesis that informs this research study as well as a tentative theoretical insight. Namely, if users were introduced to the activity at Level 3, with the full slate of interaction shortcuts, they could not appreciate these functionalities as affordances, because they would not know *what* it is that each functionality affords. At the behavioral level, these users would not be able to articulate the computer's functionalities as contributions to a co-enactment of this specialized cultural practice; as such, they would not achieve the SILOs, that is, they would not ground the conceptual content. In that sense, *constructing transparency through reverse scaffolding is the process of coming to visualize an artifact's invented functionalities as affordances*. This idea of learners perceiving technological features as enhancing their agency is closely related to Pratt and Noss's (2010) design heuristic implicating the epistemological root of mathematical concepts in children's purposeful construction of *utility* for new ideas that are instantiated into designed artifacts in the form of interaction potentialities.

Both utility and transparency speak to the notion of learning content through working with a ready-made tool. However, whereas "utility" suggests that the learner does not yet know what the tool can do, "transparency" suggests that the learner is already using the tool – already availing of its embedded utilities. With transparency you still have to figure out how the tool is doing what it is doing (i.e. how it is that you are doing-with-the-tool what you are doing). And yet in the Giant Steps for Algebra I am using "transparency" in the specific context that is neither of the above, that is, you are building the tool yourself. A key idea is that children themselves should build the situations they want to see and experience. I submit that learners will achieve this transparency optimally when they themselves have designed, or at least manifestly struggled to simulate and have explicitly wished for, the interaction constraints that constitute those utilities.

So the best way to build subjective transparency of a tool, I maintain, is to implicate the desired utilities *even before they are in fact instantiated into the device the learner is using*. One might thus colloquially name the emergent affordances in Giant Steps the "if only's," because they each fulfill the learner's wish that "if only the device could do this or that." Leveling transparency is the guided, systematic, and incremental realization of the child's rolling wish list into design features that introduce a functional relation between user input and system output. Finally, this approach might bear different, perhaps advantageous, effects for the child's sense of self-efficacy and agency. Namely, it is a different experience to work with a ready-made artifact and figure out what its features are *for*, as compared to building an artifact by implicating its desired features.

One might be tempted to describe Giant Steps as an exemplar of technological designs that *scaffold* algebra content. I hesitate to use that common term. In fact, the proposed design architecture for leveling transparency might better be described as *reverse scaffolding*. Scaffolding is the asymmetrical social co-enactment of natural or cultural practices, such as walking, cooking, or solving a mathematics problem, wherein a more able agent performs for novices elements of a complex activity. The novices' participation is thus simplified, so that they experience the activity's flow, coherence, purpose, meaning, and efficacy as well as a sense of competence. Scaffolding, as described earlier in this paper, also includes the eventual fading of co-enactment and transfer of responsibility from the more capable agent or tool to the learner. The end result of a scaffolded learning sequence is the learners' independent enactment of the task (see Figure 13). In Giant Steps, by way of contrast, the scaffolding is inherent to the design rationale but not the actual activity. Within this environment there is no co-enactment of any steps that students have not yet figured out themselves. The system co-constructs the model only after the student has discovered the necessity and functionality of each specific property of the model (see Figure 13). Thus the pedagogical system relieves users from executing what they know to do rather than what they do not know to do.

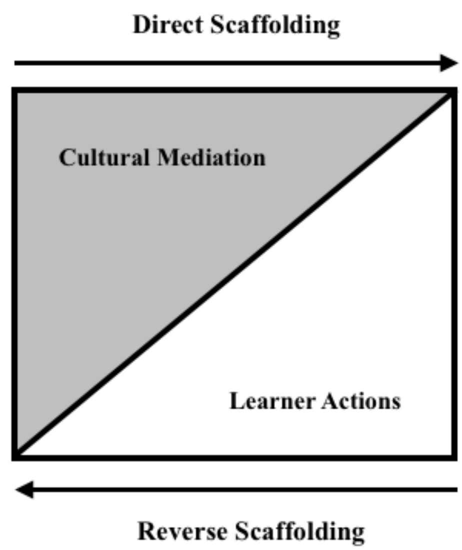


Figure 13. Two different pedagogical learning sequences for the same content demonstrates how reverse scaffolding works.

Reverse scaffolding is a pedagogical strategy birthed of constructivist theories of learning. Reverse scaffolding is embedded in a much larger world of pedagogical architectures. This dissertation seeks to more clearly delineate a small piece of turf within the larger debate. Scaffolding, by definition, is co-enactment. For socio-cultural theorists, co-enactment means scaffolding the production of a specific outcome in a "do-then-know" architecture. For constructivist theorists, co-enactment means scaffolding the discovery of a goal-oriented contribution in a "know-then-do" architecture. In classical educational practices, the target skill is co-enacted by the expert (or the tool) and novice from the get-go, with aspects of enactment gradually faded, transferred from the cultural agent and supporting technology to the learner. In reverse scaffolding, by contrast, the discovery is not scaffolded at all. Whereas some discovery-based environments contain features that co-enact the design's outcome, and the user eventually comes to realize these contributions, in GS4A the users re-organize what they know to address a problematic set of parameters, and in so doing the users imagine something that could help them. Only then do we co-enact.

3.4 A Design Synopsis

In the interest of facilitating an improved understanding of the GS4A design, and to better inform future design efforts for technology-enabled educational activities (Gaydos, 2015), I will briefly describe the significant considerations that informed the end product described above. This section captures a synopsis of the design narrative (Hoadley, 2002), in which I highlight the significant developments as they unfolded over time. Based on data collected during pilot studies (Chase & Abrahamson, 2013) the design team understood the significance of participants constructing their own meaning from the available resources. And yet, as is the case with all technologically delivered designs, we had to carefully consider how we constructed the microworld. Or put another way, we wanted participants to play in the sandbox in particular ways, so we had to build a sandbox of particular specifications. And yet, we did not get these specifications right the first time around. The team approached the design process much like the those described by Schön (1983), as a cyclical *reflection-in-action*. During our design activities, the product would often 'talk back' in unintended ways, thus revealing our assumptions and working theories (Abrahamson & Chase, 2015).

One of the interesting design conversations that we first encountered was the issue of perspective. We initially conjectured that we should intentionally design the microworld so that users had more visual clues. For example, we thought about showing a horizon line, thus the users' vantage point would be from the side. In this view, the arcs simulate a trace of the movement a giant could have taken between steps. We later realized that adding perspective introduced the common-sense assumption that giant steps or meters would appear smaller in the distance, an accurate assumption that adds considerable undo complications. Similarly, we tested the idea of establishing a birds-eye point of view with the use of footprints. The user would place footprints heel-to-toe in the 'sand' to model the giants path. While this very visual heel-to-toe feature is compelling, we realized that it became more complicated when participants modeled the second part of the narrative. In 'real life' the giant would have walked in his or her footprints the second day, therefore it would be difficult for participants to differentiate between Day 1 and Day 2. Since identifying and correcting the discrepancies between Day 1 and Day 2 is central to facilitating the discovery of the SILOs, this design consideration was abandoned.

Another feature of the current Giant Steps build that underwent discussion was the arc, either red or green, that signifies the space of either a giant step or a meter. The introduction of the arc is attributed to the origins of the number-line design, the work of Dickinson and Eade (2004). In one of the earlier builds of the microworld the arcs were programmed to have uniform heights. The result was that some arcs looked like semi-circles and some arcs looked like semi-ovals depending on how the arcs were modeled. The unintended outcome of this programming decision was that each modeled arc, whether a giant step or a meter, looked remarkably different, thus prompting the user to attempt to make them uniform. Because we wanted participants to construct transparency for this exact SILO, we opted to reprogram so that the arcs always appeared as semi-circles.

The final major design constraint that we explored was the blue line meant to indicate the giant's path. In our initial builds we did not program any indicators of directionality for the user. We considered adding a small compass to the toolbar, similar to a map, and adding directionality to the story. We concluded that this added another layer of cognitive complexity that we could not control for which may lead to unintended results. Therefore, we decided that in Level 1, the blue path line would appear as the user created his or her model, and would remain. This constraint ensured that users were not modeling an indirect path between the start and the treasure, and thus potentially calculating inadvertent irregularities.

Once the base functionalities of the microworld were in place, we began designing the automated features. Based on pilot data we knew that participants typically attended to SILO 1 and then SILO 2. Therefore, we introduced the automated meter and then the automated rescaling giant step. The automated meter, Level 2, generates meters that are of a fixed radius and can not be adjusted. This reflects the fact that a meter is of a specific size and should, therefore, also be so within the Giant Steps microworld. The automated-scaling giant steps feature, that simultaneously adjusts corresponding variables in and across both days uniformly, reflects the fact that all variables are uniform in size. Finally, the 'teacher dashboard' was developed so that the researcher could control when students transitioned between levels.

Inorder to test the efficacy of the reverse-scaffolding activity architecture, we created a separate microworld. As will be described in more detail, this baseline condition comprised of the same story narratives delivered in precisely the same sequence. The only difference was that the baseline condition had fully automated modeling tools from beginning to end.

3.5 The Research Questions

This dissertation seeks to evaluate the proposed reverse-scaffolding design architecture by comparing learning outcomes achieved by students across reverse-scaffolding- and baseline conditions of the GS4A activity. Beyond assessing for a main effect of the experimental intervention, the dissertation seeks also to understand the process by which reverse scaffolding mediates this would-be main effect.

The GS4A design offers a learning environment to address these instructional concerns within algebra. Additionally, GS4A is an opportunity to evaluate the "leveling transparency" framework—a framework that bears potential as a methodology to inform future designs of mathematics learning environments. Participants construct transparency through building the system of logico-qualitative relations using either the automated features of the baseline condition (BS), or the discovery-based progression of the experimental condition (RS). The hypothesis is that the discovery-based (RS) condition will allow for greater subjective transparency, and quality of learning.

The Reverse Scaffolding game mechanics are the activity-based implementation and concretization of Leveling Transparency. Therefore, finding the experimental (RS) condition to have better articulation of the logico-quantitative relational system, both in the activity and during the post-activity assessment questions, is an indirect evaluation of the Leveling Transparency rationale. The baseline is that the automated features of the baseline condition (BS) allow participants to interpret the relation between their input and the systems output; that the interface functionalities of the baseline condition support participants' solution strategies and, therefore, how participants construct transparency of the structural properties for constructing and maintaining relational equivalence within the GS4A interface and the subsequent post-activity assessment items.

In order to evaluate the design rationale – leveling transparency – and the pedagogical implementation of this rationale – reverse scaffolding – the following questions arise:

1. How effective is an environment that sequences learners' gradual construction of problem-solving tools?
2. To the extent that it is effective, what are the unique advantages of this environment?

3. Does the reverse-scaffolding design principle of “leveling transparency” offer theoretical traction on the learning processes it ostensibly enables?

In order to gather data that will eventually answer the above queries, this study has been designed so as to address the following research questions.

1. Does reverse scaffolding increase student outcomes, as measured by the results of the post-activity assessment?
    a. Composite scores from the New-Context items.
    b. Composite scores from the In-Context items.
    c. Composite scores from all items.

2. Does ‘leveling transparency’ for each structural property of the system of logico-quantitative relations, allow the participant to articulate thinking about the need for each structural feature?
    a. That units are consistent.
    b. That variables are uniform and variable.
    c. That expressions are the same.

3. What are the mechanisms or opportunities that each condition provides that could explain performance difference between the two groups?

CHAPTER 4: Methods

The study was conducted in the design-based approach to empirical research. This approach combines a framework for engineering products through iterative design cycles with a methodology for inferring generalizations for the science of learning (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Confrey, 2005; Edelson, 2002). The design process enabled me to explore the existing cognitive domain of algebra in search of potential explanations for why algebra is so difficult to access, and then use this to develop a conjecture that led to a proposed educational design, Giant Steps for Algebra. Through making sense of data gathered in the process of evaluating this design, I developed an *ontological innovation* that in turn fed into an iterated re-design. What is unique to the design-based approach is that the findings inform the development both of design frameworks and theories of learning. Therefore, a design-based research project stands to offer findings across multiple stakeholders.

The data were collected using a semi-structured task-based interview protocol (Clement, 2000; Ginsburg, 1997). The data were analyzed using microgenetic analysis (diSessa, 2007; Kuhn, 1995; Parnafes & diSessa, 2013; Siegler & Crowley, 1991).

4.1 Participants

The design was implemented with two groups of participants. The first group ranged in age from 8 – 10 years. This population was chosen because typical elementary school curriculum begins to introduce formal algebraic concepts around this age, i.e. 3rd or 4th grade. Consequently, the content was considered developmentally appropriate. However, while this population has some understanding of relational equivalence, they do not have formal algorithmic fluency for solving linear propositions. The second group ranged in age from 13-15 years. This population was chosen because students of this age are taking a 9th grade Algebra I class, and should already be very familiar with relational equivalence and fluent with the algorithm for solving algebraic equations (see Table 2).

Table 2. *Participants*

	Condition	
	Study	Control
Grade Level	Reverse Scaffolding	Baseline
4th grade	n = 11	n = 9
9th grade	n = 10	n = 10
Total	n = 21	n = 19

Within each age group, participants were randomly assigned to one of two conditions, RS and BS. To assure the similarity of study and control groups we ensured equal numbers of students in each group across ability levels, as measured by their teachers' report of mathematical ability. For the distribution of other demographic indicators across the study (Reverse Scaffolding) and control (Baseline) condition, see Table 3.

Table 3. Participants' Demographic Information

	4th Grade						9th Grade					
Indicator	Reverse Scaffolding			Baseline			Reverse Scaffolding			Baseline		
White		6			4			1			1	
African American		0			1			4			1	
Latina/o		3			0			0			2	
Multiple		3			2			5			3	
Asian		0			2			0			2	
American Indian		0			0			0			1	
SES	Free	Reduce	None	Free	Reduce	None	Free	Reduce	None	Free	Reduce	None
	1	0	10	0	0	9	7	2	1	7	1	2
Math Ability	Low	Med	High	Low	Med	High	Low	Med	High	Low	Med	High
	2	3	6	3	1	5	2	5	3	3	3	4

For the participants in 4th grade, there is some variance between two groups across ethnic identity, and Socio-Economic Status (SES) only varied by 1. For students in the 9th grade, there is some variance between groups across ethnic identity, and SES only varied by 1. This indicates that the groups were similar enough across demographic indicators to ensure that these indicators can not explain any variance in performance.

4.2 Materials

Each participant worked on the GS4A web-based activity progressing from question 1 to question 9. Table 4. describes the relationship between the two conditions, the activity, and the interface.

Table 4. Activity Sequence and Interface Functionality for Each Condition

Questions		Condition	
Question	Annotated narrative and algebraic expression.	Reverse Scaffolding	Baseline
1	Four giant steps forward. Then three giant steps forward and two meters forward. 4x = 3x + 2	All Manual	All Automatic
2	Three giant steps forward and three meters back. Two giant steps forward and then two meters forward 3x-3 =2x+2	All Manual	All Automatic
3	One giant steps forward and then eight meters. Two giant steps and then six meters. 1x+8=2x+6	All Manual	All Automatic
4	Three giant steps forward and then two more meters. Four giant steps forward and then one meter back. 3x+2=4x-1	Manual Giant Steps Automatic Meters	All Automatic
5	Three giant steps forward and then three meters back. One giant step forward and then one more meter. 3x-3=x+1	Manual Giant Steps Automatic Meters	All Automatic
6	Two meters forward, then two giant steps forward, then three meters forward. One giant steps forward, then one meter forward, then two giant steps forward, then one meter forward. 2+2x+3=x+1+2x+1 (2x+5=3x+2)	Manual Giant Steps Automatic Meters	All Automatic
7	Five meters forward and three giants steps back. Two giant steps forward. 5-3x=2x	All Automatic	All Automatic
8	Two giant steps forward and four meters back. The one giant steps forward and three meters back. 2x-4=x-3	All Automatic	All Automatic
9	Three meters forward, then two giant steps, then four meters. Then two meters and three giants steps. 3+2x+4=2+3x	All Automatic	All Automatic

After completing the GS4A questions, all participants were asked a series of Post Activity Assessment items.

These Post Activity Assessment items were designed to measure the participants' subjective transparency for the structural properties of the problem. The whole post-assessment activity consisted of 5 items. These were broken up into two categories; New-Context problems, in which I measured for the application of learned skills (transfer); and (b) In-Context problems that targeted the three SILOs directly within the familiar GS4A setting.

The New- Context problems (see Figure 14 and Figure 15) enabled the researcher to determine whether the participant could utilize the SILOs as construction strategies for establishing relational equivalence within new narrative contexts.

Two buildings are built next to each other and are exactly the same height. One building is 10 floors and has a spire that is 20 feet on top of it. The other building is 11 floors and has a spire that is 10 feet on top of it.
How tall are the buildings?
What do you need to do to figure this problem out?

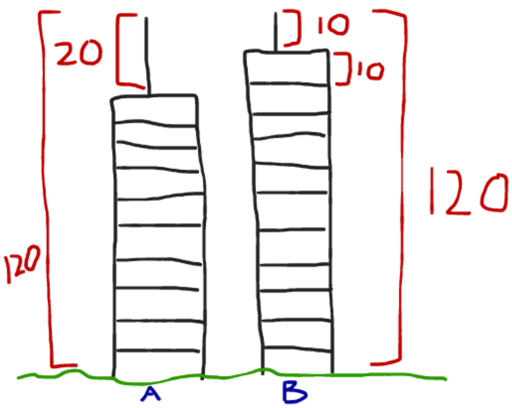


Figure 14. Two Buildings problem.

In the two-buildings problem participants were presented with the narrative. (The image on the right is the researcher's sketch of the two-buildings problem. It is included here only for clarification only and was not presented to the participants.) In this new situation the unknown magnitude was now floors, instead of giant steps. Additionally, in the two-buildings problem the logical model was vertical, while in GS4A it was horizontal.

The two-buildings problem allowed the researcher to evaluate the extent to which the participants understood the GS4A tools' (virtually) mechanical function in establishing the mathematical relationships for relational equivalence. The two-buildings problem was designed to measure whether participants were able to apply the SILOs that they had previously developed, in a new context. Namely it measured whether participants could identify and articulate the structural properties that establish relational equivalence in this new context.

Two Building SILOs

1. Units are consistent, in the two-building problem these are the height of the spires given in feet.
2. Variables are uniform and variable, in the two-buildings problem these are the floors.
3. The end points are the same, in the two-buildings problem that they are the same height.

The two-building problem allows the researcher to understand whether the participants have constructed transparency for the (virtual) mechanical constructions, comparing across conditions, RQ 1, RQ 2.

My turtle, named Yurtle, is being tricky and won't tell me how old she is.
Help me figure out how old she is in human years.

Yesterday she told me that she has lived 3 turtle years and 2 human years. She also told her 4th turtle year will begin in 3 human years.

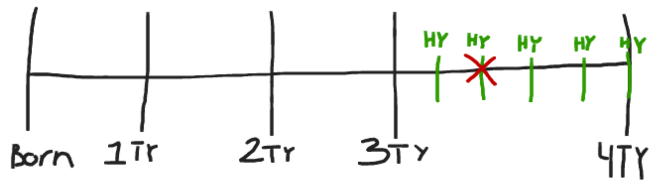


Figure 15. The Turtle Years problem.

In the turtle years problem participants were presented with the narrative and asked the included questions. (The image on the right is the researcher's sketch of the turtle years problem, included here only for clarification and is not presented to the participants). The turtle years problem provided no existing representational framework to draw from, whereas the two-building problem lends itself to an existing representational arsenal, even if participants self identify as having limited drawing skills. The turtle years problem allowed the researcher to evaluate the design rationale by highlighting the extent to which the participants understood the GS4A tools' (virtually) mechanical function in establishing the mathematical relationships for relational equivalence. The turtle problem asks the participants to identify and articulate the structural features that establish relational equivalence.

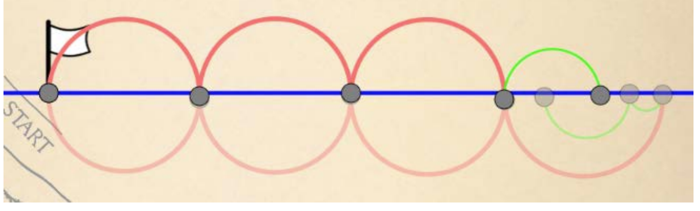
Turtle Years SILOs

1. Units are consistent, in the turtle years problem these are human years.
2. Variables are uniform and variable, in the turtle years problem these are turtle years.
3. The end points are the same, in the turtle years problem this is a point in time, namely present day.

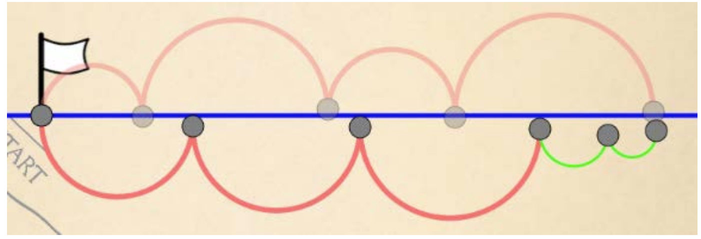
The turtle years problem allowed the researcher to understand whether the participants have constructed transparency for the (virtual) mechanical constructions, comparing across conditions, RQ 1, RQ 2.

The In-Context post-activity assessment items consisted of a series of screenshots taken from the GS4A activity. These screenshots (see Figure 16) were solutions that were incorrect and violate at least 1 of the SILOs. The participant was asked to correct the item using paper and pencil. The In-Context problem set was designed to enable the researcher to identify whether the participants could articulate how the interface's features contribute to successful problem-solving.

In this screenshot a hypothetical user created inconsistent meters. The participant must: (a) identify the user's error; (b) redraw the scenario with consistent meters; and (c) determine the treasure's location.



In this screenshot a hypothetical user created inconsistent giant steps (variables). The participant must: (a) identify the error; (b) redraw the scenario with consistent giant steps; and (c) determine the treasure's location.



In this screenshot a hypothetical user did not represent equivalence. The participant must: (a) identify this error; (b) redraw the scenario with matching end points; and (c) determine the treasure's location.

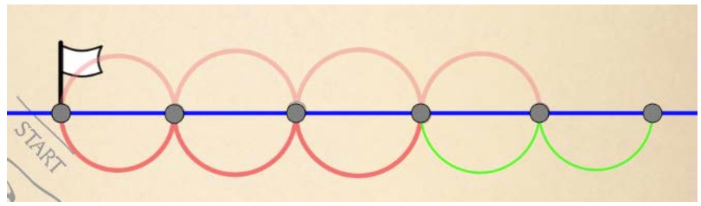


Figure 16. In-Context post-activity assessment items.

Each In-Context problem violated at least one of the three tacit axioms, namely the SILOs. The participant was asked to identify which SILO has been violated and to correct the violation. Being able to correctly isolate the function that each SILO plays in constructing equivalence, demonstrated that the user has constructed transparency for this SILO, RQ 1, RQ 2.

4.3 Procedures

The researcher engaged all participants in problem-solving sessions using a semi-structured task-based clinical interview protocol (see Appendix A). The participants worked individually during these problem-solving sessions. The interviews took place at school but outside of the math classroom, in a quiet room. During both the GS4A activity and the post-activity assessment, the researcher–interviewer interacted with the participant much as a mathematics tutor would who is informed by reform-oriented pedagogical approaches (Ginsburg, 1997). Before the participants began, the researcher–interviewer demonstrated all the suite of tools. The researcher–interviewer had the participant read the first half of question 1. Then the researcher-interviewer demonstrated how the tools could model a giant step, a meter, how it could switch between day 1 and day 2, and how the 'undo' and 'clear' functionalities work. Then the researcher–interviewer asked if there were any questions, and finally turned the mouse over to the participant.

4.4 Data collection

All interviews were video-taped for data collection purposes, and a screencasts were saved. After participants had completed all 9 questions in the GS4A activity, see Table 4., they were asked the post-activity assessment items. The complete data corpus consisted of approximately 50 hours of videography, with each participant's individual interview and post-assessment lasting between 50-70 minutes.

4.5 Data analysis

The evaluation phase of the study consisted of assessment items designed to measure participants' subjective transparency of the microworld and, as such, the proto-formal conceptual system they had developed for algebra. Using methods of interaction analysis (Jordan & Henderson, 1995) I began to identify patterns across the data corpus. Interaction analysts operate under the assumption that what a person knows and how they behave is socially constructed, and embedded in this social ecology. Therefore, measuring a participant's understanding should include any meaningful actions that the participant has engaged in as evidence of situated knowledge. Interaction analysis methodologies move beyond only coding a participant's utterances or written responses to survey questions by also coding all interactions that participants have with their environment. Using the video and screencast data, I was able to code participants interactions with and within the Giant Steps interface. Using grounded theory (Strauss & Corbin, 1990), I began to formulate a theory about how the SILOs can capture emerging understanding. Through iterations of data analysis, using both pilot data and early interviews from the data reported on here in, I created a coding scheme for measuring participants' achievement of each SILO. Thus the SILOs served as scoring criteria for measuring individual students' achievement on the post-activity assessments, with each SILO further graded into achievement levels. Two researchers scored participants' responses based on demonstration of the SILOs.

A scale of 0 – 4 was used to score participants' work on the Turtle Years and Two Buildings problems: a score of 4 marks our judgment that the participant had demonstrated all three SILOs and determined a correct solution; and a score of 0 marks our judgment that the participant had not demonstrated any of the three SILOs and had not determined a correct solution (see Table 5 and Table 6)

Table 5. Scoring Criteria for the Turtle Years Problem

Measure		Score
SILO	Applied to Turtle Years problem	
Consistent measures. All variable units (giant steps) and all fixed units (meters) are respectively uniform in size both within and between expressions (days).	*Consistent measures*. All variable units [Turtle Years (TYs)] and all fixed units [Human Years (HYs)] are respectively uniform in size both within and between expressions (both ways that age is expressed).	1

Equivalent expressions. The two expressions (Day 1 and Day 2) are of identical magnitude—they share the "start" and the "end" points, so that they subtend precisely the same linear extent (even if the total distances traveled differ between days, e.g. when a giant oversteps and then goes back).	*Equivalent expressions.* The two expressions (Age 1. 3TYs + 2HYs and Age 2. 4TYs – 3HYs) are of identical magnitude—they share the "start" and the "end" points, so that they subtend precisely the same linear extent, the same temporal length.	1
Shared frame of reference. The variable quantity (giant steps) can be described in terms of the unit quantity (meters).	*Shared frame of reference.* The variable quantity [Turtle Year (TYs)] can be described in terms of the unit quantity [Human Years (HYs)].	1
Correct Solution	*Yurtle is 17 Human Years old*	1

Table 6. Scoring Criteria for the Two Buildings Problem

Measure		Score
SILO	Applied to Two Building problem	
Consistent measures. All variable units (giant steps) and all fixed units (meters) are respectively uniform in size both within and between expressions (days).	*Consistent measures.* All variable units (Floors) and all fixed units (Feet) are respectively uniform in size both within and between expressions (buildings).	1
Equivalent expressions. The two expressions (Day 1 and Day 2) are of identical magnitude—they share the "start" and the "end" points, so that they subtend precisely the same linear extent (even if the total distances traveled differ between days, e.g. when a giant oversteps and then goes back).	*Equivalent expressions.* The two expressions (Building A and Building B) are of identical magnitude—they share the "start" and the "end" points, so that they subtend precisely the same linear extent, the same vertical length.	1

Shared frame of reference. The variable quantity (giant steps) can be described in terms of the unit quantity (meters).	*Shared frame of reference.* The variable quantity (Floors) can be described in terms of the unit quantity (Feet).	1
Correct Solution	*Both buildings are 120 feet tall*	1

For the second group of assessment items, the In-Context items, we used a scale of 0 – 3. A score of 3 marks our judgment that the participant both corrected the compromised SILOs and determined a correct solution in meters, whereas a score of 0 marks our judgment that the participant neither amended the SILOs nor determined a correct solution in meters. See Table 7Table 8Table 9.

Table 7. *Question 1 Scoring Rubric*

Criteria	Score
Participant does not determine that the meters are not represented uniformly, does not determine a correct solution.	0 points.
Participant realizes that the ends are not aligned, either verbally or through redrawing it.	1 point (a).
Participant realizes that the meters are not represented uniformly, either verbally or through redrawing them.	1 point (b).
Participant realizes that the ends are not aligned, and that the meters are not represented uniformly, either verbally or through redrawing it. BUT DOES NOT CALCULATE FINAL SOLUTION IN METERS	2 points.
Participant realizes that the meters are not represented uniformly and that ends are not aligned, and determines a correct solution in meters (i.e. how far the treasure is buried from the start).	3 points.

Table 8. *Question 2 Scoring Rubric*

Criteria	Score
Participant does not determine that the Giant Steps are not represented uniformly, or thinks that it is not problematic.	0 points.
Participant realizes that the Giant Steps are not represented uniformly, either verbally or through redrawing them, but does not attend to a Shared Frame of Reference.	1 point (a).
Participant realizes that the Shared Frame of Reference is not represent, either verbally or through redrawing them, but does not attend to the Giant Steps.	1 point (b).
Participant realizes that the Giant Steps are not represented uniformly, that there is a shared frame of reference (1GS=2M). BUT DOES NOT CALCULATE A SOLUTION.	2 points.
Participant realizes that the Giant Steps are not represented uniformly, that there is a shared frame of reference (1GS=2M) and determines a correct solution in meters (i.e. how far the treasure is buried from the start).	3 points.

Table 9. *Question 3 Scoring Rubric*

Criteria	Score
Participant does not realize that the ends are not aligned, or does not think that it is problematic.	0 points.
Participant realizes that the meter/Giant Step ratio is incorrect (SILO 3), either verbally or through redrawing them.	1 point (a).
Participant realizes that the ends are not aligned (SILO 2).	1 point (b).
Participant realizes that the ends are not aligned (SILO 2), that the meter/Giant Step ratio is incorrect (SILO 3) but does NOT CALCULATE A CORRECT SOLUTION.	2 points.
Participant realizes that the ends are not aligned (SILO 2), that the meter/Giant Step ratio is incorrect (SILO 3) and determines a correct solution in meters (i.e. how far the treasure is buried from the start).	3 points.

After one analyst had coded all the post-activity assessment problems, a second analyst independently scored 21% of this data corpus. Results from an inter-rater reliability test were Kappa = 0.822 ($p < 0.001$), 95% CI (0.646, 0.998), almost perfect agreement. Subsequent data analysis consisted of first evaluating for a main effect of the intervention by comparing post-intervention achievements of the experimental and control groups using SPSS. Once I had determined a main effect and wished better to understand *how* the experimental condition led to higher achievement, I further performed qualitative micro-genetic analysis.

CHAPTER 5: Results

This chapter reports on findings and inferences from the dissertation’s empirical effort. Reporting on results of mixed-methods analyses, I detail the effect of the two experimental conditions—Reverse-Scaffolding and Baseline—on the participants’ output, and I compare these effects. I begin with results from quantitative-analysis comparison of post-activity assessment responses across conditions and age groups and, in so doing, discern patterns and trends in the data. In the subsequent quantitative-analysis sections, I present results of the statistical tests performed on the data. Lastly, looking closely at selected case studies, I employ qualitative approaches to further interpret the mechanisms that could explain differences between the two conditions.

5.1 Quantitative Analysis

5.1.1 Patterns across conditions.

The scores for each question were grouped by frequency in order to determine patterns across participants and conditions. The scores were separated by grade level. I began by identifying patterns across the New-Context assessment items. Figure 17 presents the results of the Turtle Years New-Context problem. In the fourth grade participants in the Reverse Scaffolding (RS) group generally performed better. Two of the participants scored a 3 and six participants scored a 4. While in the ninth grade, three participants from each group received a score of 4. Yet, a majority of the RS participants received a score of 2 and the majority if the Baseline (BS) participants received a score of 1 or 2.

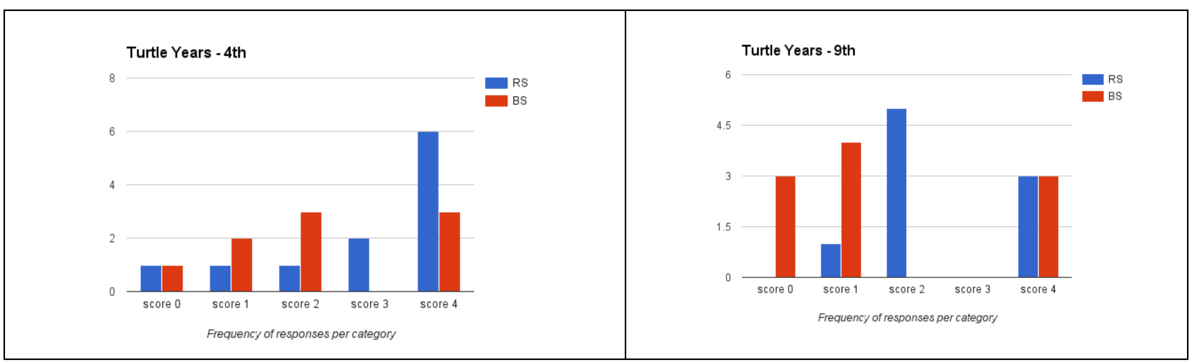


Figure 17. Frequency of responses for Turtle Years problem.

Figure 18 presents the results of the Two Buildings New-Context problem. For the participants in the fourth grade the majority of responses for both the RS and the BS groups received a score of 4, indicating that this questions was much more transparent to all. However, the RS group had more participants who received a score of 4, eight as compared to five. For the participants in the 9^{th} grade, the RS group had majority of scores either of 4 or 1. The BS group received scores across all levels with a majority receiving a score of either a 2 or a 0.

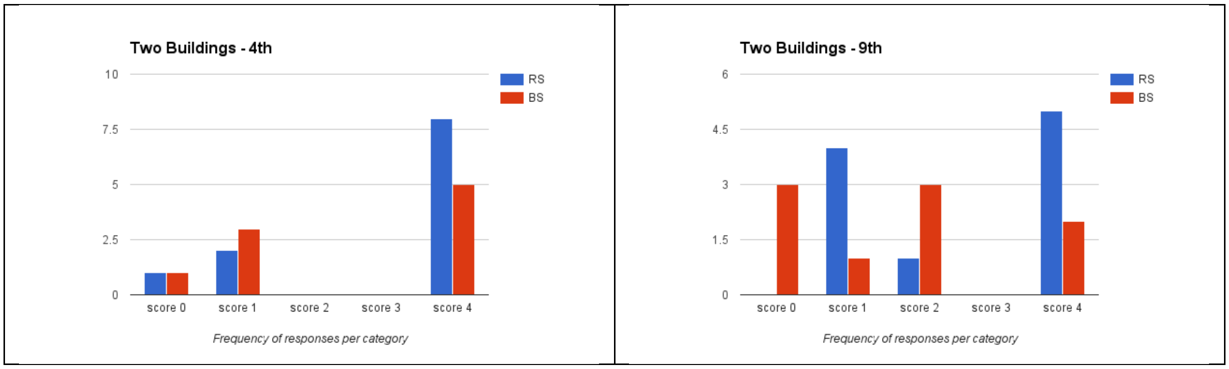


Figure 18. Frequency of responses Two Buildings problem.

Generally, for the participants in the fourth grade, across both New-Context questions, there is a concentration of receiving a score of 4, and the patterns of response frequency were similar between the RS and the BS conditions. Generally, for the participants in the ninth grade, across both New-Context questions, there is a concentration of receiving a score of 2 or less, and the pattern of response frequency were dissimilar between the RS and the BS conditions.

Using ANOVA to test the distribution of scores for the composite score for New-Context items, I determined that the distribution was not normal between the study and control groups. This can often be the case with a small sample size, therefore, I used a non-parametric ANOVA in further analyses.

Secondly, I wanted to compare the frequency of responses for each of the In-Context problems. Figure 19 presents the results for In-Context question 1. For participants in the fourth grade the concentration of scores received, comparing the RS and the BS groups, was fairly similar: the RS group received a higher number of scores in the 1(b) category. For participants in the ninth grade, the RS group received scores that were across categories 2 and 3, while the BS group received the highest number of scores in the 1(a) category.

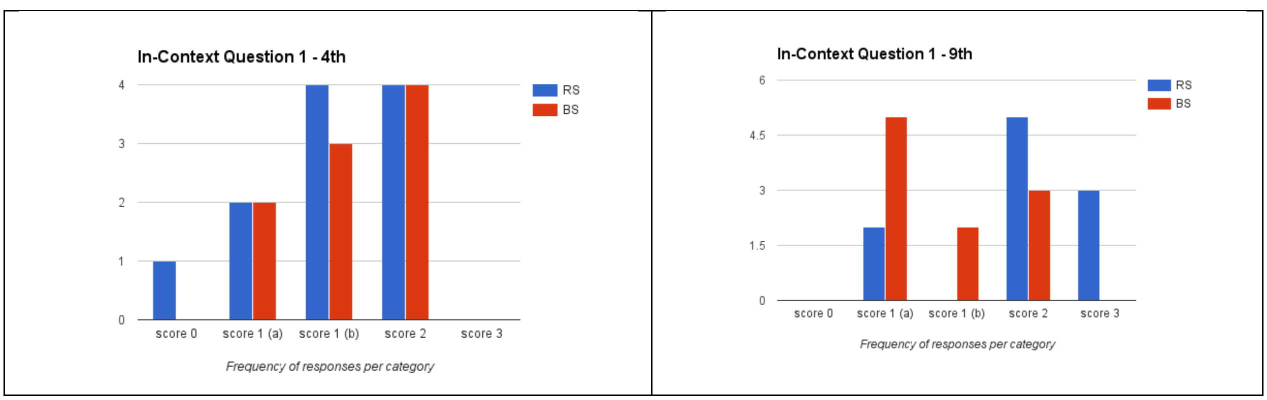


Figure 19. Frequency of responses In-Context Question 1.

Figure 20 presents the frequency of responses for In-Context question 2. For participants in the fourth grade, comparing the RS and BS groups, the RS group received a higher number of scores in the 1(a) and 3 categories. For the participants in the ninth grade, comparing the RS and BS groups, the RS group received the highest number of scores in the 3 category, while the BS group received the highest number of scores in the 1(a) category.

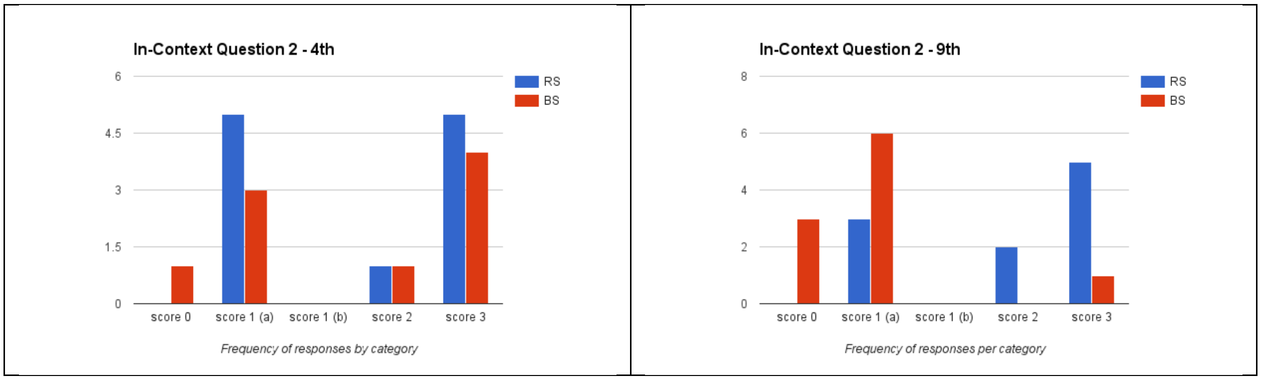


Figure 20. Frequency of responses for In-Context Question 2.

Figure 21 presents the frequency of responses for In-Context question 3. For participants in the fourth grade, comparing the RS and BS groups, the RS group received a majority of scores in the 2 category, while the BS group received a majority of scores spread across the 2 and 3 categories. For participants in the ninth grade, comparing the RS and BS groups, the RS group received the highest number of scores in the 2 category and the BS group received the highest number of scores in the 2 category.

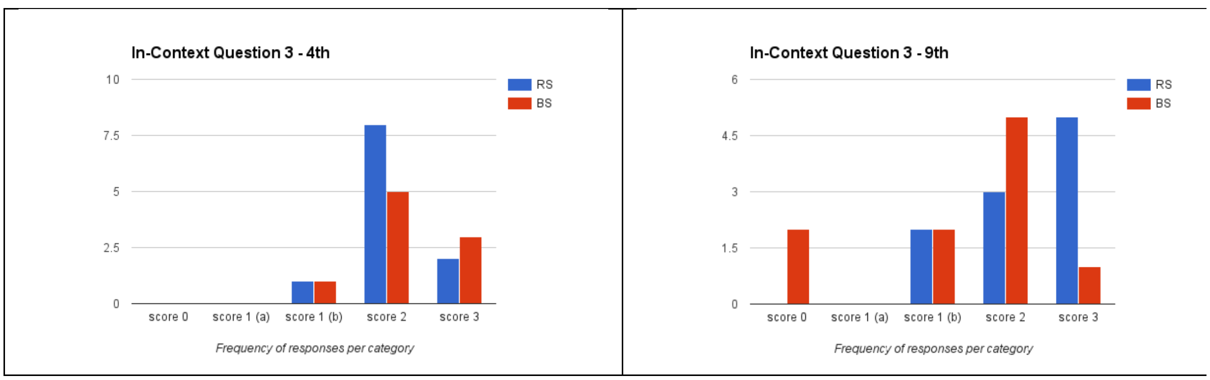


Figure 21. Frequency of responses for In-Context Question 3.

The pattern across all In-Context questions for participants in the fourth grade indicates that the RS and BS groups tended to share high or low frequencies of scores in the same categories, with the exception of Question 3. The pattern across all In-Context questions for participants in the ninth grade indicates that the frequency of scores for the RS and BS groups tended to be concentrated in different categories for each question.

Using ANOVA, I tested the variance between the results on the post-activity assessment items across conditions, see Table 10. On the New-Context items, the reverse-scaffolding (RS) experimental group (M=5.17, SD=2.34) scored significantly higher than the Baseline (BS) control group (M=4.10, SD=2.76); t (38)=1.98; p=0.02. On the In-Context tasks, RS (M=5.88, SD=2.10) scored significantly higher than BS (M=4.60, SD=1.90); t (38) = 2.00; p = 0.02. Combining results from both post-activity assessment item categories for a Total score, RS (M=11.59, SD=3.57) scored significantly higher than the BS (M=8.71, SD=3.81); t (38) = 2.46; $p < 0.01$

Table 10. Post-Activity Assessment Results

	Reverse Scaffolding		Direct Scaffolding	
	Mean	S.D.	Mean	S.D.
New-Context*	5.17	2.34	4.10	2.76
In-Context*	5.88	2.10	4.60	1.90
Total**	11.59	3.57	8.71	3.81

* t (38)=1.98; p=0.02

** t (38) = 2.46; p < 0.01

However, using ANOVA to test the distribution of the composite score for the In-Context items I determined that this distribution was not normal between the study and control groups. Therefore I used a non-parametric ANCOVA in further analyses.

In order to determine the overall main effect a composite score for all post-activity assessment item was also calculated, I call this dependent variable Total. Using ANOVA to test the distribution of Total across the study and control condition I determined that these scores were normally distributed. Therefore I used ANCOVA in further analyses.

5.2 ANCOVA Results for Reverse-Scaffolding Versus Baseline Conditions

In order to evaluate the experimental instructional methodology, I wished to compare the post-intervention competence of participants in the study condition (RS—reverse scaffolding) and control condition (BS—baseline). I expected to receive a positive difference indicating greater mean learning for the experimental condition as compared to the control condition.

Before performing the further analyses I also checked whether the three dependent variables, TOTAL, New-Context items, and In-Context items, were or were not significantly different across reported math ability levels. TOTAL was significantly different among math ability levels. New-Context was marginally significantly different among math ability levels. In-Context was significantly different among math ability levels. Given that I want to ensure that any differences between how the experimental and control groups performed can only be explained by the intervention, in further analyses I also controlled for math ability levels.

For dependent variable TOTAL, controlling for math ability levels the study condition scored significantly higher than the control condition R = 0.271, p= 0.015.

For the dependent variable New-Context, controlling for math ability levels the study condition scored significantly higher than the control condition, R = 0.187, p = 0.053.

For the dependent variable In-Context, controlling for math ability levels the study condition scored marginally significantly higher than the control condition, R = 0.148, p = 0.055.

Table 11. *ANCOVA Results*

	df	f	Adjusted R^2	p
New-Context	1	3.99	0.19	0.05
In-Context	1	3.92	0.15	0.05
TOTAL	1	6.57	0.27	0.01

These results address RQ 1, which was:

Does reverse scaffolding increase student outcomes, as measured by the results of the post-activity assessment?

1. Composite scores from the New-Context items.
2. Composite scores from the In-Context items.
3. Composite scores from all items.

The results clearly demonstrate that participants in the reverse scaffolding condition achieved better results on the post-activity assessment items. These results were persistent across all three post-assessment categories. I can conclude that reverse scaffolding does increase student outcomes.

These results also address RQ 2, indicating that 'leveling transparency' enables participants to better articulate the structural properties of the model. In particular, the evidence gathered from the In-Context post-assessment items demonstrates that the participants who received the 'leveling' activity architecture better articulated the importance of each structural property.

Does 'leveling transparency' for each structural property of the system of logico-quantitative relations, allow the participant to articulate thinking about the need for each structural feature?:

1. That units are consistent.
2. That variables are uniform and variable.
3. That expressions are the same.

5.3 Discussion

I am heartened by these results for several reasons. To begin with, the positive measured difference between study and control participants on the challenging transfer problems suggests the potential to further demonstrate a statistically significant effect given greater power, such as by working with more participants. Yet perhaps far more important, I view as an encouraging result the fact that at the very least experimental implementation of an instructional activity based on the innovative reverse-scaffolding condition resulted in learning gains that are as large, if not greater, than the baseline condition. These findings should urge researchers to question our implicit assumptions regarding best pedagogical practices for supporting mathematics content learning: perhaps, counter to the wisdom of the ages, direct scaffolding is not necessary. At the least, the findings suggest that more research should be conducted to develop and evaluate the reverse-scaffolding instructional methodology. These further studies could continue to investigate the hypothetical construct of SILO so as better to understand its mediating psychological effect on student performance on mathematical tasks as well as implications of this effect for the design of instruction and assessment.

Despite these encouraging results RQ3 remains unexplained.

What are the mechanisms or opportunities that each condition provides that could explain performance difference between the two groups?

In order to better understand the nuances of how each condition contributed to how students performed, I proceeded to perform qualitative analysis of the data corpus. Using micro-genetic analysis, I hoped to be able to pinpoint subtle differences between the actions and utterances of participants that might help me characterize the cognitive affordances and tradeoffs of each condition's activity architecture.

5.4 Qualitative Results: Emergence of the SILOs

The vignettes in this sub-section are organized as matched pairs, with compatible reverse-scaffolding (RS) and Baseline (BS) study participants juxtaposed so as to bring out critical differences. I begin by featuring vignettes of two 4th-grade participants, an RS participant and a BS participant, both rated by their teacher as having "high" mathematical abilities.

Susan (all names are pseudonyms) is working in the RS condition (study group). She is at Level #1, working on an informal narrative corresponding to the formal proposition "4*x* = 3*x* + 2." She has completed the Day 2 travel diagram (see in Figure 22 the four red loops above the horizontal line) and is now working on the Day 2 travel diagram below the line.

Res.: Ok. So she goes…
Susan: 3 giant steps and…..
Res.: …and then....
Susan: 2 meters. (Susan switches an interface feature to "meters" and draws below the line 2 equivalent meters that subtend the 4th giant step immediately above the line.)
Res.: So she goes 2 meters and then she finds the right spot.
Susan: Yeah
Res.: So in your drawing did she find the right spot?
Susan: Hmmm well yeah.

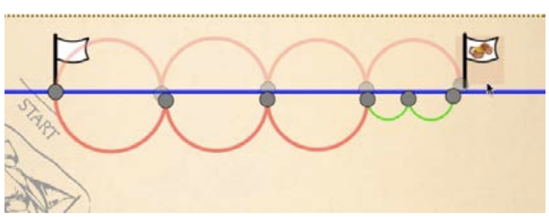


Figure 22. Susan's construction for a Giants Steps story corresponding to the algebra proposition "4x = 3x + 2."

Immediately, Susan has identified that the end point for both days is in the same screen location (see the treasure flag in Figure 22, on the right) and that, consequently, the 2 meters on Day 2 will subtend the same distance as the 4^{th} giant step on Day 1. Despite some imprecision in her modeling execution, for example the meters are not of precisely the same screen size, Susan has constructed the transparency of equivalent expressions (SILO 2).

I now turn to Karrie, a participant in the DS condition (control group), who is working on the same item (see Figure 23).

Karrie: It says she walks 2 steps further ahead and finds the treasure. But that doesn't make sense because it is more back than the other treasure. (Karrie has drawn a model in which the giant steps are too large, so that the respective ends of Day 1 and Day 2 are not co-located.)

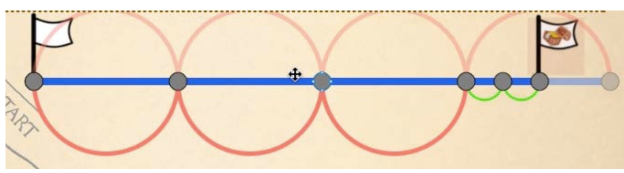


Figure 23. Karrie's construction for a Giant Steps story corresponding to the algebra problem "4x = 3x + 2."

Concerned by this misalignment between the end points of Days 1 and 2, Karrie suggests inserting additional meters. The interviewer responds by stating that doing so would violate the information in the story. The conversation ensues as follows.

Karrie: I can change the size of the giant steps.

Pursuing on her new idea, Karrie attempts to stretch the Day 2 travel diagram toward the right so that it reach the treasure flag. Specifically, she stretches the giant steps in Day 2 (the red loops below the blue line). Recall that in the direct-scaffolding condition the variable distances (all the red loops) are automatically interlinked, both within- and between days. Consequently, the variables in both Day 2 *and* Day 1 all stretched uniformly, and the two misaligned ends only became farther apart! Karrie then attempted the same maneuver by decreasing the step size in Day 1, but she stopped before the two ends met.

Karrie: It moves the whole thing?

Karrie was surprised to witness the automated-scaling feature that simultaneously adjusts corresponding variables in and across both days uniformly. Karrie had had meant to equalize the linear extents of the two days by first adjusting Day 2 and only then Day 1. Thus whereas Karrie was demonstrating SILO 3, equivalent expressions, she was doing so with disregard to SILO 1, consistent measures. Moreover, Karrie did not appear to appreciate the implication of uniform variable size for the fidelity of her story model. To Karrie, this feature is not transparent.

We now turn to our second comparison, two 9^{th}-grade participants, both rated by their teacher as having "medium" mathematical abilities. Taylor is working in the RS condition. He is at Level #2, working on the narrative corresponding to the formal proposition "$2 + 2x + 3 = x + 1 + 2x + 1$." He has just begun reading the problem.

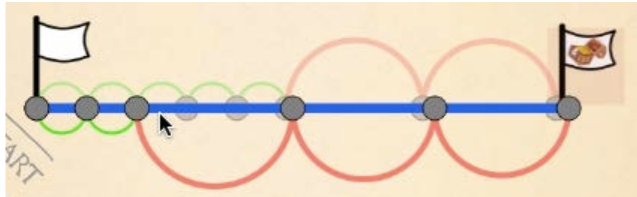


Figure 24. Taylor's completed model with the components reorganized for simplicity.

Taylor: Ok. Two meters, (begins by drawing 5 meters, see Figure 24).
Res: Wait, what did you do?
Taylor: I put all the meters first. ‘Cause, like, they’re all going to go to the same place. (Taylor performs a sweeping hand gesture from left, the start, to right, the treasure location; see Figure 25). It doesn’t really matter, the order.

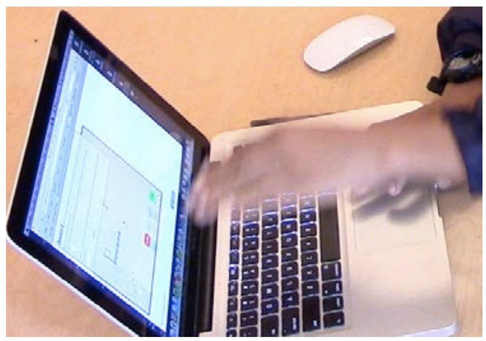


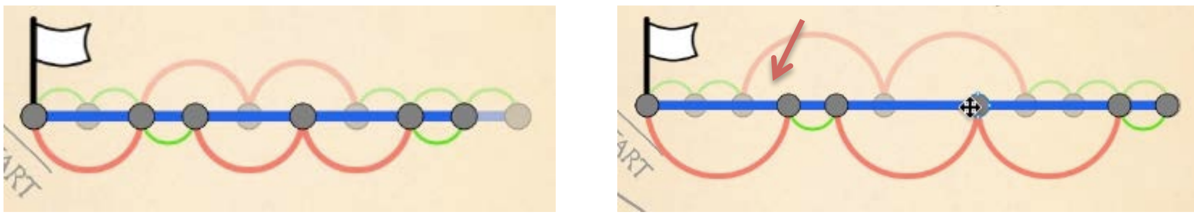
Figure 25. Taylor moves his hand from the start (left side of the screen) to the treasure (right side of the screen).

Taylor’s insight captures an auxiliary objective for the design of this particular problem item. We intentionally created this item so as to foster an opportunity for participants to combine units (meters) and variables (giant steps). And yet the participants would need to be motivated to discover the utility of such combining. By stating that the order of distances traversed (the situated addends) does not change the final destination (the sum) Taylor is taking advantage of the commutative property of number, as instantiated in the form of a string of concatenated segments, to create a model that better utilizes the number-line solution form. Taylor clearly demonstrates that he has a flexible understanding of the model he is creating and has achieved all of the SILOs. In fact, Taylor is thoughtful in discussing how he can utilize this know-how so as to improve and interpret his model. His flexibility is reflected in his scores on the New-Context post-intervention assessment, where he received a score of 6 out of 8.

I now turn to Irene, a BS participant working on the same problem as Taylor. Recall that the BS condition automatically generates fixed meters and automatically rescales all of the giant steps for the participant. Irene has just completed her model of the story narrative and realizes that the ends are not aligned (see Figure 26a).

Irene: Umm, So you need to make it bigger (she stretches the model so that the ends meet, see Figure 26b). There.
Res: Ok, so now they meet?

Irene: And I think each step is worth, ... not *worth* exactly (Irene appears to be groping for the word “equivalent”) ...3 meters?
Res: hmmm
Irene: So….
Res: What makes you say that?
Irene: Like first on day 1 (switches the interface so that Day 1 is highlighted) it is 2 meters (scrolls over the first giant step on Day 2, which corresponds to 2 meters and a gap on Day 1). And I think if you had 1 more meter it would be 3 (scrolls over the gap, see Figure 26b – *After stretching*).



a. Before stretching ***b. After stretching***

Figure 26. Irene's model before and after stretching.

Irene has achieved SILO 3, “shared frame of reference,” as observed through her actions to align the ends of her Day 1 and Day 2 models in an attempt to determine how many meters make up one giant step. However, when it comes to determining a solution, Irene warrants her claims based on available visual information rather than the narrative information. She states, “I think if you had 1 more meter,” yet she does not cross-check with the situation narrative. Furthermore, Irene’s solution strategies do not exemplify the same level of sophistication and flexibility as her classmate Taylor. The results of this lack of flexibility are reflected in her scores on the New-Context post-activity question where she received a total score of only 3 out of 8.

Whereas there are moments in Irene’s intervention that indicate her thoughtfulness, this thoughtfulness was not apparent later in the New-Context post-intervention assessment. The baseline task-flow architecture of the intervention had enabled Irene to develop an effective yet inflexible and non-transferable modeling routine: (a) model each of the two Day narratives, respectively above and below the line; (b) stretch or shrink one or both day diagrams until the ends meet; and (c) calculate the meter value of a step. Importantly, the uniform stretching/shrinking of the variable quantity was a given automatic feature of the interaction. Irene never had to discover, challenge, or monitor this feature, and so this feature remained opaque—the feature did not appear to be grounded in any insight on the modeling system as relating to the narrative situation.

Note that we are not critiquing Irene. Her reasoning was logical, rational, and consistent. Rather, we underscore that Irene’s reasoning was bound the particular contexts whence it developed. Irene’s hands-on problem-solving algorithm appears markedly different from the varied strategies Taylor employed. In the reverse-scaffolding task-flow architecture (gradual automatization) the user must modify the solution algorithm with the introduction of each new level. Doing so, I believe, offers the user opportunities to devise new and adaptive forms of manipulating the model’s structural features as well as opportunities to interpret the emerging structural systems from multiple perspectives. Thus the user develops subjective transparency of the modeling system by exercising flexible visualization and manipulation. In particular, the user devises new operatory schemes that become articulated as the SILOs.

Consider the case of Taylor. Recall that the task-flow change from Level 2 to Level 3 introduces the automatization of uniform Giant Steps. The moment this feature was enabled, Taylor recognized its utility, exclaiming, "Oh, I need that!" He immediately knew how this new control would function, understanding that it would generate and maintain consistent yet automatically scalable giant steps (SILO 1). In turn, the transparency of this Level 3 utility enabled Taylor to instantiate SILO 3, the shared frame of reference between variable and known quantities.

Now compare Taylor's case to that of Irene. During the post-intervention In-Context assessment, Irene is asked to interpret and possibly fix an incorrect model created by a hypothetical participant (see Figure 27).

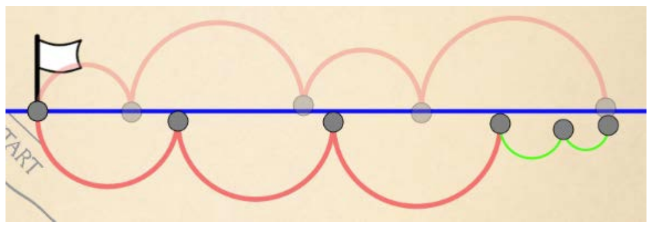


Figure 27. In-Context Question 2.

Irene correctly identifies that the giant steps and the meters in this item are each modeled as non-uniform. In response, she wishes to enact her three-step solution strategy—model, stretch/shrink, calculate—so as to amend the apparent irregularity. Irene makes the following suggestions: (a) Pointing to the end nodes on the right that are not perfectly aligned, she says, "The giant steps on the bottom should be moved (toward the left) so that the ends (on the right) meet"; (b) "They should put big meters in big giant steps, and small meters in small giant steps…"; and finally (c) "…so that a giant step is 3 meters." Irene's suggestions for fixing the model intimate that she does not view the interaction affordances as instantiating critical features of an emerging conceptual system. At no point in her proposed solution does she directly address the non-uniform size of the giant steps. Her second solution step violates SILO 1, consistent meters. Her last solution step, while adhering to SILO 3, shared frame of reference, is incorrect.

Unlike Taylor, who expressly predicted the interface's affordances for the modeling the problem situation, Irene never wondered about the interface's action capabilities that were present in the construction of the model, namely that the interaction was manual. Irene is process oriented—she has developed an effective protocol for solving a particular class of problems under particular interaction conditions, and yet she never had to *will* those interactions and then acknowledge their arrival. She cannot appreciate how the model maintains or violates the SILOs, because the model's functions are opaque to her.

5.5 Emergence of the SILOs via Guided Problem-Solving Interaction with Educational Technology

Qualitative analysis of the videotapes is enabling the research team to develop deeper understandings of the experimental activity. In this subsection we present excerpts from two contrasting sample interviews, one with Lucy (study group: reverse scaffolding), and one with

Mary (control group: direct scaffolding). (Both are pseudonyms.) These samples are arguably comparable in that the participants were of similar age (9[9], 9[11]) and both were considered by their teacher as on the high end of proficiency. Furthermore, these students scored similarly on the post-activity assessment questions, with Lucy scoring a total of 15 points, and Mary a total of 14.

Based on these analyses we cautiously claim that the experimental-condition students were more likely to develop subjective transparency of the emerging conceptual system as compared to the control-group students, because the reverse-scaffolding condition created for them greater need to figure out and articulate for themselves the fundamental principles of effective mathematical models for the problem scenarios.

5.5.1 Emergence of SILO 1 “consistent measures,” the case of the unit “meters.”

Lucy (reverse-scaffolding study condition, RS) is working within the instructional phase on Question 2. This item is composed of a Day-1-and-Day-2 narrative corresponding to the formal proposition “3*x* - 3 = 2*x* + 2”. Lucy is at Level 1 of the activity regime, and so the software is not providing her with any automation for the SILOs. With respect specifically to SILO 1, consistent measures, Level 1 means that Lucy must proactively build and maintain consistent measures as she interacts with the interface. In just over 4 minutes, Lucy has completed modeling both the Day 1 and Day 2 travel narratives (see Figure 28).

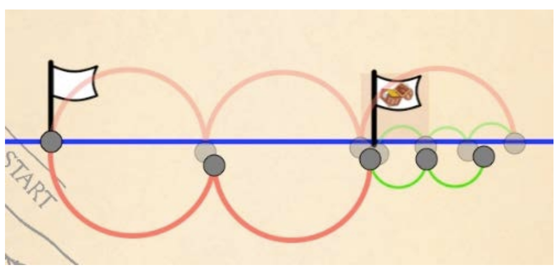


Figure 28. Screenshot from Lucy's work-in-progress on the "3x - 3 = 2x + 2" Giant Steps narrative. Day 1, above the line, models “3x - 3” as running from left to right with three large red arcs (3x) and then retracing back toward the left with three small green arcs (“- 3”). Day 2, below the line, models “2x + 2” as two large red arcs (2x) followed by two small green arcs (“+ 2”). Lucy has modeled meters as consistent measures both within each of these two fragments—above the line and between these two fragments (compare the large arcs above and below, compare the small arcs above and below). However the Day 1 and Day 2 model fragments do not bring the giant to the same destination.

We join the tutor–student dyad just as Lucy is fiddling with the size of the meter units in an attempt to make both diagrams reach the exact same location. In her initial model, she had made the meters of equal size both within- and between-days, however upon reflection she realized she must adjust all these sizes so that Day 1 and Day 2 diagrams bring the giant to the same destination. Figure 28 shows a screenshot from the work in progress. Note how the “3*x* - 3” model fragment (Day 1, above the line) extends further to the right as compared to the “2*x* + 2” model fragment (Day 1, below the line). Lucy believes that she could make ends meet by taking measures to adjust the sizes either of the giant steps or the meters. She begins by adjusting the meters on Day 2. In the transcription below, parenthetical texts, such as “(Day 1)” serve to clarify for the reader the interlocutors’ communicative intent as suggested by their non-verbal multi-modal utterance, including screen actions as well as various deictic gestures (e.g., pointing toward elements on the shared visual display).

Res.: Well I noticed that you made meters smaller on this day (Day 2).
Lucy: Yeah. So I should make them smaller on this day (Day 1) too probably.
Res.: Yeah? Why don't you try that. I think that could probably make sense. Why do you think that you should probably do that?
Lucy: Because then they will all be the same size, and then you will be, umm… I'll be able to the see whether it's still on that point (Lucy points to where she expects the treasure flag to be after the adjustments, which is slightly to the right of it's current placement, see Figure 29).

Figure 29. Lucy indicates on the computer monitor where she anticipates Day 1 and Day 2 travel destinations will be co-located once she adjusts the meters.

We interpret Lucy's multimodal utterances (i.e., her speech that is ecologically coupled to the screen via deictic gesture) as indicating an achievement of SILO 1—she apparently knows that meters should be of consistent size both within and across days. Moreover, she anticipates that by executing this with fidelity she will have access to pertinent information that will help her solve the problem at hand, as demonstrated through her projection of the new co-located end point.

We now turn to Mary, a participant in the Baseline condition (BS). Mary is working on the same item as Lucy, "$3x - 3 = 2x + 2$". However, Mary's BS interface *a priori* produces meters of fixed size that are therefore automatically consistent both within and across Day 1 and Day 2, and the interface also recalibrates all giant steps when any of them is resized. Mary has been working on this problem for almost 10 minutes. She completed both Day 1 and Day 2 travel models and was unable to determine a solution (see Figure 30). Mary has just erased her diagram and is starting over. We join the researcher–student dyad as Mary begins modeling the first 3 giant steps (large red arcs) and is adjusting their size exploratively: she stretches, shrinks, and finally leaves them slightly extended as compared to her erased diagram.

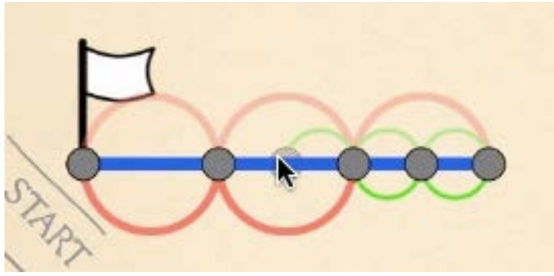


Figure 30. Working on "3x - 3 = 2x + 2", Mary has modeled the Day 1 and Day 2 narratives. The Day 1 and Day 2 journeys do not end at the same location, and so Mary has not solved the problem.

Mary: Maybe the giant got bigger.
Res.: Yeah, maybe the giant got bigger, or smaller, I don't know. It's a different giant, that's all I know
Mary: So Day 1 and Day 2 are different giants?
Res.: No no no, it's a different giant from Question 1.
Mary: (Slightly stretches the giant steps again, then creates three small green arcs running back toward the left for the "-3", see Figure 31) Ohhh, so…
Res.: What happened?
Mary: So the meters are always the same.
Res.: Does that make sense?
Mary: Yeah, so the meters are always the same, you can move the giant steps but not the meters.

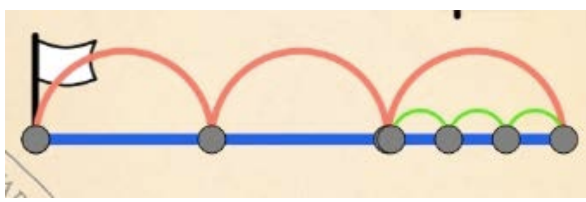


Figure 31. Mary is still working on modeling the "3x - 3 = 2x + 2" story. She has now stretched the giant steps so that they have each become 3 'meters' in length.

Recall that Mary is working under a condition where all measures are always uniform in size—all giant steps are always equal to each other, and all meters are always equal to each other—only that whereas the giant steps are adjustable (they all adjust simultaneously) the meters are not adjustable (they are of fixed size). Mary's insight into the consistent measure of the fixed meters unit in this interface is apparently facilitated by way of juxtaposition with the variable size of the giant-step unit measure. Her inflection indicates surprise at this discovery, and she reiterates her discovery thrice, as if memorizing it as a new rule.

In each of the episodes the participant has arrived at the conclusion that meters, within this microworld, should remain consistent in size both within and across days. What differs between the two episodes is how this knowledge surfaces, and how it is used. Lucy constructs a model in which her meters are fairly consistent, and yet, when she is unable to determine a solution, she can imagine how shrinking them uniformly, will solve the problem without violating what she has established as a viable measuring practice. Being able flexibly to adjust meters and giant steps vis-à-vis each other and the narrative indicates that Lucy has constructed subjective transparency for consistent measures by experiencing an opportunity to exercise agency in making equal units. Mary, too, is reasoning about the consistency of meter units, and yet she does so only as a feature of the interface. Mary, like Lucy, appreciates that the meter units behave differently from the variable units, and yet she conceptualizes the relation between meters and variables not as an instrumental function advancing the solution process but as some arbitrary interface feature she has detected. Thus Mary did not experience an opportunity to appreciate the potential utility of this embedded feature; she did not develop subjective transparency for this received feature. Per Freudenthal, Mary is a victim of a didactical crime, wherein the interface constraints, which were designed to scaffold her learning activity, in fact robbed her of an opportunity for discovery by imposing a mathematical notion as an immutable rule.

5.5.2 Emergence of SILO 1 "consistent measures," the case of the variable unit "giant step."

We turn now to two excerpts from the interview videography when our participants Lucy and Mary each articulate their emerging understanding that the variable unit measure, a giant step, should be consistent across the diagram model of the narrative.

Lucy is working at Level 1, Question 1—a story narrative that corresponds to the algebraic proposition "4*x* = 3*x* + 2". Recall that Lucy is working with an interface that provides no automation of her modeling activities. She has completed diagramming both Day 1 and Day 2 and is using her diagram to determine how many meters away from the start point the treasure is buried (see Figure 32). Lucy's diagram indicates that she has understood that Day 1 and Day 2 share the same end point, and that the corresponding giant steps in her Day 1 and Day 2 model fragments (above and below the line) should match in size, that is, they should subtend the same distances along the "story line". It is precisely this coordination that is of interest.

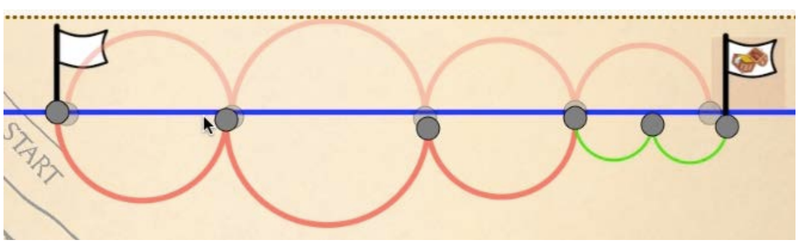


Figure 32. Screenshot of Lucy's completed diagram for the expression "4x = 3x + 2."

Res.: So I noticed the other thing that you're doing, that is interesting to me, is that on Day 2 you put the giant steps in the same place that you put the giant steps on Day 1.
Lucy: Yeah.
Res.: Why did you do that?
Lucy: Umm, I don't know. I just wanted to start at the same point (scrolls over the giant steps from left to right), so that it gets to the same spot (scrolls over the treasure flag)
Res.: Right. And you feel like it would make sense that she would, if she was trying to get back to the same place, she would walk in the same spots.
Lucy: Yeah (nods her head several times).
Res.: So do we know how big a giant step is?

Lucy: About this big (scrolls over the first giant step, alternating between the corresponding giant steps above and below the path line).
Res.: About that big.
Lucy: Yeah.
Res.: But it doesn't look like you drew them all the same size. Did you do that on purpose? Or were you just approximating?
Lucy: Yeah. I was trying to get, (moves her hands, with the palms facing each other, together and apart a few times, using small movements) just the average.

Lucy has expressed in two different forms her knowledge that giant steps should be of uniform size: (a) her *between*-day giant-steps constructions on the interface are paired above and below the line (their respective start- and end-points literally share the same node locations), and upon interrogation she vigorously asserts that this is logical; and (b) she discounts apparent *within*-day variability in giant-step size as construction flukes bearing no pragmatics intent—they are still conceptualized as constant.

By comparison, Mary is working on Question 1, a story narrative that corresponds to the proposition "$4x = 3x + 2$". She has modeled Day 1 of her diagram with 4 giant steps. The transcription begins with Mary's asserting that the total distance from the starting point to the treasure is 8 meters.

Mary: 8 meters.
Res.: 8 meters. Why do you think 8 meters?
Mary: Because, if 3 giant steps (scrolls over the first 3 giant steps in her model) and 2 meters (scrolls over the 4th giant step in her model) is the same as 4 giant steps. So it would be like you can take away 3 giant steps from both of them. And then you're left with, it's like the same amount of distance, like right there (she "holds" the size of a giant step between her thumb and index, see Figure 33). One is 2 meters, and one is a giant step.
Res.: Got it. So how does is end up being 8 in the end?
Mary: Because it's like 2 (scrolls over the 4th giant step), and 2 times 4 (scrolls over each giant step from right to left) is 8.

Figure 33. Mary uses her thumb and index fingers to measure the screen space that she believes is 2 meters long. Then, assuming that all other giant steps in her model are of equivalent size, she calculates 4 giant steps as 8 meters.

While Mary does not formally articulate that all giant steps in her model must be consistent in size. Rather, she calculates them as such and solves the problem correctly.

Both participants solve the problem and both numerically translate giant steps into meters, therefore the point of comparison here is very subtle. Lucy's articulation indicates intentionality, she made a decision to generate an ‘average,’ and consequently she has cognitively produced this construction property. Lucy enacted an internal thought process so as to regulate her end product. Mary, on the other hand, receives this regulation from the interface itself and never questions the validity of this construction property. Returning to the Freudenthal for a moment, it appears that the interfaces constraints, designed to scaffold Mary’s production, may have robbed her of discovery and simply introduced consistency among giant steps as a rule.

5.5.3 Emergence of SILO 2 “equivalent expressions” and SILO 3 “shared frame of reference.”

Lucy is working on Level 1, Question 3. The item depicts a story corresponding to the algebraic proposition “1*x* + 8 = 2*x* + 6”. Three minutes into her work on this item, Lucy’s diagram was non-normative in two ways (see Figure 34): (a) the model presented each giant step as 4 meters long (it should be 2 meters long); and (b) the Day 1 and Day 2 model fragments did not end at the same destination as they should.

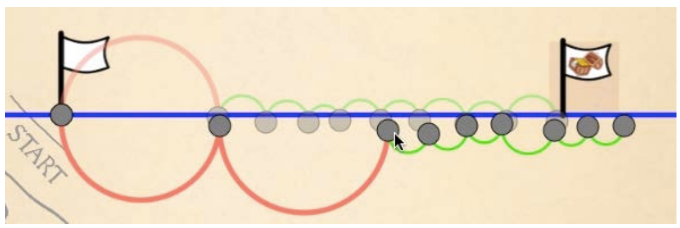


Figure 34. Lucy is working on "1x + 8 = 2x + 6". This is a screenshot of her initial yet faulty model, where giant steps are each equivalent to 4 meters, and the journey ends are not co-located.

Lucy attempts to repair this mismatch by shrinking the giant steps on Day 2, so that they are each 3 meters long (see Figure 35). Lucy is talking about how the giant steps and the meters need to match across days, what parts of the story are the same across Day 1 and 2, but is not sure what to do next.

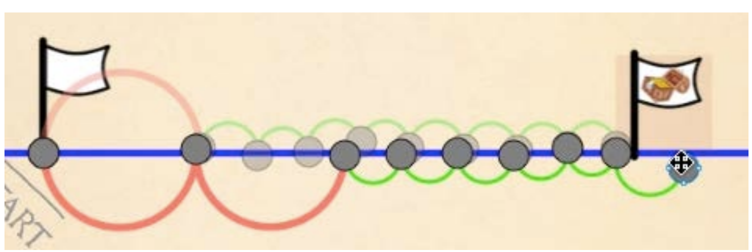


Figure 35. Lucy is still working on "1x + 8 = 2x + 6". This is a screenshot of her model after micro-corrections. Now each giant step is equivalent to 3 meters, and the ends still are not co-located.

Res.: You’re right, the meters always match up from Day 1 and Day 2. So what’s different?
Lucy: This (scrolls over the first giant step).
Res.: The giant step.
Lucy: Yeah.

Res.: And so what's different?
Lucy: This (scrolls over the second giant step on Day 2, which currently corresponds to 3 meters on Day 1).
Res.: And in the story what part is that?
Lucy: That is in Day 2. (She marks another 1 giant step and 6 meters.)

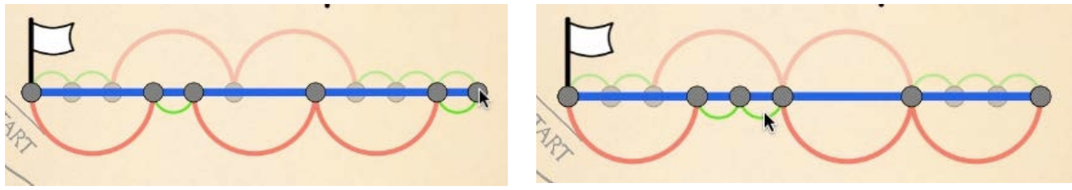
Lucy is talking about the difference between Day 1, 1 giant step and 8 meters, and Day 2, 1 giant step and 6 meters, but the researcher probes for more clarification.

Res.: She takes 1 giant step. Do you mean like an extra giant step?
Lucy: No.
Res.: She takes 2 giant steps.
Lucy: Oh yeah, she takes 2 giant steps. And then she takes 6 more meters (scrolls over the meters).
Res.: Right. How is that different from Day 1?
Lucy: On Day 1 she only takes 1 giant step and then 8 more meters.
Res.: 8 more meters. So what's the difference?
Lucy: It's 1 that she is taking…She took 1 more giant step but 2 less meters (she scrolls over the 2nd giant step on Day 2, where she is focusing her comparison). So the giant steps are supposed to be 2 meters, I think.
Res.: Oh, what makes you say that?
Lucy: Because this is (scrolls over the treasure site), 8 is 2 meters more than 6. And this (scrolling over the Day 2 part of the diagram) needs to be 2 meters more.

Lucy has used her faulty diagram to examine characteristics of the story that allow her to think about which elements are equivalent. She is grounding her thought process in the knowledge that the two expressions are equivalent, they must end at the same location, and she can therefore manipulate her faulty diagram in small ways, even in her imagination, so as to achieve this parity. Simultaneously, Lucy is establishing a shared frame of reference by linking the difference in meters from Day 1 and Day 2 to the additional giant step on Day 2. We now turn to Mary.

Mary is working on Question 6, a story narrative corresponding to the proposition "$2 + 2x + 3 = 1x + 1 + 2x + 1$". She begins by generating a complete model (see Figure 36a). She then deletes her Day 2 diagram and re-models it, changing the order of the story's features (see Figure 36b). We join the dyad as the interviewer asks Mary about this rearrangement.

Res.: So why do you think it works better to put the meter there?
Mary: Because then you can tell (scrolls over the first 2 giant steps in her model) how much… Oh it doesn't really work that much better. Oh wait, because then you can tell how many meters are in a giant step right here (scrolls over the final giant step).
Res.: Oh, OK.
Mary: Because it lines up.
Res.: And how do you know that you're allowed to do that, to change the order?
Mary: Because you still walk the same amount of distance. Like, if I walked 1 miles and then 1 foot, it would be the same as if I walked 1 foot and then I walked 1 mile.



a. *b.*

Figure 36. Screenshots of Mary's model for the problem story that corresponds to the algebraic proposition "2 + 2x + 3 = 1x + 1 + 2x + 1": (a) a completed model exactly following the narrative, (b) a reordering of the narrative's components that enable her better to determine visually the size of each giant step—note that the Day 2 giant step, below the line and on the extreme right, subtends 3 meters above the line.

Mary thus first performed a diagrammatic reordering of the narrative components and then cited her findings as unequivocal evidence supporting her conclusion. We interpret this behavior as indicating that Mary clearly understands that the two narratives depict journeys of equivalent extent and that the variable and fixed quantities mutually reference each other.

While both participants clearly demonstrate an understanding that the expressions are equivalent and that enacting this will result in finding a shared frame of reference, the quality of this understanding is instrumentalized by the tool that they are using. Lucy, using a more flexible interface, uses a combination of her imagination and her model to 'see' the completed narrative. While Mary, using a less flexible interface, must physically reproduce the model. It would seem that the more rigid interface garners a more rigid response, while the more flexible interface allows for a more flexible, if not even incomplete, model.

5.6 Discussion

The qualitative analysis reported herein sought to capture and convey nuances of the process by which participants developed an emerging understanding of algebra. This understanding was manifest in evidence that the students were building subjective transparency for the proto-algebraic logico-quantitative system embedded within the Giant Steps for Algebra problem scenarios. In particular, I was able to depict the participant's understanding as it emerged along successive moments of the interview. The qualitative analysis thus addressed the final research question (R3):

What are the mechanisms or opportunities that each condition [reverse-scaffolding vs. baseline] provides that could explain performance difference between the two groups?

The data corpus exemplifies how the reverse-scaffolding design architecture, and in particular the paced introduction of interaction features into the activity, offered the study participants unique learning opportunities. In the absence of ready-made interaction functions, the study participants were obliged *themselves* to intuit, infer, determine, and construct features of the mathematical system. By way of contrast, participants in the baseline condition, who received the interaction features ready made, did not experience as much insight into the embedded mathematical principles. Corroborating earlier constructionist research, the qualitative analysis thus indicates that students can develop subjective transparency of mathematical concepts by creating, articulating, and generalizing the structural properties of the models they use to solve situated problems. Furthermore, an activity that is designed so that students have agency in the discovery and articulation process yields greater subjective transparency as compared to a compatible activity where students do not experience this agency.

Finally, the results of this study highlight the importance of differentiating between *learning to use a tool* and *using a tool to learn*. In many cases, baseline participants developed a know-how (Ryle, 1945) that was situated in the particular immediate context and subsumed the tool. What more, they could articulate how they were using this available tool to solve the problem at hand: they used the ready-made interface features and often understood the relationship between their mousing actions and changes on the screen. However their understanding was only 'screen deep.' For example, baseline participants understood that the tool could stretch or shrink giant steps, but not how this action functions within the larger mathematical system. Learning how to operate a tool did not lead to compatible opportunities to develop subjective transparency of the mathematical system. The tool's ready-made utilities were never interrogated with respect to problems of practice they each solved. The utilities were conceptualized as manipulative features, not as solutions. Just as in the case of bicycle gears, one can become highly skilled in using an artifact's utilities without ever questioning their rationale or build, without looking under the hood. Using a ready-made tool does not necessitate the development of situated intermediary learning objectives (SILOs).

By contrast, participants in the reverse-scaffolding condition had to enact and formulate interaction strategies that compensated for the tool's shortcomings; only once they had articulated these compensatory strategies did the experimenter supplement those strategies into the tool as built-in features. Through this process, the participants came to understand their ideas with more clarity and, in so doing, achieve greater fidelity with the target concepts. Thus the reverse-scaffolding condition enabled participants to encounter a problem of practice, enact their solution, and ultimately articulate and confirm it. The process of first needing a particular tool and only then receiving it created increased opportunities to develop subjective transparency of the emergent mathematical system.

Thus, *learning to use a tool* is a learning activity that does not necessitate that the participants encounter the problem of practice, nor articulate how the built-in features of an artifact enable their success. Furthermore, knowing how to operate a tool does not guarantee a seeing and understanding of the cultural–historical disciplinary knowledge embedded in the tool (see also Meira, 2002). On the other hand *using a tool to learn* implies that users build new knowledge by engineering improvements to imperfect tools, where these engineering micro-solutions embody the design's learning objective.

5.7 Human Computer Interactions: Reflections on the User Interface, Usability, and Accessibility

The Giant Steps for Algebra platform was designed for delivery via the internet. Therefore, it was optimized for use via a desktop or laptop computer. Some participants struggled when using the laptop's built-in trackpad. Inorder to avoid undo frustration, I brought a Bluetooth mouse to all data collection sessions. All participants were familiar with using a mouse to select and drag onscreen modeling features. The platform layout proved to be intuitive to the users. Participants quickly located the story, identified that there were two components and quickly toggled between these two components. The participants also quickly identified the features in the modeling toolbox and associated them with the story, even prior to modeling.

The usability of the platform was somewhat problematic. The tolerance levels for selecting one of the toolbox features were not optimal. In particular, the cursor had to be entirely over the desired ‘button,’ or else the program responded as if there was no desired change in selection. What often happened was that a participant would intend to switch between Day 1 and Day 2, but they would inadvertently create a new Giant Step, for example. Because of these unintended, and somewhat glitchy, interface responses users had to spend time correcting. This tampered the ease of usability, making the process rather cumbersome at times. Notwithstanding, there was only one instance of a participant abandoning the modeling interface for another modeling modality – this participant quickly sketched her ideas for the researcher on a piece of paper while explaining her thinking.

After interacting with the GS4A platform, many participants commented that it lacked certain game-like features. Some participants suggested a more robust and flashy acknowledgement for a correct response, some suggested the addition of sound effects, some suggested that the appearance be made to look more realistic. I categorize these comments under the participants’ appreciation of, and comfort level with, more traditional computer game’s appearance and embedded motivational components. This category of game mechanics was not an explicit focus of this first build of GS4A.

Finally, when reflecting on the accessability of the technological tool, there were a few interface features that could be improved upon. Firstly, some students had a difficult time differentiating the hue of the green meter arc from the background. It would greatly improve the accessability if the color palatte were adjusted. Secondly, the current build of GS4A contains no formal web accessability features, in accordance with ADA, built in at this time.

CHAPTER 6: Conclusion

This dissertation investigated the hypothetical phenomenon of discovery-based mathematics learning. The investigation took form as a conjecture-driven design-based research project centered on creating an empirical context for pursuing a set of emergent research problems. The learning environment was a technology embedded early-algebra modeling activity, Giant Steps for Algebra (GS4A), which was designed so as to foster student discovery of the qualitative principles governing the solution of unknown values. The project addressed the following broader research questions:

1. What are effective design heuristics for creating discovery learning activities? How might such activities avail of technological functionalities?
2. In particular, is it possible to create a microworld based on constructionist principles, wherein students learn mathematics content through building artifacts? Can a learning design balance constructionist principles with specific curricular goals? What particular activity architecture would achieve this balance?
3. To the extent that the activity is effective, how exactly does building artifacts lead to content learning?

For its theory, the GS4A activity architecture design drew heavily on on the theoretical construct of *transparency* (Meira, 1998, 2002) so as to envision a new approach to scaffolding. Transparency is a psychological construct related to objects and procedures inherent in cultural practice—it is the social agent's understanding of how purposeful artifact-mediated actions, such as manipulating the GS4A interface features to solve a situated problem, accomplishes their objective. Transparency captures novices' understanding of how features of artifacts they are using function in the accomplishment of situated goals. Transparency is the key hypothetical construct of this dissertation. The design of educational materials and activities as well as the design architecture for discovery-oriented scaffolding were all oriented on student development of subjective transparency for the algebra artifact.

For its design, GS4A utilized the double-number-line visualization of algebraic propositions, that is, two equivalent expressions (Dickinson & Eade, 2004). GS4A students read a story about a giant who travels from Point A to Point B on two consecutive days, and then they use features of a microworld to model this story. Each leg of the journey unfolds with different combinations of giants steps (the variable) and meters, and students are to determine how many meters one giant step comprises.

As they tinker with available microworld features so as to represent the story on the screen, students unwittingly achieve a set of pragmatic principles for coordinating the construction of the mathematical model. I named these principles *Situated Intermediary Learning Objective* (SILOs). The three GS4A SILOs were as follows,:

- *Consistent measures* coordinates among units, so that all spatial diagrammatic instantiations of a variable are equal to each other (and the same goes for the known units)
- *Equivalent expressions* coordinates across two expressions of an algebraic proposition
- *Shared frame of reference* coordinates between two unit systems.

This study set out to investigate whether, and how, participants developed subjective transparency of the target content, namely the SILOs for foundational algebra. I measured the development of subjective transparency under conditions where I assign the participants a task and provide them with relevant resources but do not tell them how these resources should all come together to get the task done. I called this minimal-interventional approach *reverse scaffolding*. As opposed to the standard idea of scaffolding, which is a robust idea in the parlance of educational research and practice, the phrase "reverse scaffolding" implies a form of instruction in which the expert does not perform for the novice what the novice *cannot* yet do but only what the novice *can* already do.

In technological environments, reverse-scaffolding is implemented by way of gradually introducing into the interface automated functionalities designed to support students' diagrammatic construction. By way of receiving new interaction tools only on demand, that is, only after a particular construction function has been discerned and possibly articulated, students develop subjective transparency of these new tools. That is, students understand the logic, mechanism, and purpose of these tools, not only their use; they know what precisely these tools accomplish, what problems of practice the tools solve, what pragmatic principles of assembly the tools actuate, how the tools enhance a grip on the world. Short of literally building the tools, students reinvent the tools.

Note that the new tools are introduced not in a single event but over a sequence of events, whereby increased proficiency is iteratively rewarded with increased functionality. As such, the pedagogical approach of reverse-scaffolding is implemented technologically by way of an interaction activity architecture I call *leveling transparency*. The term "leveling" was selected specifically so as to conjure computer games, wherein new levels are attained, with all rights and privileges thereto pertaining.

To evaluate whether the proposed reverse scaffolding pedagogical approach and leveling transparency activity architecture enable the development of subjective transparency, I conducted an empirical study that compared the implementation of an instructional activity under both reverse-scaffolding and control protocols. Statistical analyses of the two groups' mean scores on post-intervention assessments demonstrated a positive main effect: Participants in the reverse-scaffolding condition significantly outperformed those in the control condition.

Qualitative analyses of the intervention process implicated psychological mechanisms apparently causative of the main effect: Reverse-scaffolding students struggle more than baseline students to manage structural properties of the modeling system, and therefore these properties become evident to them—they develope more transparent structural understanding of the modeling procedure as well as greater facility in articulating the emerging conceptual system. In particular, I submit, the system of construction principles that students discover by tinkering with elements of a modeling environment (i.e. the SILOs) are tantamount to the core mathematical content of the instructional design.

I thus conclude that students can reinvent mathematical knowledge through engaging in modeling-based activities, and reinvention positively correlates with the development of subjective transparency. To do so, participants should attend to the structure of their own constructions; they should apprehend this structure as reifying their tacit knowledge—knowledge that is brought forth as material form through the dialectics of shaping construction resources to tell a story; they should come to see latent structural features of their own spontaneous models of problem situations as properties of purposeful functions. Tasks, construction resources, facilitation, and activity flow can and should be designed to enable this apprehension of structure. In particular, students can develop subjective transparency of mathematical concepts by creating, articulating, and generalizing the structural properties of the models they use to solve situated problems. The findings support previous arguments for learning through discovery (Martin & Schwartz, 2005; Noss & Hoyles, 1996; Radford, 2003; Schneider et al. 2015) and raise questions for skeptics (Alfieri et al. 2011; Kirschner et al. 2006). Moreover, I have offered and validated reverse-scaffolding, a pedagogical design architecture for discovery-based mathematics learning in technological environments (see also Holmes et al. 2014).

6.1 Limitations

The quantitative results reported herein are somewhat limited by the nature of the small sample size, only *n*=40. Given a larger sample size, it would be advantageous to test whether the significant learning gains experienced by the experimental group is consistent across all reported mathematics ability levels.

The results must be considered only as a preliminary proof of concept within the context of a larger ongoing design-based research process. This was the first empirical study using the GS4A interface, and further refinement of the technology is still needed so as to increase usability. Additionally, whereas this experimental technology embeds some of the human tutor's facilitation actions in the form of "intelligent" interactive software, the current build cannot as yet eliminate the human tutor—we still rely heavily on a human tutor to monitor that the participants indeed build subjective transparency as they progress through the activity levels. In the interest of eventually scaling up the project, that is, creating empirically evaluated and internationally accessible online learning tools for early algebra, it would be necessary to continue refining the software's interactions as well as to extend students' work into the symbolic semiotic register.

Lastly, this study makes no direct claims regarding the impact that leveling transparency for the GS4A SILOs can have on proficiency in formal algebra. The scope of this study did not include a pre-post assessment of formal algebra skills. This should be considered for further studies of the leveling transparency theoretical approach.

6.2 Future Design Trajectories

The results from this initital build of the GS4A platform, as measured through student outcomes, were promising. I would like to take this opportunity to outline potential next steps for the GS4A project and the aspirations I have for GS4A 2.0. We have begun working on a tablet application. The hope is to explore the affordances of the touchscreen environment as a more direct and manipulatable medium. Additionally, future work on the GS4A web platform should address the specific shortcomings described earlier.

Furthermore, the GS4A platform serves to demonstrate a new pedagogical approach to instructional design for technology-enabled activities. Future design work should explore how the reverse scaffolding activity architecture can be leveraged in other areas of STEM learning.

6.3 Implications

I have put forth a framework for building educational technology that facilitates students' development of subjective transparency—the framework calls for a pedagogical approach that embeds within the interactive technology an activity sequence designed to foster incremental development of transparency. Leveling transparency, as an activity-design principle for creating technology-based learning activities, could stand to significantly inform the design of pedagogical tools for discovery-based learning.

This dissertation has corroborated the fundamental tenet of Constructionist pedagogical philosophy: What children learned about algebra was the sum total of the systemic relations they figured out *through* and *about* a model that they built. The learning environment—its structure, rationale, and resources—was supportive of the various discoveries and coordinations students achieved,—the “nuts and bolts” of building a workable model that both bears functional parity to the situation it represents and enables drawing inferences about latent properties of that situation. The logic of this educational design as well as its formative construct of a situated intermediate learning objective (SILO) may prove valuable in creating environments for other mathematical concepts as well as, more broadly, other STEM concepts.

You know what you build. Granted, you can learn to use ready-made tools, but when you yourself build these tools your learning will be of a different quality altogether.

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APPENDIX A: INTERVIEW PROTOCOL

Digital Interview Protocol for GS4A - Leveling up

Prior to beginning the Giant Steps for Algebra activity the researcher should ensure that the participant can use the computer mouse. The interviewer briefly introduces that participant to the context.

Int: “Egbert the Giant has stolen the elves’ treasure. Imagine that he has escaped their land and voyaged to a desert island. Imagine that this is the desert island (point to the computer screen). After getting off the boat, Egbert wants to hide the stolen treasure. You are an Elf. You are positioned on this island, and you are spying on the giant to find out what he does with the treasure. Here is the story that tells how the giant buried the treasure (point to the story box). These are the tools that can help you find the treasure (point to the toolbox). Do you have any questions?”

Int = Interviewer
Part = Participant

LEVEL 1 – Freeform

Task	**Why**	**Anticipated Response**	**Possible Follow up**
Problem 1: 4x=3x+2 Int: "What do you need to figure out?"	Assure user can establish problem-solving method Interviewer can show the participant how the interface works by demonstrating how to 'draw' a giant step and a meter, that the nodes can be moved, how the undo and delete work, that there is day 1 and day 2.	Part: "How big a giant step is." Part: "How big a meter is." Part: "Where the treasure is."	Int: "Do we know how big a giant step (or meter) is?" Int: "Do we need to know right now how big a giant step (or meter) is?" Int:" Can we just pretend it is this [demonstrate a random size] for now, or any size really?"
Problem 2: 3x-3=2x+2 Int: "What do you need to figure out?"	Assure user can recreate a problem-solving method	Part: "well the last time the giant step was _ # of meters" Part: "Is it the same giant?" *Or the participant will replicate the same sequence of responses as above.*	Int: "I think that it is a different giant this time, so can we imagine that this giant walks differently?"

Assessment problems for leveling up - From level 1 (freeform) - level 2 (fixed meter)

Task	**Why**	**Anticipated Response**	**Follow up**
- **Problem 3:** **1x + 8 = 2x + 6** Int: "How could drawing (building) this model be easier?"	*Participant demonstrates that s/he understands the representational intentionality of maintaining uniform constant units (in this case meters) despite perhaps representing them in a non-uniform manner.*	Part: "The computer does it" Part: "Not so many meters"	Int: "Does what exactly" "Creates something (meters) for you?"

LEVEL 2 - Fixed Meter

Task	**Why**	**Anticipated Response**	**Possible Follow up**
- **Problem 1:** **http://art.visheshk.net/gs/meter/1.html** **3x+2=4x-1** *precursor for stretchy. Uniformity of meter, and giant step (within days)* Int: "So what do you need to know?"	*Participant constructs representations using a fixed meter. Can zoom in and out so that representation remains on the canvas* Similar question structure as in level 1 - freeform	Part:"The meter won't change." Part:"I don't have enough room now that the meter is fixed"	Int:" the game has decided how big a meter should look" Int: "You can zoom in/out to make more space"
- **Problem 2:** **http://art.visheshk.net/gs/meter/2.html** **3x – 3 = x + 1** *Uniformity of giant steps (between days)'* Int: "So what do you need to know?"	Similar question structure as in level 1 - freeform		

Assessment problems for leveling up - from level 2 (fixed meter) - level 3 (stretchy)

Task	Why	Anticipated Response	Possible Follow up
- Problem 3 http://art.visheshk.net/gs/meter/3.html *Day 1 a giant walks 2 meter, then 2 giant steps, then 3 meters. Day 2 a giant walks 1 giant step, then 4 meters then 1 giant step then 1 meter.*	*Because they have to retain the uniformity of giant steps as well as the shared start and end point.*	Part: " It's hard changing all of the giant steps to match." Part: "How do I know that all the giant steps are exactly the same size?"	Int: "It is important to use the meter, which a size that you know and that is given to you, to see if you can make sure all of the giant steps are exactly the same size."

LEVEL 3 - Stretchy

Task	Why	Anticipated Response	Possible Follow up
- Problem 1 http://art.visheshk.net/gs/new/1.html ***5 – 3x = 2x*** *Giant steps backwards* Int: "What do you need to figure out?" Int: "So what do you notice?"	*Participant experiments with negative integers*	Part: "I need to figure out how to make the ends meet (how to over lap Day 1 and Day 2 end point) Part: " I notice that all of the giant steps change (move) together"	Int: " What can you move so that the ends meet?" Int: " Can this help you figure out where the treasure is buried?"
- Problem 2: http://art.visheshk.net/gs/new/2.html **2x – 4 = x – 3** *Behind the starting point*	*Participant increases the range that they are manipulating the giant steps*	Part: "But it seems to be behind the start?" Part: "Can a giant move in any direction?"	Int: " A giant can move in any direct."
- Problem 3: http://art.visheshk.net/gs/new/3.html **3 + 2x + 4 = 2 + 3x**	*Participant rearranges the order of the story components either manually or mentally to create accurate*	Part: "But the giant steps/meters don't match up." Part: "Is it possible to	Int: "What can you do so that the drawings match up?" Int: "It is possible to switch the order of

Constants and giant steps in different orders (requires the functionality of removal of intermediate nodes or user rearranges) 3 + 2x + 4 = 2 + 3x (combining like terms, is introduced/demanded here)	*matches. Maintaining internal consistency but simplifying the equation.*	switch the order that I draw thing in?”	the drawing without changing anything.”