Body of Knowledge:
Practicing Mathematics in Instrumented Fields of Promoted Action

by

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Abstract

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A central issue in education in general, and mathematics education in particular, is the relationship between skills and concepts, between performing procedures and understanding content. This dissertation draws on recent research on the embodiment of cognition to cast doubt on the accepted separation of bodily practice and mental understanding. What are the implications of embodiment perspectives for designing mathematics learning environments? Can conceptual understanding sprout from procedural fluency? To answer these questions, I partook in two interrelated strands of research: (1) investigating pedagogical traditions of explicitly embodied disciplines; and (2) implementing and analyzing embodied-interaction mathematics designs. The first strand involves immersive ethnographic investigations of pedagogical practices in surfing and martial arts, corroborated via interviews and archival research. It focuses on *enactive artifacts*, or disciplinary routines through which students develops a *felt sense* as grounding for disciplinary concepts. This felt sense, I argue, cannot be taught directly via demonstration or verbal instruction. Instead, it must be personally experienced via practice. The second strand involves video analysis of task-based interviews situated in technology-enabled motion-sensitive learning environments. In the focal design, students who have not formally studied the mathematical concept of proportion first learn to enact a dynamical bimanual coordination, in which they move their hands proportionally, and only later signify this felt sense mathematically using symbolic artifacts interpolated into the problem space. I claim that conceptual understandings can sprout from practicing mathematics in *instrumented fields of promoted action*. Therein, practice serves as a form of exploration rather than drill. Ultimately, I argue for an account of learning across disciplines as explorative problem solving, where students find themselves moving in new ways and, upon appropriating available disciplinary frames of reference, recognize in their own actions its disciplinary significance. Regardless of the discipline, one's body of knowledge is built through the labor of practice.

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Finally, it is customary to dedicate one's dissertation to something or someone the author is grateful for. I dedicate mine to the small hill on the far, quiet end of Komogovina, and all those who climbed it with me.

Chapter 1: Introduction

> Thinking, or knowledge getting, is far from being the armchair thing it is often supposed to be.
> (Dewey, 1916, p. 13)

1.1 Introduction, or, a Motivation for a Novel Framework for the Study of Actions and Concepts in Mathematics Education

1.1.1 Stating the problem, or, What no one knows about conceptual understanding

In contemporary research on cognition and learning, the constructs of procedure and concept play a curious role.

On the one hand, they are commonplace to the point of ubiquity. One frequently encounters education articles with explicit mention of procedures or concepts. Among those, a common approach is to advocate conceptual understanding even at the expense of procedural skill. An alternative to this, more common in mainstream media publications than academic journals, is the emphasis on "back to basics"—that is, an emphasis on acquiring procedural skills by rote drills. Ultimately, however, most scholars and pundits argue that "both are needed." In the United States, the influential National Council of Teachers of Mathematics Standards argues for a marriage of "procedural proficiency" (or "fluency") and "conceptual understanding." Common Core State Standards (CCSS), the latest national initiative, uses nearly identical language, frequently referring to "core conceptual understanding" and "procedures" as two distinct (and disjoint) constructs. For example, an earlier version of the CCSM Key Points in Math stated: "The standards stress not only procedural skill but also conceptual understanding, to make sure students are learning and absorbing the critical information they need to succeed at higher levels" ("Key Points in Mathematics," 2012, p. 1). Granted, political forces and astute diplomacy are steering some of this conciliatory phrasing, and yet the great divide remains.

On the other hand, there is a persistent disagreement over what, exactly, the nature of these constructs and their relation is beyond the simplistic, implicit notion that procedures are something you *do* and concepts are something you *have* "in the head."[1] Evidence of this disagreement can be located in the heavily politicized, so-called math wars (Schoenfeld, 2004) waged over issues involving perceived advantages of going back to procedural "basics" versus aiming for conceptual "learning for understanding." Indeed, whenever thought and action are seen as *separate*, educational reform invites political interrogation, as seen in the case of Hannah Arendt's (1958) deeply skeptical essay on educational reform, *The Crisis in Education*.

[1] I use the quotation marks around this metaphor for reasons that will become more evident in Chapter 4.
[2] This meta-review involved substantial work—it was Crooks' dissertation.
[3] Martha Alibali (personal communication) suggested that there might be cases of learners with disabilities who have a grasp of mathematics principles but are, due to their specific disability, unable to perform the mathematics in question. An empirical question might be whether they are able to internally simulate procedural practices, perhaps through watching someone else perform the work. Regardless, it is telling that this was the only (potential) example of "understanding but not doing" that I came across.
[4] Thus we may find ourselves in an awkward position of having to argue that, sometimes, our bodies do our thinking

This lack of agreement holds for educational theoreticians, too. What is agreed upon, by and large, is that mathematical knowledge is divisible into two general types: procedures and concepts. Yet while some scholars begin by noting a difference between “inflexible procedures” and “fluent thinking using conceptual knowledge” (see Gray & Tall, 1994), others conceptualize these constructs as the duality of mathematical “processes” and “products” (see Sfard, 1991).

In 2014, Noelle Cooks and Martha Alibali authored a comprehensive review[2]—*Defining and Measuring Conceptual Knowledge in Mathematics*—of how the field has characterized this divide in the literature. The review, overall, is critical, noting that “few investigators provide explicit definitions of conceptual knowledge” and “the definitions that are provided are often vague” (p. 344).

Drawing on this text, I provide only a few examples of how researchers have conceptualized conceptual understanding, and refer the reader to the Crooks and Alibali article for many more. One common metaphor is to label conceptual understanding as a “connected web of knowledge” (Hiebert & Lefevre, 1986, pp. 3-4). While I am most sympathetic to this definition, it is vague on what these pieces of knowledge are and how they might be connected. Another well-cited definition has conceptual understanding be “knowledge about facts, concepts, and principles” (de Jong & Ferguson-Hessler, 1996, p. 107), which is unspecific to the point of uselessness. Finally, some researchers define conceptual knowledge as “the awareness of what mathematical symbols mean” (Ploger & Hecht, 2009, p. 268) and “understanding of the underlying structures of mathematics” (Robinson & Dube, 2009, p. 193), which appear merely to shift the burden from “concepts” to “awareness” and “understanding.” In short, Crooks and Alibali’s conclusion that “there does not appear to be a clear consensus in the literature as to what exactly conceptual knowledge is” (p. 345) is an understatement.

Despite the lack of agreement on what, exactly, conceptual understanding is, “the general consensus, in research on mathematical thinking and in mathematics education, is that having conceptual knowledge confers benefits above and beyond having procedural skill” (p. 345), a notion that Crooks and Alibali explicitly agree with when they state that “it is important to identify…conceptual knowledge, *as distinct from procedural knowledge*” (p. 346, emphasis mine).

In other words, despite the fact that our field does not agree on what conceptual understanding is, a plurality of approaches in mathematics education (including Crooks & Alibali, 2014) proceed without questioning the very notion that there *is* a dichotomy. I offer that this is problematic: how can we be certain of properties of conceptual understanding when we cannot even agree on what it is? Furthermore, I offer that many scholars involved in education or educational research *already* know this to be problematic. Indeed, my own review of literature indicates it is difficult, if not altogether impossible, to find a scholar who argues that mathematical knowledge belongs exclusively to either procedural skills or conceptual understanding. “Surely, both are needed,” argues virtually everyone if pressed, despite the apparent state of the literature.

[2] This meta-review involved substantial work—it was Crooks’ dissertation.

My interest in this dichotomy rose out of my observation that across the diverse practices in which I engage, my procedural practice is by no means “altogether separate” from my understanding. Indeed, I would characterize my understanding as emerging from my practice.

The aim of this dissertation is to offer scholarship on cognition and learning a theoretical framework alternative to those grounded in concept/procedure dichotomy.

I have so far suggested that this dichotomy is deeply entrenched in our discourse. And while it is tempting to write a dissertation focused on delivering a deathblow critique to this discourse, my overarching goal is different: to open up new vantage point for discussion, from which the tension inherent in this dichotomy is either resolved or, at least, lessened.

Yet it is useful, as a means of charting out the various territories covered in this discussion, briefly to explore *why* this dichotomy of action/concept remains so persistent, and why I find those reasons unconvincing.

1.1.2 The persistence of procedure versus concept dichotomy in education research

A first conjecture for the puzzling endurance of this dichotomy is that it must, in some way, be perceived as *useful* in order to have carved out such a prominent position in our discourse.

Specifically, it can be argued that it is pedagogically useful to notice that a student can achieve a degree of procedural competency *without* conceptual understanding. The ubiquity of this argument is evident in informal discourse on mathematics, such as oft-repeated claims that “yes, the students did well on the test, *but do they actually understand the material*?” I am unconvinced that this is, in general, a useful categorization of students’ knowledge.

A paradigmatic example of this dichotomy can be located in Erlwanger’s 1973 *Benny’s Conception of Rules and Answers in IPI Mathematics*. There, Erlwanger presents a case study of Benny, a twelve-year-old high-achieving student who, based on his scores, appears to be making excellent progress with mathematics. This appearance notwithstanding, Erlwanger shows a number of problems with Benny’s understanding of mathematics. Most damaging is that Benny characterizes mathematics as a game with haphazard rules that need not—and typically do not—make sense. To him, it was “magic” that certain rules work across different examples, and consequently there was no attempt to make sense of why the same rule applied across cases.

One way to interpret this finding is to claim that a student like Benny can exhibit procedural skill *without* conceptual understanding. An extension of this claim would be that teaching conceptual understanding should take precedence over procedural skill. I take a contrarian stance, suggesting there are problems with both the interpretation and its implication.

First, I take issue with the word “understanding.” It is generally accepted, for example, that students can perform the division algorithm without understanding it. However, I offer that “understanding it” in this case is more accurately (and honestly) written as “without understanding it in a way we desire.” Understanding, or “perceiving the meaning of,” is necessarily dependent on the subject’s point of view. A theoretical physicist has a deep and profound understanding of the physical world, yet I prefer plumbers, electricians, and carpenters when it comes to fixing my apartment.

My point is that understanding of the sort we value is of little concern to students who are implicitly taught that *answer-getting* is the appropriate way of understanding mathematics. I am reminded of a former student of mine, an undergraduate at UC Berkeley, writing:

> Growing up, I was taught math by just learning the correct procedures and formulas to find the answer. I just practiced so many problems that I was able to memorize/recognize the types of problems that would arise and combine that with my memory of procedures and always get the right answer; that's why I enjoyed math so much, because it was just doing steps until you got to the end—a simple puzzle.

Later in the semester this student made another comment, one I am unlikely to forget: "I actually think that the student is the wise person in figuring out that he/she can pass without actually learning."

Let's return to Benny. It is, I offer, dishonest to categorically state that Benny did not understand mathematics. After all, the fact that he developed an understanding of mathematics as a game to be played is not an indication of his *lack* of understanding, but an indication of his *correct* understanding of the learning environment he was thrust into. This is not a pedagogically desirable outcome, but an altogether expected outcome.

In other words, I am arguing that Benny developed an understanding of mathematics as a particular sort of a game *because it was presented to him and practiced by him as a particular sort of a game*, a game in which understanding of the sort we value is of little consequence. That is to say, mis-conceptions are *not* non-conceptions (see Borovcnik & Bentz, 1991; Smith, diSessa, Rochelle, 1993). Indeed, Shaughnessy, in a perspective on Benny, writes: "I think that Erlwanger would agree that the issue is not concepts *or* skills but concepts *to support* skills" (in Carpenter, Dossey, & Koehler, 2004, p. 48). And, I add, *skills to support concepts*.

Furthermore, even if we were forced to concede that Benny exhibits procedural skill without desirable conceptual understanding, then what follows? This shows that certain procedural performances can *occur* without pedagogically desirable conceptual understanding but says nothing whatsoever on whether desirable conceptual understanding can *develop* without procedural performance. And indeed it is difficult—even as a thought experiment—to conceive of a student who has a conceptual understanding of some aspect of mathematics without *at least some* relevant procedural competency.[3]

In short, I suggest that even the apparent utility of the procedural/conceptual dichotomy is pedagogically pernicious if pursued uncritically. This is especially the case when we confuse a

[3] Martha Alibali (personal communication) suggested that there might be cases of learners with disabilities who have a grasp of mathematics principles but are, due to their specific disability, unable to perform the mathematics in question. An empirical question might be whether they are able to internally simulate procedural practices, perhaps through watching someone else perform the work. Regardless, it is telling that this was the only (potential) example of "understanding but not doing" that I came across.

description of student's knowledge with an educational *prescription* for how learning should take place. I mean this in the following sense. Once we say, "This student is able to do procedures, but lacks conceptual understanding" the danger is in following with, "Thus, let us teach her concepts." This assumes that the "cure" for her lack of conceptual understanding is something that can be accomplished in a linear, straightforward fashion—simply teach more concepts! I highlight this assumption because, for reasons that will become more apparent in later chapters, I do not believe that concepts can be taught directly. What if they emerge from and demand practice? This, in essence, is what I shall argue.

A second conjecture on the persistence of dichotomy under discussion is that a plurality of approaches in education are grounded in scholarly investigations of *schooling* and *formal* education. As I now argue, the formal–informal dichotomy itself is suspect and potentially deleterious.

Formal education or schooling oftentimes involves disciplines—such as mathematics—wherein the transition from procedural to conceptual is viewed through a Piagetian lens of "encapsulation" (explicitly or not). From this perspective, according to Beth and Piaget, procedural performance is "reorganized on a higher plane of thought and so comes to be understood" (1966, p. 247). In the same text, a remarkable claim is put forth about correspondence between the evolving cognitive structures within an individual's mind and those of the structures espoused by Nicolas Bourbaki.

Bourbaki was collective pseudonym of a group of European (mostly French) mathematicians who influenced mathematical discourse in mid-20th century, extolling the importance of rigor and logic *at the expense of* mathematical imagery and intuition. Bourbaki cast a long shadow, arguably lasting to this day, over the more intuitive approaches to mathematics, advocated by mathematicians such as Henri Poincaré. Consequently we see that Piaget strove to align his model of cognition with abstract mathematical structures celebrated in his time. In turn, the primary value of embodied performance, according to Piaget, is that it allows us to reach a "richer, yet purer form" of abstract mental thoughts (Piaget & Inhelder, 1956). Mature cognition, as it were, was seen as more akin to mathematical structures than to dancing.

Arguably, one can trace this perspective back to the influence Immanuel Kant cast on Piaget, and even further back to Cartesian dualism, where mind is not only made of a different substance than the body, but the superior substance—a notion that itself may be traced to Judeo-Christian religious beliefs.

In contrast to the above, the pedagogical traditions of disciplines less studied in the learning sciences, such as dance, seldom if ever speak of anything resembling Piagetian "higher planes of thought" as necessary (or even desired) for the acquisition of understanding. Instead, understanding is viewed, matter-of-factly, as emerging *in* and *through* performance. Consequently one might ask: Why has this seemingly unique characteristic of dance—and other "physical" disciplines—been largely ignored in education and human development research? The answer seems to be that scholarship on learning has historically made a distinction between disciplines observably physical and disciplines observably mental. The former are typically labeled *motor skill learning* and concern themselves with improving bodily performance. The later are typically labeled *conceptual learning* and concern themselves with mental knowing. And never, as the saying goes, the twain shall meet!

Yet the last few decades of scholarship suggest these barriers to be more culturally imposed and accepted than naturally evident or necessary. For one, the field of cognitive sciences at large saw growing interest in the theories of embodied cognition—a topic I shall consider in the next section, for embodied cognition challenges directly the procedural/conceptual dichotomy under discussion. For the moment it suffices to say that one implication of re-framing cognition as *embodied* is that action, whether physical or simulated, is viewed not as epiphenomenal but as supportive of, or even constitutive to "abstract" mental activity. One implication is that learning of mathematics and of dance may be more alike than previously believed.

With this, the next section turns to the increasingly prominent frameworks of embodied cognition in cognitive sciences.

1.2 Embodied Cognition

Why might one entertain the notion that thinking is, in some sense, embodied?

It may be useful to start from the beginning, with an evolutionary argument. A strong evolutionary case can be made that cognition is for directing physical and simulated action rather than abstract symbol processing. In presenting a case for the primacy of action, biologist Llinas (2002) describes the life of a sea squirt. The sea squirt begins life as a tadpole-like creature with a rudimentary nervous system, including a brain-like ganglion on top of a neural tube homologous to a spinal cord. It swims until it finds a stable place to which to attach, where it stays for the remainder of its life. Shortly after attaching, the sea squirt begins digesting its nervous system. Without a need ever to move again, its brain becomes worthless—or, at least, good for food only.

The peculiar fate of a sea squirt's brain is a particular case of a broader biological claim that we—brains included—evolved for action. And since evolution tends to be conservative in developing faculties, a reasonable conjecture is that our ability to conduct mental actions is built on our ability to conduct physical action. The appeal of this conjecture in the cognitive sciences is illustrated well by the symbol-grounding problem (Harnad, 1990; Glenberg, 1997; also see Searle, 1980, on the "Chinese Room" thought experiment). The grounding problem, put in language of mathematics education, might be: "How do abstract, mathematical ideas gain their meaning—what grounds them?"

An increasingly accepted argument in the cognitive sciences is that concepts are grounded in action (see Kiefer & Barsalou, 2013, for a recent review). This stands in contrast to the mid-20th century cognitive science paradigm inspired by the more formal theories of logic, language, and computation. In "traditional cognitivist" views, *a*modal symbol representations support language, thinking, attention, memory, and meaning—all this, done independently from the sensorimotor processes. Because embodied cognition emerged historically as a reactionary framework to the above, it can be characterized as what it argues against. In this sense, embodied cognition is the perspective that *denies* that knowledge is independent from perceptual systems, bodily action, and their operations. Roughly speaking, the opposite of this is the embodied thesis, which states that cognition is bound to what our bodies can do (see Dreyfus & Dreyfus, 1999). Specifically, our bodies—both by their inherent properties and how they incorporate prosthetics (e.g., glasses, walking canes)—restrict and regulate our cognition.[4]

So, it is a fact of embodiment that we have hands that grasp, and the claim coming from the emerging field of embodied cognition is that bodily facts, such as our prehensile hands, affect our thoughts. Again, this contrasts with traditional accounts of cognition, where the body merely provides inputs and outputs for what goes on "inside the head."

Over the last two decades, strict dichotomies of mind and body have been put into question by empirical evidence to the contrary. For example, contradicting predictions by the traditional models of cognition, research indicates that hills appear steeper when wearing heavy backpacks or when tired (see Proffitt, Bhalla, Gossweiler, & Midgett, 1995; also see Eves, 2014). This suggests that we process distances not in abstract representations like meters or feet, but in terms of bodily exertion. Examples of findings indicating this sort of embodied effect on cognition are myriad (see Barsalou, 2008; 2010; Kiefer & Barsalou, 2013; Nemirovsky, 2003 for additional examples). Rather than list them all, I will only mention a few more, so as to provide some detail of the current landscape.

Merely thinking about a hammer activates the same areas of the brain used when actually grasping a hammer (Martin, 2007). This neural activation of a hammer is neither symbolic nor abstract: it concerns our everyday, intimate, and practical knowledge of hammer use. Indeed, even abstract thinking activates the same neural resources that are also active in bodily perception and action (Soylu, 2011).[5]

Furthermore, even symbolic manipulations such as "moving" symbols from one side of an equation to another have been shown to be cognitively processed as literally moving elements from one side of the equation to the other (Goldstone, Landy, & Son, 2009). Because of this, embodied cognition is sometimes characterized as the account that cognition is embodied in the sense of being grounded in neural simulation of perceptuomotor action (Landy, Trninic, Soylu, Kehoe, & Fishwick, 2014). Nearly a century ago, Soviet psychologist Lev Vygotsky anticipated these empirical findings:

> Even the most abstract thoughts of relations that are difficult to convey in the language of movement, like various mathematical formulas, philosophical maxims, or abstract logical laws, even they are related ultimately to particular residues of former movements now reproduced anew. (1997, p. 162)

And, finally, research has shown that the tools we shape, shape us *literally* by modifying our brains and our interactions with the world in fundamental ways (Kirsh, 2013; arguably another empirical result Vygotsky anticipated).

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[4] Thus we may find ourselves in an awkward position of having to argue that, sometimes, our bodies do our thinking for us. One solution, addressed in Chapter 4, is to consider "thinking with our bodies" a genuine form of thinking.
[5] It may also be worth considering that the part of the brain primarily associated with abstract thinking and problem solving, the frontal lobe, is also associated with coordinated movements and fine motor skill.

In short, the field of cognitive sciences is increasingly beginning to accept Haugeland's 1998 claim that

> If we are to understand mind as the locus of intelligence, we cannot follow Descartes in regarding it as separable in principle from the body and the world….Broader approaches, freed of that prejudice, can look again at perception and action, at skillful involvement with public equipment and social organization, and see not principled separation but all sorts of close coupling and functional unity…Mind, therefore, is not incidentally but *intimately* embodied and *intimately* embedded in its world. (pp. 236-237, as cited in Kiverstein & Clark, 2009).

Whereas there exist multiple accounts of embodied cognition (see Kiverstein & Clark, 2009), all accounts view the dichotomy between mind and body, thinking and doing, as a historical error, or a mistake, to be remedied by articulating how these two seemingly disparate aspects of our lives are in fact deeply interrelated.

For the purposes of this dissertation, I make use of a specific subfield of embodied cognition which may be labeled *radical embodied cognition* (Abrahamson & Trninic, 2015; Chemero, 2010; Hutto & Myin, 2013; Melser, 2004; Nemirovsky, Kelton, & Rhodehamel, 2013; Roth & Thom, 2009; Verela, Thompson, & Rosch, 1991). Whereas embodied cognition in general claims that embodiment and interaction affect or reveal our thoughts, radical embodied cognition holds that embodiment and interaction not only shape and reveal but constitute our thoughts. This approach holds that our cognition is created through interaction; consequently, interaction literally enacts aspects of the world: "Cognition does not represent a world, it creates one. This point is one of the earliest insights driving enactivism" (Reid & Mgombelo, 2015; also see Kieren, 1994). This does not mean that the practice literally creates something new in the physical world, but that through "a history of recurrent interactions" (Maturana & Varela, 1992) something is enacted that wasn't there before.

I return to embodied cognition in general, and radical embodied cognition in particular throughout this dissertation, and at length in Chapter 4.

For now, I hope I have offered enough convincing reasons that embodiment may be a topic worth investigating in the context of pursuing fundamental educational questions regarding conceptual development. If the reader agrees with this statement, then it follows that traditional dichotomies of mind and body should be looked into anew.

1.3 Incorporating Embodied Cognition into Mathematics Education Research

In this section I first address a concern regarding the application of embodied cognition to mathematics in particular, and then outline some reasons as to why a researcher in mathematics education might wish to incorporate the framework of embodied cognition.

1.3.1 Symbolic, discursive, logical—Why might embodiment apply to mathematics?

What remains to be addressed is an elephant-in-the-room argument *against* the applicability of embodied cognition to mathematics. It goes as follows: practice in mathematics, by and large, involves symbolic manipulation, logic, and discourse. Hence, the argument goes, mathematics is more akin to and should be understood as something like a symbolic *language* rather than a physical activity. Variants of this argument make up the most common concerns I have heard from other scholars (typically at conferences) over the last four years. In response, I will follow an intellectual strain, which I trace to Wittgenstein, by which mathematical symbolism refers to mundane situated actions of people engaged in enacting the cultural practice of mathematics.

It seems to me that an intellectually honest approach begins with acknowledging that out present understanding of cognition is limited. Yet although the science moves slowly, the educator must teach, and the designer must dream up better learning environments. In other words, while we wait for final word on scientific understanding of how the brain works—and, more importantly, how we humans work—we have to continue with our daily jobs. Due to this practical constraint, it will not do to wait for evidence that embodied cognition is found generally applicable to mathematics education *beyond a reasonable doubt*, to borrow a legal term. Practically speaking, it must be enough to consider *where the preponderance of evidence lies*: to pursue what is convincing rather than what is undoubtedly true. In what follows, I present some arguments that I found convincing.

In resolving this tension of mathematics as an activity versus mathematics as a language, I find it helpful to draw on Ludwig Wittgenstein, arguably the most influential philosopher of the previous century, and particularly on his account of language and symbol use. While my training in philosophy is limited to my undergraduate degree, I have supplemented my understanding of Wittgenstein—at least, as his work relates to the issues at hand—through personal communications with two Wittgensteinian scholars, Derek Melser and Daniel Hutto.

Wittgenstein's overall body of work is difficult to pin down. His earlier writing concern the hidden nature of logical meaning behind language use (*Tractatus Logico-Philosophicus*, 1921), yet his later writings concern the meanings of language hidden in plain view, that is, meanings in context (e.g., *Philosophical Investigations*, 1953). Here I will be mostly concerned with his later writings, namely Wittgenstein's articulation of the role symbols and language play in our lives.[6]

But first, what does Wittgenstein have to do with embodied cognition? Recently, scholars have made arguments that much of Wittgenstein's later works can be rightly understood as early accounts of enactivism. For example, Hutto writes:

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[6] A brief note on Wittgensteinian argumentation and citation. Wittgenstein wrote with little regard for the standard academic conventions of writing. By many accounts, including his own, Wittgenstein was not interested in *logically* convincing the reader of anything, but rather in impressing upon the reader that something important was going on, and that this was worth pursuing. Consequently, Wittgenstein's writing—most of it not published during his lifetime—typically took the form of enumerated short paragraphs, which he called *remarks*, of which there is sometimes a fairly long chain about the same subject, and sometimes a sudden change, jumping from one topic to another. These remarks were an invitation for the reader to work through the problems presented.
The accepted method of citing Wittgenstein is to list the name of the work (e.g., Philosophical Investigations, or PI) and provide a number corresponding not to the page but the remark. I follow that convention here.

> [B]oth Wittgenstein and enactivists prioritise and highlight the primacy of ways of acting over ways of thinking when it comes to understanding our basic psychological and epistemic situation. Both give pride of place to what is done in the world over what is thought about the world, or how the world is represented. And both are committed to the idea that natural ways of acting both foster and come to be shaped and developed by customs, practices and institutions. For both enactivists and Wittgenstein, it seems, recognizing these general facts about minded beings is necessary if one is to understand correctly how humans and other animals are situated in the world and how they relate to one another (2013, p. 37)

In later works, Wittgenstein move away from his earlier position that meaning can be understood through careful attention to the hidden logic of things (e.g., by analyzing grammatical structures). Instead, he argued that meaning is hidden in plain sight: it is what we do. "I go *through* the proof and then accept its result. – I mean: this is simply what we *do*. This is use and custom among us, or a fact of our natural history" (*Remarks on the Foundations of Mathematics*, 1956, 63).

Wittgenstein emphasized that we play all sorts of games with words, yet he took these games seriously, criticizing accounts of language that treat as unimportant the activities in which our words are embedded. All of us use language as part of living—as part of trying to get things done. Melser writes:

> Maybe the main thing Wittgenstein says (albeit indirectly) that has not yet been widely understood, is that the primary function of language is not referential and fact-stating but imperative. It's to get people to do things…The imperative theory makes normativity fundamental in our lives. (personal communication, 2015; also see Melser, 2004)

Thus the meaning of mathematical symbols is not found in their reference to some Platonic form, but in what we do with them—and this doing cannot be logically reduced or analyzed precisely because it is couched in human normativity. Consider parallel lines: whether they meet or not depends on which set of rules we choose to endorse. This is what Wittgenstein meant by normativity—even mathematics is bound to whatever rules we *agree to follow*.

In the final analysis, Wittgenstein holds that even in the domain of producing and evaluating mathematics, "I act, without reasons" and this lack of *logical* reasons "does not trouble me" (*Philosophical Investigations*, 211-212). According to him, there is no secret, rational explanations of what it is that mathematicians do. Mathematicians do mathematics, a collective activity shaped by our social participation, same as a myriad of other human activities. It involves symbols, yet Wittgenstein—quoting Geothe's Faust—claims: "in the beginning was the deed" (*Culture and Value*, 1980, p. 31). Learning to do mathematics is better understood as immersion in activities that give meaning to mathematics, as evident in the fact that we can *choose* sets of rules among those with competing mathematical meanings.[7]

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[7] For example, do we endorse the axiom of choice or the axiom of determinacy? Ordinary Euclidean plane geometry or projective geometry? Whichever we choose, we chose not logically, as by deduction, but based on which we found more contextually useful, productive, or interesting (or aesthetically more pleasurable).

In short, Wittgenstein reminds us that language-use *is* action, or rather a communal activity we participate in. Mathematical symbols do not point to idealized Platonic forms but to actions actually undertaken by the practicing mathematician.

1.3.2 A few additional considerations for the incorporation of embodied cognition into mathematics education

One line of arguments *for* the incorporation of embodied cognition in mathematics cognition and learning research comes from personal accounts of scientists and mathematicians on how they experience mathematical knowledge. For example, in a letter to mathematician Jacques Hadamard, Albert Einstein wrote:

> Thoughts do not come in any verbal formulation. Words and language, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thought are certain signs or images, more or less clear, that I can reproduce and recombine at will…*The above mentioned elements are, in my case, of visual and some of muscular type.* Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will (Hadamard, 1945, pp. 142-143, emphasis mine).

In similar vein, Israeli mathematics education scholar Anna Sfard (1994) questioned mathematicians about mathematical concepts. In particular, she investigated if they operate on mathematical concepts in a way that is similar to physical objects. When asked to describe the sense of having a deep understanding of a mathematical idea, mathematicians responded with: "a structure [one is] able to grasp somehow", "to see an image", and "to play with some unclear images of things." One mathematician reported, "In those regions where I feel an expert….the concepts, the [mathematical] objects turned tangible for me" (p. 48). Another mathematician stated:

> To understand a new concept I must create an appropriate metaphor. A personification. Or a spatial metaphor. A metaphor of structure. Only then can I answer questions, solve problems. I may even be able then to perform some manipulations on the concept. Only when I have the metaphor. Without the metaphor I just can't do it. (p. 48)

This same mathematician also reported that the structure he uses has to have some spatial elements *no matter how abstract the mathematical idea*.

The final example of this sort comes from MathOverflow, a question and answer site for professional mathematicians. There, prominent mathematician and Fields Medal awardee Terrence Tao addressed the following question: "How big a gap is there between how you think about mathematics and what you say to others? Do you say what you're thinking? Please give either personal examples of how your thoughts and words differ, or describe how they are connected for you." Tao writes:

> For evolutionary PDEs in particular, I find there is a rich zoo of colourful physical analogies that one can use to get a grip on a problem. I've used the metaphor of an egg yolk frying in a pool of oil, or a jetski riding ocean waves, to understand the behaviour of a fine-scaled or high-frequency component of a wave when under the influence of a lower frequency field, and how it exchanges mass, energy, or momentum with its environment. In one extreme case, *I ended up rolling around on the floor with my eyes closed* in order to understand the effect of a gauge transformation that was based on this type of interaction between different frequencies. (Incidentally, that particular gauge transformation won me a Bocher prize, once I understood how it worked.) I guess this last example is one that I would have difficulty communicating to even my closest collaborators. Needless to say, none of these analogies show up in my published papers, although I did try to convey some of them in my PDE book eventually. (2010, emphasis mine)

Drawing on my background in mathematics, I too add my voice to the above examples. While I never rolled on the floor to understand advanced mathematics, I vividly recall paced around an empty room to make sense of a mathematical game involving a hunter catching a rabbit across the nodes of an infinite grid; the rabbit is able to start from any node, choose a random direction and from there continues deterministically and, most importantly, the hunter has limited information about the rabbit's location at any given time. This is a rough non-technical description of the problem, yet I hope sufficient to illustrate that I made use of everyday activity (walking) to work out a rather abstract mathematical problem.

I could provide further personal examples, or draw on ones in literature. For example, I refer the reader to Marghetis and Núñez (2013) for a study of the embodied sensations we draw upon when thinking of limits and continuity; or Nemirovsky, Rasmussen, Sweeney, and Wawro (2011) on the same for complex numbers; or de Freitas and Sinclair (2014) on *reading* mathematical texts as a form of re-embodiment. However, my main point is that, as Tao hinted above, although mathematical *products* are typically purged of such notions, what mathematicians actually do readily involves bodily sensations of all sorts—enough for us to take it seriously, I think.

1.4 Orientation of Present Work

We now return to the work at hand. Historically, I view my investigative approach as one branching off from certain historic proposed (re)directions of scholarship in mathematics education. I am referring to those occasional times when the field of mathematics education proper entertained analogies between mathematics and other, more evidently physical, disciplines. For example, as far back as 1983, at the fifth Annual Meeting of the North American Group of Psychology in Mathematics Education, von Glasersfeld stated:

> Ten or 15 years ago….[e]ducators were concerned with getting knowledge into the heads of their students, and educational researchers were concerned with finding better ways of doing it….Something, apparently, went wrong. Things did not work out as expected. Now there is disappointment, and this disappointment – I want to emphasize this – is not restricted to mathematics education but has come to involve teaching and the didactic

> methods in virtually all disciplines. To my knowledge, there is only one exception that forms a remarkable contrast: the teaching of physical and, especially, athletic skills. There is no cause for disappointment in that area. (pp. 41-42)

Later in his talk, von Glasersfeld urges mathematics education researchers to look to the teaching of physical skills for insight and inspiration which may be applied to mathematics education. It is telling of the spirit of times—and indeed this holds even today—that von Glasersfeld felt it necessary to voice caution that a comparison between physical and mathematical learning "might seem utterly absurd" (p. 42). If a researcher of von Glasersfeld's stature felt the need to make such qualifications, it should be unsurprising that few (if any) heeded his call.

And so, further steps in this broad direction were left to other established leaders in the field. One example is Alan Schoenfeld's 1997 *Making Mathematics and Making Pasta: From Cookbook Procedures to Really Cooking*. There, in a spirited response to Jim Greeno's theory of situated cognition, Schoenfeld looked to the discipline of cooking for insight on mathematics education, concluding:

> What is clear, however, is that I didn't spend the first *n* years of my culinary life memorizing recipes—or doing things like practicing boiling, or poaching, or sautéing, or broiling, or grilling, or… before I was allowed to make full recipes. Basic skills were learned, sometimes with drill, in the context of meaningful work; and the work, like the work of writing this paper, was a pleasure. I can only hope that we will learn to create learning environments for mathematics that have the same properties (p. 318).

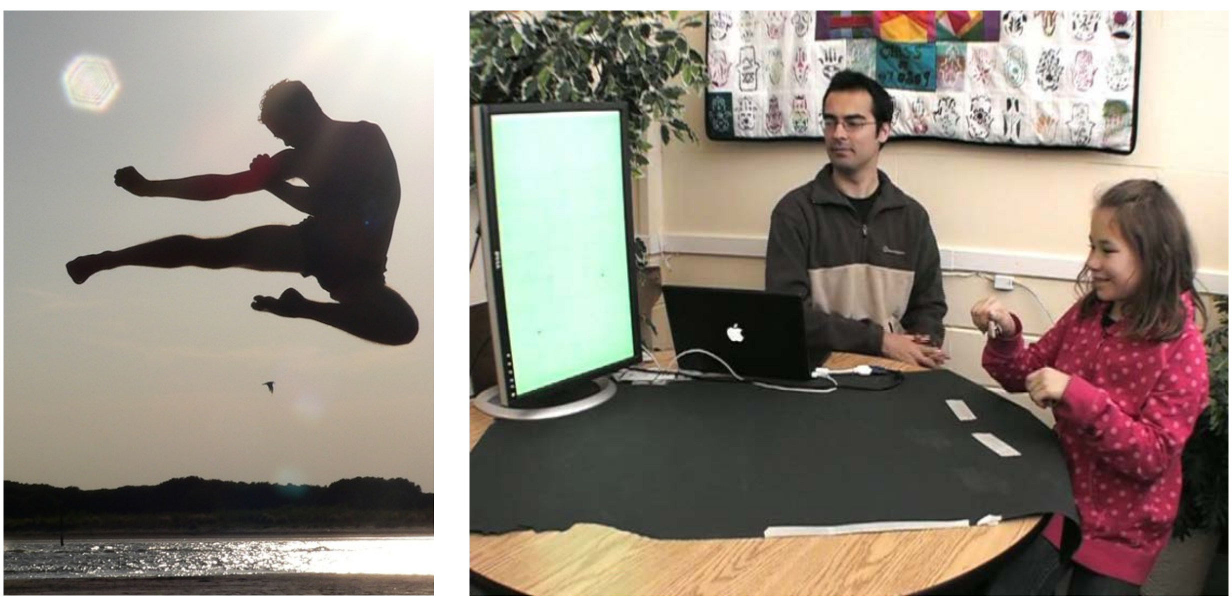
Similar to Schoenfeld, I look to disciplines outside of mathematics in order to characterize areas for improvement within mathematics education.

Specifically, this text revisits the apparent dichotomy of procedural performances and conceptual understanding from a novel vantage point afforded by investigations of pedagogical practices found in *explicitly embodied domains*. The term explicitly (or overly) embodied domains is used to refer to those domains wherein the core practices are directly visible to interested observers—as compared to, for example, what Rodin's *le Penseur* is doing. These domains include arts, such as traditional dances, and crafts, such as herbalism. The arts and crafts literally "embody the knowledge artisans have had with the materials of nature and the circumstances of their communities" (Borgmann, 1992, p. 121). In these domains there is no strong tradition of distinguishing between procedural performances and conceptual understanding. Dancers, as a characteristic example, "move to think" (see Ken Robinson's 2006 TED talk) and "use their body to think with" (Kirsh, 2011).

These investigations of explicitly embodied domains are one of two interwoven strands of research pursued in this dissertation. The first strand, mentioned above, involves ethnographic research in the explicitly embodied disciplines. Some disciplines have pedagogical traditions spanning centuries; to a researcher, these serve as a time capsule granting a peek at time-tested ways in which our ancestors interacted with their worlds and also at how they taught others to do the same. This strand of research, discussed at length in Chapter 2, aims at the generation of conjectures regarding pedagogical approaches to the teaching of procedures and concepts. Since

2007, I have conducted investigations of learning and teaching in explicitly embodied domains of Tai Ji Quan (California, Hong Kong, and Beijing); surfing (Hawaii, California); and traditional herbalist practices (the Western Balkans). In most cases, I document not only the practices of other participants as an insider, but also narrate my own (see Harris, 1976, on *emic* accounts of cultural practices). The research approach taken here was not in search of universal laws but, rather, an interpretive, immersive one in search of meaning and generating conjectures (see Geertz, 1973; Wacquant, 2011).

The second yet interrelated research strand involves interrogating instantiations of these conjectures via task-based clinical interviews (Clement, 2000; Ginsburg, 1997) centered around a technology-enabled learning environment for mathematics. This strand of work involves turning theory into design, or design-based research (see Abrahamson, 2009; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003); that is, the process and product of design serve as a practical lab for thinking about the interplay of procedural performances and conceptual understanding. In these learning environments, described at length in Chapter 3, students entrain motions in the service of performances that can be interpreted as mathematical. As they master the physical movements, students are asked to interpret their action and offered various mathematical tools to do so *without* being explicitly asked to make use of them. Yet the interviewer is not entirely hands-off during these interactions; in line with pedagogical practices observed in explicitly embodied disciplines, interviewers served also as a tutors or coaches. Importantly, the results at this stage of research can evaluate the robustness of earlier conjectures (see Hoadley, 2004, on interweaving social and design research) on the relation of procedures and concepts and, thus, influence further investigations on the topic.



a. b.

Figure 1. Pedagogical traditions in (a) explicitly embodied disciplines are oftentimes profoundly different from those found in disciplines of formal education, like mathematics. Yet if also mathematics reasoning is embodied, then the present dichotomy of bodily skill and mental concept may be pedagogically deleterious. Rather than ignore, how might (b) mathematics education embrace embodiment?

1.5 Summary and Dissertation Outline

In sum, this work is motivated by a desire to make sense of the constructs of action and understanding, or procedure and concept, as they apply to mathematics education practice and research. As such, my dissertation grew from a perceived need to reconceptualize the procedure/understanding dichotomy that underlies, and I argue undermines, much discussion in mathematics education. I take the study of embodiment as a promising avenue towards resolving this dichotomy. To generate new conjectures regarding the pedagogical roles of procedures and concepts, I look to the pedagogical practice of explicitly embodied disciplines. One major reason for this is that motor learning is an easier research problem than mathematics learning—for within *explicitly* embodied domains of practice, expert and novice actions are physical and typically pragmatic, affording researchers greater transparency onto those subjects' reasoning processes as compared to the case of implicitly embodied mathematical activity (Trninic & Abrahamson, 2012).

Findings emerging from these investigations are then used to generate conjectures informing the design and analysis of embodied, mathematics learning environments. The two strands—studies of (1) explicitly embodied disciplines and of (2) embodied design for the learning of mathematics—complement each other: conjectures from one are pursued in the other, and so on, iteratively.

In short, this program's overarching intellectual gambit is to sidestep from mathematics to motor skill, learn over there about practices and processes of teaching and learning, and then sidestep back to mathematics, where parallels are put into practical and theoretical use. I argue that if we are to take the premise of embodied cognition seriously, then doing mathematics is more akin to dancing than to the seemingly disembodied products of mathematics we call proofs, definitions, and theorems (but see Thurston, 1994). To this end, I am interested in how, if at all, actions become concepts and how insight into this can inform educational practice in the mathematical discipline. Thus I hope to avail of humanity's ancient pedagogical traditions while struggling to fashion a contemporary mathematics pedagogy.

Next I will provide an outline of the remainder of the dissertation. The following chapter, Chapter 2, makes forays towards resolving the action/concept dichotomy by introducing an alternative way of seeing the educational process. Earlier I argued that understanding in explicitly embodied disciplines is characterized *without* invoking the procedure/understanding dichotomy. There, routines are initially learned *on trust* yet subsequently—only toward perfecting the procedures toward mastery and further dissemination—are interpreted by experts as embodying disciplinary knowledge and understanding. I chart out this process in traditional Chinese martial arts as well as surfing, emphasizing that certain routines are taught and practiced not for their own sake but because they provide the practitioner with a felt sense of the underlying disciplinary principles and thus opportunities for further insight into the discipline. The central claim coming from this strand of research is that the "secrets" of a discipline are not taught directly, because they cannot be acquired via mimicry, mirroring, or following a set of instructions, but indirectly through participating in particular forms of practice. These "secrets" are understood as the learner's embodied responses to emerging contingencies (Bernstein, 1996);

as such, they necessarily emerge through varied practice. (Later I contrast this with the emphasis on drills in mathematics education, where drills are practiced merely for the sake of later performing nearly identical drills with increased speed and accuracy, rather than developing new insights.)

Whereas Chapter 2 focuses on education in explicitly embodied domains, Chapter 3 investigates how ideas emerging from these findings may be pursued in mathematics education. To that end, I describe the creation and implementation of a novel technology-enabled learning environment for the concept of proportion, the Mathematical Imagery Trainer for Proportion (MIT-P). With the MIT-P we attempted to design a learning environment that entrains routines which, once signified mathematically, provide insights into the mathematical principle of proportion. Attention is placed on a phenomenon called *hook-and-shift*, wherein disciplinary "secrets" emerge as an (unexpected) discovery during student's interaction with the learning environment. Specifically, I show cases of students who *find themselves* making use of mathematical forms ultimately to make sense of the concept of proportion. One major finding coming from this chapter concerns highlighting this unanticipated, opportunistic, yet serendipitous role of discovery. The short of it is that the introduction of new forms into a given interaction space results in novel yet unexpected actions, observations, and intentions.

Broadly speaking, both Chapters 2 and 3 deal with learning as a sort of unintended consequence brought about by the introduction of novel cultural forms. In general, studies that take a serious look at inadvertent discovery and unintended consequences are rare (but see Saxe, 2012; and Trninic, Gutiérrez, & Abrahamson, 2011). In this sense, my interests are similar to that of scholars who study the introduction of novel forms into social groups over periods of time (e.g., Saxe & Esmonde, 2005); however, my work is more focused on individuals and their moment-to-moment shifts introduced by interacting with artifacts.

In the final chapter, I revisit the original question concerning dichotomies of action and concept, performance and understanding. Positioning my work as an extended generative case study (Clement, 2000), I offer a conjectural alternative to the aforementioned dichotomy. A theoretical perspective is outlined wherein the heart of the relation between procedures and concepts lies in something as ordinary as repeated action. That is, I present and argue for a framework where conceptual understanding emerges from and is interwoven with particular forms of practice. I view this finding as aligned with Artur Schnabel's proclamation that practice "should be experiment, not drill" (Bamberger, 2013, p. 3). Though Schnabel used to say this in the context of music education, I extend it to mean that one primary role of practice *in any discipline* is in providing opportunities for students to develop a felt sense of the underlying principles as a step towards the sorts of reasoning we call conceptual understanding.

This perspective provides an alternative conceptualization of procedure/routine/practice versus knowing/understanding/concept; so doing, it also reorients the learning paradox (see Bereiter, 1985), which has intrigued scholars since Plato's *Meno*. The paradox can be posed as a question: "How can a student *intend* to acquire knowledge he is *not aware of* in the first place?" Framing practice as experiment shows a way out: a student's understanding, I offer, is not *intended,* but an emergent phenomenon that arises in and through situated action (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011; Roth & Thom, 2009; Thelen & Smith, 1996). This type of learning— characterized as much by the fortuitous as by forethought, and emerging from seemingly unexceptional doing—is by its nature unpredictable or even uninteresting. In spite of this, or rather, because of it, I submit that this process should be further investigated if our field is to provide a rich and robust account of learning.

Chapter 2: How to Teach Secrets

> We focus on the ways in which the information conveyed by instructions could facilitate learning through searching.
> (Newell & Ranganathan, 2010, p. 20)
>
> Let the use *teach* you the meaning.
> (Wittgenstein, 1953, p. 212)

Chen Fake, a Chen style taijiquan grandmaster is reported to have said, "There are three steps to learn taijiquan: first to learn the moves, then to practice often, and finally understand the details." I think it worthwhile to consider the account of learning implied by this statement.

We can start by noticing that the learning trajectory *begins* not ends with "learning the moves." Taijiquan—a Chinese martial art, better known in the United States under its Wade-Giles transliteration Tai Chi (I will use taiji)—is best known for its forms, or choreographed sequences of movements. Contrasting this with the above quotation would imply that taiji is about something more than these movements: it is about "understanding the details." Putting aside just what those details are for the moment, we see that students transition from memorized sequences of *moves* to *understanding* through *practice*.

In this chapter, I turn to explicitly embodied disciplines in order to coin and elaborate on a novel construct, *enactive artifacts*. Enactive artifacts are disciplinary routines such as, for example, an arabesque penchée in ballet or Karate *kata*. The central argument I will present is that enactive artifacts foster not only local skill development per se, that is, becoming better at some particular movement, but are methods for gaining insight into disciplinary knowledge. In other words, the central argument is that enactive artifacts encapsulate not only particular cultural ways of coping with problem situations, but that, through their practice, these routines serve as *methods* of developing disciplinary knowledge. Thus we find enactive artifacts not practiced for their own sake, but for what is gained through practice.

Why go about education in this way? Why not simply teach, directly, what the student needs to learn? A number of reasons. One is that some knowledge may be simply too dangerous to practice "realistically"—for example, defending against a knife attack. However, the primary reason behind this practice—one I wish to explore here—is that some aspects of knowledge simply cannot be taught directly, that is via expert modeling or exposition. Instead, practice becomes *an exploration and recognition* of possibilities too nuanced and too context-dependent to convey via language. In the course of this exploration, enactive artifacts both bring forth and highlight features that may otherwise remain invisible, features that are then disciplinarily signified as instantiations of principles. Hence—in explicitly embodied disciplines—understanding is seen as obviously, and necessarily, emerging through one's practice.

2.1 Through practice to Practice

In the previous chapter I suggested that understanding in explicitly embodied disciplines, such as surfing and martial arts, is characterized *without* invoking the dichotomy of procedural fluency versus conceptual understanding. How? This chapter answers this question by investigating pedagogical practices where procedures are initially learned *on trust* yet subsequently—once the student *investigates* or *explores* the procedure through practice and reflection—are interpreted as embodying disciplinary knowledge and understanding. My research group coined this phenomenon *Action Before Concept*, or ABC (Abrahamson & Howison, 2008, used *[Dynamical] Image Before Concept*; later *Action Before Concept* in Trninic & Abrahamson, 2013).

To elaborate on these pedagogical methods, emerging from my research in this area is an articulation of a novel construct. Here, the use of new theoretical constructs is indicative of a need to reorient the discussion away from the traditional action/concept dichotomy. The construct is *enactive artifact*—rehearseable action sequences that serve as resources for encountering and coping with particular problem situations in the world (see Trninic & Abrahamson, 2012). The name of this construct is meant to evoke reference to both embodied cognition, by emphasizing the enactive nature of knowing (e.g., Varela, Thompson, & Rosch, 1991), and sociocultural theory of learning, where artifact mediation plays a central role (e.g., Vygotsky, 1982).

Acting with and through enactive artifacts, the practitioner is able to solve previously intractable or even invisible problems in the world, a process through which these enactive artifacts reveal *what they are for* and thus acquire meaning. Thus even in explicitly embodied disciplines, it is not the case that *knowledge* per se is transmitted from teacher to student, but rather the *method* for acquiring that knowledge. Enactive artifacts are the vehicle of this method.

To further illustrate this point I will explore a disparity between taiji the historical discipline and taiji as it is currently perceived in the United States.

Taijiquan is literally translated as *ultimate supreme fist*; a more accurate translation might be *supreme boxing* or *supreme way of fighting*. Originally it was called *zhan quan*, or *touch boxing*, which happens to be an accurate description of its martial method. It is a martial art, often said to be the most popular martial art in the world, practiced by over 250 million people worldwide. And it is a discipline I will refer to most extensively in this chapter.

Like any complex, culturally rich ancient practice, taiji is difficult to summarize in a sentence, or even a few paragraphs. And yet anything beyond a cursory account is unfortunately beyond the scope of this dissertation. Perhaps the best plain-language summary is to describe taiji as a sort of practice in which awareness and manipulating of one's balance and touch-sensitivity (i.e., physically making contact with another person) are utilized for a martial purpose. By *balance* I literally mean what is commonly understood as balance, or rather the ability to stay balanced. Simply imagine another person pushing you, and you stumbling and regaining balance—this is what I mean.

Through physical contact, such as placing one's hands on another person, or even blocking a punch (*redirecting*, because taiji has no blocks per se), a taiji practitioner senses the opponent's balance. Tactile sensitivity offers full-body information, including the relative strength of the opponent, his state of balance, the direction he is moving in, and so on.

Why would this sort of reading of an opponent's balance matter for a martial purpose? Imagine walking in the dark and, unexpectedly, stepping on a sheet of ice. You manage to catch your balance, just barely. Now, imagine that someone pushed you ever so gently in the opposite direction at the precise moment that you were attempting to regain your balance. Likely, you would fall, even if this same push would have had little or no effect on you had you been standing on solid ground. One way to talk about taiji's martial training is to say it consists of learning how to place opponents into this precarious position—even without a sheet of ice—and then helping them fall.

To consider a rough example, if I sense that my opponent's weight is mostly on the heels of his feet and that his upper body is leaning or moving backwards, a sudden downward and forward press will throw him on the ground—he will be unable to take a step back to regain his balance. Sensitivity and awareness are paramount here, because the same downward press executed at the wrong time would have little or no effect and may even place me at a disadvantage. This emphasis on fighting from a position of physical contact also explains the reason behind taiji's original name, *touch boxing*.

We are arriving at the central point. The fact that taiji is a martial art, a "*supreme boxing*" art at that, seems at odds with the most public aspect of the practice: slow-motion choreographed movement forms. Indeed, when taiji is featured in the western world, it is most often presented as a relaxing exercise or a slow-motion dance—in other words, decidedly *not* as a martial art. For example, a 2008 Jeep Liberty TV commercial features a young attractive couple engaged in a relaxing dance reminiscent of Yang style taiji. They smile at each other as they (unwittingly, one presumes) execute mock versions of martial takedowns, and the narrator intones:

> Move with ease. Enjoy your freedom. And find serenity… within.

An observer would be readily forgiven for assuming that taiji was a gentle form of partner dance, nothing at all to do with martial combat. In a similar vein, consider the following joke repeated by a number of American stand-up comedians:

> During my lunch break, I saw a taiji guy stop a mugger. He threw this really beautiful punch, I stopped walking just to admire it. Unfortunately I had to go back to work before he finished, so I've no idea how the fight ended.

Likewise, Jimmy Fallon, the host of *Late Night with Jimmy Fallon* (Season 5, Episode 212, October 2014), had this to say in his conversation with actor Keanu Reeves: "I see a lot of older people, in the park, getting their flow on… it's all about flow and balance and focus, and I don't really see it as a [martial art]."

I mention these to emphasize that the gentleness and relaxation evident in taiji practice *is* its public face in the Western world. And it is precisely this disparity I wish to consider, between taiji the relaxing exercise of today and historical taiji being labeled the "supreme boxing" *because its practitioners were peerless fighters*. The same art whose practitioners beat all comers at the Chinese royal court is now renowned for lowering blood pressure. How is it that the same discipline is both a relaxing health exercise and an effective martial art?

The answer is simple yet illustrates my central argument: taiji is a martial art that includes relaxing exercises as part of its practice. These exercises undoubtedly have health benefits. However, the original aim of these exercises was to develop a particular way of moving and interacting, the knowledge of which would then be useful in a martial situation. That is, the slow-motion choreographed movement forms so often seen as taiji were developed not as an end in and of themselves, but *as a method of acquiring disciplinary knowledge via practice*. To put it in another way, two hundred years ago taiji forms were practiced not because they helped the elderly with balance (although they do that as well—I certainly intend to continue practicing taiji in my senior years), but because this practice was part and parcel of an overall training regime that led to development of desired martial arts prowess.

To recap, martial-arts forms, such as forms performed by taiji practitioners, were originally practiced[8] not merely for the sake of becoming better at performing those forms (as a sort of dance), but for the sake of developing competences deemed valuables by masters of the discipline. This approach is not unique to taiji.

Consider yoga, with its roots in the practices of Indian religious ascetics, once done for the sake of spiritual enlightenment, now done as a trendy workout in the Western world, complete with merchandizing and celebrity advertisements. It is, of course, true that yoga can be a powerful workout; the point is that its original purpose was aimed elsewhere. Or simply consider all the time boxers spend jumping rope and working the speed bag. Jumping over a rope and rhythmically hitting a small suspended bag very fast are *decidedly not* the same as fighting another human being. Nonetheless, boxers consider these exercises vital in their training.

The point is that many disciplines involve routines, I call them enactive artifacts, methods of training or practice, that are done not only for the sake of improving say, one's overall health[9], or even for the sake of improving those particular routines, which are authentically germane to the practice, but because they serve to develop understandings of disciplinary knowledge that cannot be conveyed via verbal instruction or modeling. I call these sorts of understandings a *felt sense* (Gendling, 1991) of the underlying principles. It is a pre-reflective, embodied sort of knowing which may then become disciplinarily signified (but more on this later).

The idea that learners engage practices, particular routines which I call enactive artifacts, for the sake of developing disciplinary knowledge is a point I will come back to again and again throughout this chapter. The central argument I am attempting to make is that some aspects of knowing are far too nuanced and situation-dependent to be taught via verbal instruction or modeling. Instead, students are taught enactive artifacts, which are generally more

[8] It is, however, worth noting that at least some movements found in these forms in turn came from non-martial practices, for example religious ceremonies and performances.

[9] It is important to note that I am also *not* merely speaking of body conditioning, that is, increasing one's endurance or flexibility. Consider a swimmer: as she practices, the body becomes conditioned. She can hold her breath for longer periods of time, her shoulder and arm movements become more agile. But beyond endurance or flexibility, she also develops a way of moving the body that a novice simply cannot match, no matter how flexible or otherwise *fit* the novice might be.

straightforward and even demonstrable, and encouraged to practice these across a variety of contexts. Here, one's practice is a form of exploration in the sense of investigating how the practice necessarily changes across context, rather than a drill.

The following section will lay out a theoretical foundation for this claim, and the section after that introduces the methodology used in this investigation. Then I present examples of how enactive artifacts are pedagogically used in martial arts and surfing, after which I provide summative discussion and closing remarks.

Because this is an unorthodox direction for a dissertation on mathematics education, it may be useful to recall where this discussion is heading. I offer that this approach of practicing routines as *exploration* for the sake of developing personal meanings, present in explicitly embodied disciplines, is conspicuously absent in mathematics education wherein "answer getting" techniques, such as FOIL, are taught and practiced only for the sake of getting an answer.

My point is that practice can be valuable to students. Not only as drills to "get something cold" but as methods of exploration and developing a felt sense of the underlying principles. Yet, in mathematics, much of said practice is done only for the sake of answer getting—that is, as 'application problems' of pre-defined knowledge—rather than providing opportunity for the emergence of disciplinary insights.

Later, in Chapters 3 and 4, I will argue that it is possible and even desirable to practice mathematics without resorting to the notion of practice as menial drills (of the FOIL sort). I will argue for a *third way*, an alternative to both the account that children should begin learning a discipline through getting the drills down cold and the account that children ought first to focus on conceptual meanings. Instead, I argue that children should, whenever possible, practice engagement with disciplinary notions even before these ideas are formalized. Yet this engagement need not be in the form of drills, but explorative and instigative practices. In the present chapter, I outline how this third pedagogical approach lives in martial arts.

2.2 Repetition-without-repetition

In mathematics, students' practice often takes the form of drills. For example, while studying for a geometry exam, students might hear, "You'll have to know all your constructions cold" (Schoenfeld, 1988, p. 155). I argued that, in disciplines such as martial arts, practice often takes a form of exploration (that involves meaning-making), and the goal of practicing a routine is often about more than the routine. In this section introduce a theoretical account of skill development I have found useful in my thinking about these sorts of routines.

Independently of the work of Piaget or, for that matter, Vygotsky, the Soviet neurophysiologist Nikolai Bernstein (1896-1966) developed a highly original account of skill development, summarized by his claim that skill development is best understood as "repetition without repetition" (Bernstein, 1996). On this, Bernstein wrote:

> The actual importance of repetitions is quite different [than what has formerly been believed]. *Repetitions* of a movement or action are necessary in order to *solve a motor*

> *problem* many times (better and better) and *to find the best ways* of solving it. Repetitive solutions of a problem are also necessary because, in natural conditions, external conditions never repeat themselves and the course of the movement is never ideally reproduced. Consequently, it is necessary *to gain experience relevant to all various modifications* of a task, primarily, to all the impressions that underlie the sensory corrections of a movement. (Bernstein, 1996, p. 176)

He further elaborates:

> A trained athlete's consecutive running steps are as identical as coins of the same value, but this identity results, not from the brain's ability to send absolutely identical motor impulses to the muscles, but only from the faultless work of sensory corrections. (p. 180)

Thus skill development is not in perfecting an ideal motor routine but in developing an implicit ability to modify the routine across situations and, so doing, developing a repertory of agile fixes to emerging contingencies. This repertory takes the form not of internal mental representations but bodily sensations which Bernstein called automatisms.

Bernstein's theory of skill development predicts that practice is best understood as exploration of potential bodily responses when attempting to execute a routine across changing circumstances. This prediction is validated by the "varied practice" literature in motor skill development. Specifically, Shea and Morgan's influential study (1979), replicated many times since its publication, analyzed the effects of "contextual interference" on learning. The central finding was that people develop more robust skills and understandings when training in an environment or task context that encourages a more explorative approach to movement repetition (rather than an environment in which each repetition *feels* the same as the last).

Bernstein warns educators, "The fact that the 'secrets' of swimming or cycling *are not in some special body movements but in special sensations and corrections* explains why these secrets are impossible to teach by demonstration" (p. 187). While it is impractical to go into Bernstein's entire theoretical framework here, for our purposes it is important to note that he distinguishes between different stages of learning a skill. Specifically, the earlier stage of learning a motor skill—labeled *motor coordination*—can be taught directly, either by means of verbal instruction or modeling. Bernstein states that all *movement* can be learned directly, that is, via mimicry, demonstration, or verbal instruction. So a student can be directly taught to move his left foot here and place his right hand there. Yet the later stage—that of *error correction*—can only be taught indirectly, that is, by providing students with situations where they can develop the necessary sensitivity to recognize and make such error corrections (see above on "repertory of agile fixes to emerging contingencies"). On this, Bernstein writes:

> We can observe how the craftsman performs seeming clear and simply movements, but we cannot see from outside the concealed corrections…that take place in his brain. The difference between motor composition and the creation of correction is exactly in that: During the motor composition phase, a novice decides how the movements involved in a motor skill look from the outside; during the third phase, the novice learns how these movements and their sensory corrections *feel inside*. (pp. 184-185)

This theoretical perspective supports the notion that practicing enactive artifacts—a form of repetition-without-repetition—is about more than improving that particular routine. It is, I posit, about providing students with a method of developing their own understanding. I will contextualize these claims in the later section where I present three enactive artifacts—ranging from simple to more complex—and examine their role in the development of disciplinary knowledge.

2.3 Methodology

> The concept of habitus supplied at once the anchor, the compass, and the course of the ethnographic journey… [it] is the *topic* of investigation…. but it is also the *tool* of investigation: the practical acquisition of dispositions by the analyst serves as technical vehicle for better penetrating their social production and assembly. In other words, the apprenticeship of the sociologist [into the very practice that is the focus of ethnographic investigation] is a methodological mirror of the apprenticeship undergone by the empirical subjects of the study; the former is mined to dig deeper into the latter and unearth its inner logic and subterranean properties. (Wacquant, 2011, pp. 81-82)

This dissertation is grounded in extensive fieldwork and in the understandings and intuitions produced *by* and *in* me during my experiences in the field. In this process I have relied on my fieldnotes and on video-recorded speeches, conversations, interactions, interviews, and archival data. Over the years, I have investigated surfing, a number of martial arts (including taiji, Yiquan, Systema), scythe harvesting, and medicinal herb gathering. For the purposes of my dissertation, I am focusing primarily on taiji (Berkeley, California), as well as peripherally on surfing (Honolulu, Hawaii).

Across both surfing and taiji, my primary mode of data collection has been one of immersive (or carnal) ethnography (Wacquant, 2011, 2013). While this approach guides my work, it is admittedly unorthodox in education (but see Sánchez García & Spencer, 2013). Consequently, I will discuss immersive ethnography at some length here in order to better explain why I choose this particular approach.

The central premise of immersive ethnography, captured in the above quote at the start of this section, is that the ethnographer's body is deployed as both empirical object (*explanandum*) and method of inquiry (*modus cognitionis*). In other words, in this approach, an ethnographic study of learning a practice involves personally going through all the motions of *actually learning those disciplines*. The experiences are not only used while analyzing data, they *are* also data. In practice, this means that field notes are written as a practitioner rather than a researcher; afterwards, these field notes are data to be analyzed. While I took occasional and brief videos (using a hand-held smart phone device), these videos were again taken as a practitioner rather than with an eye on recording researchable moments. Other members of the community experienced my presence as that of a fellow practitioner rather than a friendly scholar.

Clearly, this sort of data of one's own learning progression is less objective, than, say, a videography. Yet it is not "anything goes." Care is taken in order to be honest to one's experiences *as they were* rather than veer off into fantasy about how they could have been. For example, if I struggled with a disciplinary notion that later, to my chagrin, appeared trivial, I would nonetheless be honest to my experience of that struggle as it unfolded rather than how it appeared after the fact.

In a nutshell, immersive ethnography can be summarized by declaring that, if the goal is understanding a practice, watching from the sidelines is not enough:

> A carnal sociology that seeks to situate itself not outside or above practice but at its "point of production" *requires that we immerse ourselves as deeply and as durably as possible into the cosmos under examination*; that we submit ourselves to its specific temporality and contingencies; that we acquire the embodied dispositions it demands and nurtures, so that we may grasp it via the prethetic [pre-reflective] understanding that defines the native relation to that world—not as one world among many but as "home". (Wacquant, 2005; p. 466, emphasis mine)

Wacquant developed immersive ethnography to advance certain theoretical constructs in sociology (namely *habitus*, see Bourdieu's landmark *The Logic of Practice*, 1980), and I make use of it because of its applicability to understanding the role of embodiment in the learning of a discipline. To understand how people become practitioners of a discipline and come to understand its "secrets," become a practitioner of that discipline and understand its secrets! As might be already evident, Wacquant himself is a supporter of the embodied turn in cognitive sciences:

> Academics live under the comforting illusion that "physicality" is a property of a restricted class of practices that does not concern them because the specificity of scholarly embodiment resides in the radical effacement of the body proper from the phenomenological foreground: the scholastic condition as withdrawal from practical urgency intensifies the modal experience of bodily absence. But the most "mental" of actors, such as the mathematician or the philosopher, are incarnate beings; and thinking itself is a deeply corporeal activity, as the embodied cognition movement is now showing from within cognitive science. (2013, p. 13)

I was initially exposed to this ethnographic perspective through Raúl Sánchez–García, a sociologist of sport who spent half a year in Berkeley, working with my research lab during the 2012-2013 academic year. He, along with Dale Spencer, put together *Fighting Scholars* (2013), an interdisciplinary and international suite of field studies of martial arts and combat sports by researchers who learned and practice the bodily craft they dissect. Fighting Scholars, in turn, was inspired by Wacquant's earlier study of boxing. As I was already studying martial arts, and interested in the role of embodiment in learning, this approach immediately drew me in.

A note on validity. Because the observer becomes a practitioner through-and-through, this methodological approach is particularly well-suited in addressing the standard question of validity: Do researchers actually observe what they think they observe? Thus, what I believe I observed in other practitioners I corroborate with my own experiences. Some additional strategies to enhance design validity include interviewing other participants and occasional mechanical recordings (video taken via unobtrusive hand-held smartphone). Yet, while I can be reasonably confident that I genuinely observed the meanings and practices I describe here, this sort of work also is weak with respect to external validity, or generalization. As is evident in the following section, no attempts were made to become involved with “representative” groups of practitioners (whatever that might look like). Quite the opposite—I sought out *exceptional* teachers of martial arts. However, external validity was not the explicit aim of this research, and may be considered a future goal for this sort of work.

2.4 Research Sites and Disciplines

When choosing which disciplines to investigate for my dissertation work, I elected surfing and martial arts. The reasons were largely pragmatic, in the sense that I had excellent access to both these disciplines (and not some of the others I mentioned earlier; for example, the only experts in medicinal herb gathering I know personally live in remote regions of the Balkans). Initially, I only intended to use my studies of martial arts. By including surfing—an explicitly embodied discipline in which I am an absolute beginner—the hope was that the introduced differentiation would allow for a richer data set. That is, I picked these disciplines because I am a novice at surfing yet an intermediate martial artist. As such, my role as a surfer is different (a novice learning the ropes) than my role as a martial artist (one capable of helping novices).

Surfing fieldwork was collected over four separate instances: July 2010, January 2011, August 2013, and August 2014, in two-week stretches. Surfing is a surface water sport in which a surfing board is used to “ride” the forward face of a wave. The primary site in each case was the Waikiki beach in Honolulu, Oahu, Hawaii, specifically Queens and Canoes surfing area(s) (see Figure 1a, below). During this time, I would stay with a local contact and friend, Marcus, an experienced surfer in residence approximately 15 minutes (walking) from the beach. Queens and Canoes are well-known Waikiki reefs that attract experienced surfers and are considered an excellent spot for novices as well. This, alongside its location, contributes to what the locals consider to be the most crowded surfing line-up in the world.

I participated in surfing in the following manner. First, I attended a lesson with a local instructor (see Figure 1b, below). Through my local contact, this lesson was arranged to be semi-private and the instructor was available for brief informal interview immediately after the lesson (difficult to achieve on one of the busiest beaches in the world). After this experience, I practiced surfing with Marcus and, during later visits, occasionally by myself, typically surfing on an 8-foot foam-soft-top surfboard, well-suited for beginners. When spending time with my local contact and his group of surfing friends, all of whom were experienced surfers, I was provided with informal lessons and advice. Because some of these individuals were surfing instructors, the lessons were of high quality; as a novice I found them immensely useful.

a. b.

Figure 1. (a) Waikiki beach with Diamond Head volcanic cone in the background, and (b) a surfing instructor, seated, prepares to propel a neophyte surfer forward (note Diamond Head in the background). I trained with this instructor the first time I attempted surfing.

In addition, on weekends, when my contact was off work, we would drive around the island, visiting well-known surfing locations, such as the Banzai Pipeline on the North Shore. I did not surf in most of these locations—they are too dangerous for the inexperienced (and even experienced). During these outings, I would find myself surrounded by amateur and professional surfers, trying to make sense of their vast and, to me, often confusing vocabulary dealing with the various aspects of waves, surfing techniques, and even surfing etiquette. As one might expect, I remain very much a beginner, and my account of surfing is appropriately an account of a novice “learning the ropes.”

Martial arts fieldwork was collected continuously from the beginning of October 2011 towards the end of December 2013. The primary site was a small, core group of taiji and yiquan practitioners meeting four to six times a week under the guidance of a martial arts master, Fong. Taiji was mentioned before, and yiquan is likewise an “internal” martial art relying primarily on cultivating one’s ability to use balance and sensitivity for a martial purpose. If taiji practice is characterized by slow choreographed movements, yiquan is characterized by holding static postures over periods of time lasting from a few minutes to, potentially, a few hours.

I was involved with this group of practitioners for almost year before I “officially” decided to use the site to collect data. In this work, I also drew on my previous experiences with taiji (2004 onward, intermittently) and my experience as a martial artist in general (1997 onward, on and off; most notably I studied Soo Bahk Do, a traditional Korean martial art, continuously for seven years 1997-2004). Specifically, for a few months in the summer of 2004 I had the opportunity to begin the study of Wu style taiji. (There are a number of taiji styles, with five of the most popular ones being, in descending order: Yang, Wu, Chen, Hao, and Sun styles. Despite differences in forms, each style emphasizes cultivating one’s ability to use balance for a martial purpose.)

My organized practice of taiji then stopped until the summer of 2011, when I spent approximately a week training Yang style short form, sometimes labeled the Beijing 24 form. This simplified taiji form was created in 1956 by the National Physical Culture and Sports

Commission of the People's Republic of China as part of the drive to standardize martial arts. I was initially exposed to this form in Hong Kong in July, 2011, and again observed it in Beijing some weeks later. Upon my return to the United States, I studied a variant of this form with the UC Berkeley Tai Chi Club for six months. It was during this time I met Fong, the aforementioned martial arts master, at a park in Berkeley.

In his youth, Fong was a student of the Yang style taiji masters Dong Yingjie and Yang Shouzhong (the latter being a great-grandson of Yang Luchan, the founder of Yang style taiji).

When I met him, Fong was sitting on a park bench enjoying a drink. When I told him I had some experience with martial arts, he held out one arm and, still sitting, asked me to push his arm, using as much strength as I wanted. This sort of "testing" is common and typically good natured among martial artists (it was in this case). Despite Fong's age at the time (early 70s; I was 28), our difference in heights (Fong stood at slightly over 5 feet; I am slightly over 6 feet), and my not altogether insignificant training background, I was unsuccessful in moving his arm any significant distance. Next, he stood up and declared he was about to push me, inviting me to stop him. Although Fong used only one arm, I found myself moved backwards without any obvious effort on his part. The sensation puzzled me—I experienced it as if *I* had moved backward, rather than him pushing me (later, I came upon a realization that this was not an inaccurate first impression). From this simple 'push hands' demonstration (see below), I understood that his knowledge was extraordinary, and began to study with his group in October 2011.[10]

Fong emphasized *practice* to what seemed to me an extreme degree, sometimes outright refusing to verbally elaborate on underlying principles or theory. He was also the greatest proponent for practice as personal exploration I have ever encountered. He expected all beginners to spend at least one hundred hours practicing standing postures, "That's the minimum to change your thinking." Early on in my training, I asked him to elaborate on how it is possible for a taiji student to strike if we are told to always follow the maxim of "no collision," one of his preferred expressions. After all, I told him, taiji is supposed to be soft, so how could anyone actually strike while being relaxed and without colliding? His response was, "You want to figure everything by understanding it intellectually. That won't help here. Do the work for at least one hundred hours and you'll start to understand." A couple of years later, when I asked him the same thing, he told me to brace myself, and struck me. It made sense, then: a taiji strike is not a collision strike like the one used by a boxer. There was no contradiction in Fong's teachings, but it was not something that could be accurately put into words, either.

The core group consisted of six students who attended most or all meetings every week. We met Monday and Wednesday evenings, and Tuesday, Thursday, Saturday, and Sunday mornings. Typical practice consisted of holding Yiquan stationary postures, working the taiji *chih*—a gentle, slow motion practice involving a small "ruler" held between one's palms, performing Yang style long form (often said to contain 108 moves, though the exact number depends on what counts as a move), and engaging in various cooperative and competitive partner exercises,

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[10] And so I was exposed to my teacher's skill the very first day I trained with him. I invite the reader to contrast this to students taking mathematics classes. How often do they see their teacher *actually* demonstrate mathematics rather than copy already-solved problems onto the board? How many students have to respect their teachers as "knowing mathematics" based *entirely* on his or her authority as a teacher?

including *push hands*, a two-person training routine that is sometimes described as a gentler form of sparring. On most days, our practice lasted two hours. On Saturdays, we trained for two hours, talked for an hour, then practiced for another hour or two.

2.5 Enactive Artifact or, How to Teach Secrets

Here I outline three explorative accounts of embodied artifacts, ranging from relatively simple to increasingly complex. As a reminder, enactive artifacts are routinized practices through which the practitioner, it is hoped, discovers the underlying principle, developing a felt sense of what this practice is *about*.

Though nobody can force this discovery, it is not an accident, either. It is born of guided exploration facilitated by the enactive artifacts and the teachers who teach them.

2.5.1 Surfing on sand

To begin, imagine a first surfing lesson in Honolulu, Hawaii. Despite the endless crowds at the sun-drenched Waikiki beach, a neophyte surfer is eager to get in the water. Doing so immediately, however, is likely to invite disappointment. His inability to distinguish the many forms of waves, crowding by dozens of other nearby surfers, neuromuscular fatigue from continuous paddling (a surprisingly difficult activity for the inexperienced!), an uneasy sense of opaque, unspoken social hierarchies among the more experienced surfers, and a myriad other factors large and small all conspire to quickly dizzy and exhaust the novice. Yet the beachboys (surfing instructors) of Waikiki are famous for claiming they can make *anyone* ride a wave—at least, that is, for a second or two. How?

Even before getting in the water, the beachboy will ask the first-timer to lie down upon the surfboard *on the sand*. There, the beginner is taught the elementary Pop Up (or Stand up) sequence on the surfboard, roughly[11]:

1. prepare: from a prone (face down) paddling position on the surfboard, place hands underneath chest, feet together, on the balls of your feet;
2. pop up: propelling yourself with both hands and feet, legs are “floated” underneath the body into a sideways stance with the front knee ends up underneath the chin, back knee ends underneath the torso;
3. stand up: rotate hips and stand up into proper surfing stance.

So the introduction to surfing begins, so to speak, outside of its ultimately intended context. That is, the enactive artifact—a Pop Up sequence—is taught on sand. The novice is taught a motor sequence and given initial opportunity to practice it out of context. Why train out of context? Why not learn how to pop up in the ocean? This is a possibility. But consider the following exchange between a student and professional coaches taken from a popular surfing website surfcoaches.com:

Student:

[11] There are a number of variations to the Pop Up.

> Wait. I'm confused! I thought you had to surf in the water! Why is he doing it in the grass? Can someone please explain?
>
> Reply:
> It's a way to practice without the constant variables of catching waves…and in our combined 25 years of teaching experience, we have found that teaching beginners the fundamental movements on land, without the constant change of the ocean and the constant variables, that our first time or new surf students could achieve a higher level of success by taking a few moments to practice their fundamentals on dry land so that they can accomplish more once they enter the ocean.

Only once the beachboy determines the neophyte is capable of executing this basic sequence with confidence does the surfer take to the water. There, the instructor will wait for an appropriate wave, a selection process beyond the novice's capacity, then push his charge into this wave at the appropriate moment. At this point, all the neophyte must do is paddle hard into the wave—and (attempt to) pop up on the surfboard. A complex activity is thus partitioned into: (1) select a wave; (2) approach a wave; and (3) pop up and attempt to stay balanced on board. Hence the beachboys accomplish their claim of getting anyone to "surf" by performing (1) and (2) on behalf of their charge and having given the neophyte an enactive artifact, the elementary pop up sequence (3). From the neophyte's perspective, the event looks like this:

> "Up!" yells my instructor after he pushes me into the wave once more. "Up!" Like we practiced, I pop up, this time manage to stay up, and glide towards the hotels looming large over the Waikiki beach. I feel wobbly, but I crouch low, reach my arms outward, and try to relax. I suspect I look silly in this pose, but the experience of surfing is amazing—a completely novel sensation, unlike anything I have ever experienced. Peripherally, I become aware of other surfers around me, gliding along. Then the wave I'm riding changes somehow and, unable to adjust, I flail my arms, falling.

It was the instructor who did most of the work, yet the neophyte's—that is, my own—feeling of success was unmistakable. Importantly, I genuinely felt like I experienced surfing. From that day on, when people talked about surfing, I had a sense—a felt sense—of what it is they meant.

I chose to start my exploration of enactive artifacts with Pop Up training because training on the sand to surf on waves strongly illustrates an important dimension of enactive artifacts. Enactive artifacts are oftentimes not practiced for their sake *per se*. Pop Up is a variation on this: the aim of practicing pop ups on sand is not to get good at doing it on sand, but to develop particular insights that are more broadly applicable to the discipline.

One argument against this might be that practicing on land is all about "acquiring some muscle memory" for later use. In other words, there are no insights here, just physical skill. This seems to me an unfair reduction, as if physical skill does not benefit from insight. The Pop Up training *does* entrain a movement, but that is not all it does.

Consider the following excerpt taken from my notes following my first surfing lesson and attempt to surf (I use the term Stand Up for the Pop Up):

> Executing Stand Up in water was both harder and easier than on sand. It was harder, because the waves, unlike sand, didn't stay still. I had to adjust to account for what was happening around me. [This] was difficult. How do I avoid other surfers and swimmers? But it was also easier [in water]: when I pushed down, I pushed the board down. I was jumping even farther than I anticipated….Still the movement felt smoother [than when done on land].

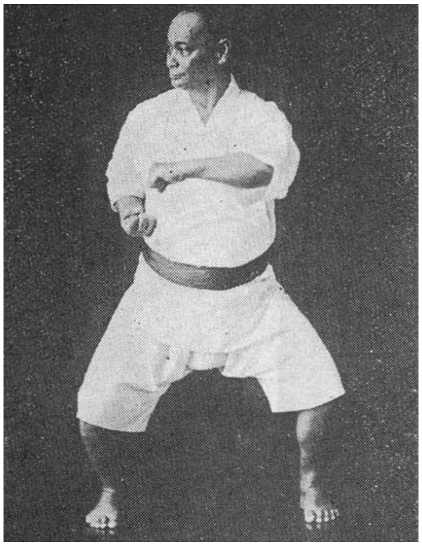
Above I laid out my observations about practicing the Pop Up on sand and executing it in water. During my practice on sand, I noticed it is important to press down hard—somewhat like a push-up motion—in order to build an explosive upward momentum necessary to bring my feet underneath my hips in a single smooth motion. Yet I also noticed a difference in sensations between pushing down on the surfboard in sand and this same motion executed in water. When pushing down on the board in the ocean, the water underneath me yielded and provided more room than I had otherwise come to anticipate while training on sand. Similarly, though not recorded above, practicing in water helped me understand where to place my feet after popping up in order to balance—it is difficult to gauge this when on solid land. Thus training in sand, and then in water, involves not only "muscle memory" but also various observations, recognitions, and adjustments.

Finally, it may be worth considering that at least some experienced surfers still practice the pop up on sand, at home, and otherwise outside of water. This is, apparently, not only because they cannot be in water at all times. "You need explosive power," commented a surfer who befriended me during my brief visits to Honolulu. I even heard about practicing pop up onto a foam roller, a sort of hard-foam tube that would be unstable when stood on, or using other similarly unstable surfaces. Obviously this sort of a move would be much harder to execute, and thus, presumably, more suited for a more experienced practitioner. Yet I never advanced sufficiently to experience what it is that more experienced surfers might gain out of practicing pop ups on sand, or even in water.

To see what enactive artifacts provide even experienced practitioners, I now turn to martial arts.

2.5.2 The horse-riding stance in martial arts

One illustrative example of an enactive artifact in martial arts is the *horse riding stance* or simply horse stance. The horse stance is an important bodily posture in virtually all Asian martial arts and takes its name from the position similar to the one assumed when riding a horse (see Figure 2, below). It is called *mabu* in Chinese, *kiba-dachi* in Japanese, and *juchum seogi* in Korean martial arts. I picked horse stance as an example because it is not immediately obvious *what it is for*. I will elaborate.



a.

b.

Figure 2. Records of horse riding stance as found in martial practices in (a) Japan (taken in first half of 20th century) and (b) China (early 1980s).

In appearance, horse stance appears somewhat like a high squat position: legs straddled a bit wider than shoulder length, toes facing forward, knees bent, butt dropped behind the ankles as opposed to in front, torso erect, slight posterior pelvic rotation. The depth and width (length) of the stance varies from discipline to discipline, but never so deep and wide that one's lower back drastically hyper-extends, or knees collapse inward.

As may be evidenced from pictures alone, the stance as it is usually trained is not amenable to mobility. Additionally, this stance is weak in the sagittal plane; an opponent only has to shove you from the front to seriously challenge your balance—obviously, there is no third leg coming out of the spine to stabilize against this. The groin is also invitingly open to being kicked. Approaching an opponent this way from the side creates a mobility problem, as any movement is necessarily slow, and the front knee is also open for a strike to render it unusable. Almost without exception, a higher, more mobile stance is a superior posture in most encounters, or aspects of an encounter. (That said, the horse stance does have direct applicability, for example against standing grappling maneuvers.)

Thus, practical instances for the application of this form seem rather limited compared to the amount of practice the form sees. And yet, to recall, the horse-riding stance *is* a very commonly used practice across many martial arts.

This brings to mind the question—Why is the horse stance practiced? I believe the answer isn't found in looking for its direct martial applications, or even in its obvious utility as a leg-muscle-building exercise. Instead, I offer that horse stance keeps the practitioner *rooted*, or stable in the face of incoming force seeking to off-balance him, such as in the case of an opponent attempting a takedown or a leg trip. Although the horse stance is weak from the front (i.e., sagittal plane), it is extremely stable from the sides. So, when pressed from the left, the pressure can be directed through and across the body, through the right leg, and into the ground. This is a relatively easy method of experiencing *rooting*.

Some other terms for rooting are grounding, nullifying, or channeling. Because rooting is an important aspect of martial-arts practice, one I will use again in a taiji example, it is worth discussing it briefly. To do so, I will draw on my own experience as a martial artist, but will also, on occasion, draw on writings by Steven Pearlman, the Smithsonian Institute martial arts historian and expert, and author of *The Book of Martial Power* (2008).

Rooting can be thought of as directing force into and from the ground through the body. Properly speaking, it is not a single thing one does, but the *outcome* of proper physiokinetic alignment of the body and proper attention to relaxation, centeredness, and various other somatic principles. By and large, these principles are neither esoteric nor arbitrary: for example, “Don’t lift with your lower back!” is one modern admonition pointing to humans having a body structure that is better used in some ways than others.

There are various interrelated reasons for rooting as a martial activity. Defensively, rooting allows the force to pass through us, into the ground, which is preferable for absorbing force (than our spine, for example). Offensively, if properly rooted, we can push ourselves into the ground as we push against an external object. Another aspect of rooting is heaviness. When rooted, one feels heavy. The sensation is felt somewhere deep and low in one’s belly. Of course, this doesn’t mean we *literally* become heavier. Rather, when we feel heavy, it is difficult for our opponent to pick us up because we make it difficult.

At any length, I hope it is becoming evident that being grounded is a useful martial skill, and I will return to it when discussing taiji. My argument, then, is that martial arts employ the horse stance in part because this sense of rootedness is a highly desirable attribute. Next I will lay out the process of this instruction.

I provide an idealized chain of instructional events concerning the practice of horse stance. These are drawn from my practice as a martial artist and interviews with martial artists, but are also simplified and reductive for the explicit purpose of providing the reader with a clear sense of what I mean by stages of instruction. In other words, consider the following as an aid in making sense of this practice rather than a description of reality. In practice, stages are never this clearly delineated.

1. *Instruction of motor coordination*
   Students are taught the horse stance, specifically the movements required to hold the posture. By this I mean, if asked to assume the horse stance, students know to assume a stance similar to the one demonstrated by an expert, “legs straddled a bit wider than shoulder length, toes facing forward, knees bent...”

2. *“Decontextualized” practice*
   Horse stance is practiced out of a “normative” context similar to how Pop Up is practiced on sand. I mean that the practice is done outside of a martial or otherwise dynamic context. Posture training leads the student to develop not only flexibility and leg

musculature, that is, the material body proper accommodating to better enable the sheer physical enactment of the pose, but also provides him or her with glimpses of what it is to be rooted, initially felt as a sort of stability. One student I interviewed described this as "feeling like I'm sitting or leaning against a wall… even though I am standing." Another said he started feeling like he was "sitting back on [his] legs."

This sense of stability and the adjustments necessary to achieve it gradually become a part of students' bodily repertoire. The process is gradual, and depends on students developing a felt sense of what being grounded or rooted *feels like*, and then attempting to maintain this sense even as they execute techniques.

3. *Practice in and across a variety of contexts*
   Next, students are placed into a training situation where stability is required to accomplish some task. That is, students encounter a situation in the world that can be solved by using what they have been practicing. Because of (2), this does not require *thinking about what to do* so much as *feeling, or having a sense of what to do*.

   The students now learn what it feels like to remain stable or rooted across various contexts. So doing, they find themselves moving in new ways as they adjust to emerging circumstances. When they recognize these new ways of moving as advantageous, we might say that they have gained further insight into the practice.

   Typically, these opportunities to practice "in context" (or, rather, across a variety of contexts) would be provided by the teacher; however, it could range from a mugger attempting a takedown to pushing a dead car on an icy road to standing unsupported on a moving train.

4. *Practice across contexts leads to understanding*
   Because of (3), horse stance now has a meaning as more than an isolated exercise. Its meaning is inseparable from its usefulness; as students encounter and solve further problem situations in the world, we say that they develop understanding of what it is to be rooted. We might say that each practice is an exploration of how to solve a given problem situation with the sort of rootedness or stability developed via the horse stance.

   So practicing the horse stance is no more *yet no less* than using it across a variety of situations and being open to making adjustments contingent on varying circumstances. At some point, we may say that the students now have a deep "conceptual understanding" of rootedness, or that they understand the principle of being rooted.

For this progression to work, it is not necessary for the students to know or understand what the enactive artifact is for, disciplinarily speaking, ahead of time. Indeed, it is commonly assumed the students simply *cannot* understand the form until it is entrained and consequently applied across a variety of contexts, an important point to which we shall return later.

As indicated above, in addition to providing students with enactive artifacts, teachers create educational environments in which embodied artifacts serve as the solution to a particular problem. It is precisely this aspect of problem-solving that generates meaning for the student. In other words, because the meaning of an action *is what it is for* (see Glenberg, 1997), teachers create environments facilitating the development of meaning. Without these environments, development of meaning is left to happenstance or students' appetite for seeking or generating relevant problem situations. As an aside, I think this goes a long way of explaining why some students "get" mathematics while others do not. Those who do "get math" generated relevant problem situations through intense labor and/or happenstance. Those who do not, did not.

One paradigmatic popular cultural reference of an instructor teaching out of context and then creating context is found in Miyagi's training of Daniel in 1984 *The Karate Kid*. Daniel, a teenager interested in karate, expects rigorous martial training from the get-go; instead, Miyagi, a karate master, has him do various chores, and, importantly, shows him the precise ways he wants those chores done. After painting the house, painting the fence, sanding the decks, and waxing a small fleet of classic 1940s cars, Daniel decided to quit, believing he was merely used for free labor. Miyagi then demonstrated that the precise motions necessary for those chores were various blocking actions, useable in martial combat. Without knowing it, Daniel entrained enactive artifacts, whose martial meaning was revealed and signified when Miyagi provided him an opportunity to defend himself using those entrained action sequences. In Chapter 3 I introduce a learning environment for mathematics inspired by this account.

I will now turn to yet another enactive artifact in martial arts, a final and more elaborate example than either of the previous ones.

2.5.3 Taiji, push hands, and resisting force with force

We return to taiji to examine some features of push hands (also, pushing hands, *taishou*), two-person competitive/cooperative training routines. I begin by describing push hands in general terms and then turn to a brief case study. The focus here is on the teacher using the push hands routine to introduce recurring opportunities for the student to witness certain taiji principles in action and, so doing, become aware of what these principles *mean* (as we say, what they mean *in practice*).

Push hands is perhaps one of the more misunderstood aspects of taiji practice. It is not uncommon to hear push hands referred to as controlled sparring by some, while others think it more akin to partner dance. Push hands is cooperative in the sense that both practitioners adhere to a set of limitations of what is acceptable. The exact set of limitations varies, but typically include the prohibition of striking of any sort, prohibition of certain throws, and prohibition of certain joint locks (organized competitions naturally have more specific rules). And push hands is competitive because the two practitioners are attempting to off-balance each other. This dual nature of the activity—fiercely competitive yet fundamentally cooperative—is reflected in some of the terminology used: the person you push hands with could be your *opponent*, your *partner*, or simply a fellow *player* (see Figure 3, below).

a. b.

Figure 3. Push hands is a cooperative yet competitive activity: (a) two players attempting to off-balance each other; and (b) a master off-balancing his partner ("opponent").

Pedagogical uses of push hands are many. To give the reader a sense of this landscape: it is possible to attend a full day martial arts workshop focusing on just *one* aspect—*Nian*, or Stick—of *one* category of principles—called contact principles—emphasized during push hands practice. To illustrate, just the contact principles alone involve *Lian*, *Sui*, *Nian*, and *Zhan*. Lian, or Connect, is the fundamental contact principle, as it involves making and maintaining physical contact with the opponent; the remaining three principles concern how to manipulate the opponent via this physical contact in a martial setting.

Each one of these contact principles can be emphasized and explored during push hands practice. Yet there are many other principles that could be emphasized instead. For example, understanding of the basic four movements of taijiquan—*Peng* (Ward Off), *Lu* (Roll Back), *Ji* (Press Forward), and *An* (Push Downward)—is often evaluated through push hands exercise. To pick the most obvious case, a student unable to demonstrate even the basic level of Peng would feel either too stiff or too loose. I was alerted to a popular taiji website that lists *22 distinct principles* that can be explored through push hands practice, and adds,

> So I get all of this from standing in front of someone and pushing on them? Yes, in effect. Think of the amount of academic work that it would take to understand the above concepts. You can get there much faster just by beginning push hands. The above list may appear daunting but you can gain these skills from proper pushing hands without even knowing the concepts above exist. ("What Is Tai Chi Push Hands," 2015)

I want to emphasize that none of these aforementioned principles are specific motor coordinations (like, for example, a Karate sidekick), they are instead broad characterizations of how a body naturally moves[12] and relates to other bodies. As such, these principles may be instantiated in a wide variety of forms that may appear unrelated at first glance. The reader may appreciate from the gloss of principles, above, that push hands instills in the practitioner a vast

[12] Some practitioners say that principles are best understood as "internal energies," either metaphorically or literally. However, this is a topic I do not explore here.

"vocabulary" of effective interaction routines or, better, fundamental orientations that subsume the numerous traditional martial arts.

Because articulating even one principle at length would require a significant detour, I will focus on a fairly straightforward use of push hands as it relates to training beginners. We could call this *redirect-not-resist*. Specifically, pushing hands helps undo a person's natural instinct to resist force with force[13], teaching the practitioner to yield to force even as she redirects it, rather than to resist it.

Learning not to oppose incoming force with force is a substantial challenge for most if not all beginners. This development goes hand-in-hand with an oft-repeated admonition for beginning taiji practitioners to "relax more!" *Relax* is an imperfect translation of the Chinese word *song*, which was taught to me as an appropriate state of bodily alertness, neither slack nor tense. When in this relaxed-yet-alert state, the body acts as a sort of inflated rubber ball such that incoming force is naturally redirected rather than opposed. It is difficult to verbally describe this phenomenon (yet simple to physically demonstrate it).

A telltale sign of using force to resist force is that the defender *leans into* the incoming force in the sense that her balance becomes compromised (by being dependent on this external factor). Meaning that, if the aggressor stops applying force, the defender will stumble forward. In contrast, a skilled taiji practitioner would align her body structure with the incoming force (see discussion on *grounding*, above); if the incoming force proves too strong to be absorbed via body structure alone, she would move her body in a minimal way such that the incoming force no longer compromised her balance. Hence the earlier analogy to an inflated rubber ball: imagine using a single hand to push a large, inflated rubber ball under water of a fast flowing stream and keep it there—this is what trying to push a taiji master feels like.

During the practice of taiji push hands, the teacher will both (1) demonstrate this principle of redirect-not-resist on the student, and (2) provide opportunities for the student to do the same in return. Thus the teacher uses both first and second person perspectives to direct the student's attention to an aspect of interaction deemed as important, in effect saying: this is appropriate interaction from the perspective of our discipline, yet this is not (*hao*/*bu hao*, respectively). This change in perspective is useful, as it not only highlights to the student what she should do but—by putting her in the disadvantageous position—why such an action is martially effective. The following case study[14] will illustrate the pedagogical situation described above.

The following excerpt showcases an Eureka! moment one (beginner-to-intermediate) student experiences after the teacher guides him to redirect force. Prior to this moment, the teacher consistently (and patiently) directs the student's attention towards a particular dimension of the interaction. A more skilled observer is able to notice that, despite the master's efforts, the student continues having difficulty with redirecting incoming force.

[13] I use *force* in the everyday "common folks physics" sort of way. In general, I speak of these principles as other martial artists do, without regard to how, say, physicists would interpret such statements.

[14] I am grateful to Eton Churchill for highlighting certain dimensions of this interaction I might have otherwise missed.

The particular interaction sequence which we explore here starts when the master extends his right arm across the student's chest as if to offer it as something to work with or against (just before Figure 4a, see below). The student pushes down on the master's forearm with his left forearm. When he does this, the master freezes their movement in a pragmatic meta-comment, as if to say "Pay attention!" Then the master points several times at the intersection of their forearms (Figure 4a).

Master: When you feel force increase, change in direction. [Figure 4b; master squeezes his hand in front of Student's face, then opens hand and wiggles side-to-side while uttering the last three words, with emphasis].
Student: Change in direction.

Even as he finishes uttering the above, the student successfully redirects the incoming force ("changes in direction"). The master falls off balance (Figure 4c).

Student: Ah, yeah. Wow. [laughs] OK. Thank you.
Master: OK, See. In mind, never use force, never use force. (Figure 4d)

Figure 4a. Master (right) pointing to where his right forearm makes contact with the student.

Figure 4b. "When you feel force, change in direction." Master applies force.

Figure 4c. Student "changes in direction." As the master falls slightly off balance, student grins ("Ah, yeah. Wow.")

Figure 4d. "In mind, never use force, never use force." Practice continues.

During this interaction, the teacher introduces a problem situation into an ongoing push hands routine and states the general principle. This is done in order to provide the student with an opportunity to actually *experience* the principle of redirect-not-resist. That is, it is done for the benefit of student developing a felt sense of this principle. Once the student successfully does so, the teacher again verbally describes the principle. Thus, the felt sense is signified in a disciplinary context, and we can say that the student how has a sense of—or understands—what is meant by redirect-not-resist. If we are willing to consider that this principle might also be labeled a *concept*, the above interaction can be properly understood as the genesis or emergence of conceptual understanding.

It is worth noting that prior to the above interaction, the student had unsuccessfully attempted to redirect-not-resist a number of times. The teacher remained patient and provided additional guidance by specifically (and repeatedly) highlighting the very micro-event during which the student should redirect his force—and then allowing him to do so. The main point I wish to emphasize with this is how much effort the master put into guiding the student towards this single moment of insight. This is the purpose of repetition-without-repetition, and, I would argue, practice across all disciplines: to explore and discover.

For this discovery to take place, it is paramount for the student to experientially differentiate an instance of redirect-not-resist from his previous actions. The student's joyful utterance "Ah, yeah. Wow. OK. Thank you." indicates he was not merely thanking the teacher for allowing himself to be off-balanced. If that were the case, the student would have thanked the teacher with that same wonder *every time* this happened. So, the student was thankful not only that he "changed in direction" but that he *experienced what it means* to "change in direction."

Thus what is absent from the excerpt above is that this particular student "failed" many times. Yet his teacher showed patience. Why? Because "failure" is unavoidable when one explores a new territory[15]—and discovery cannot be forced.

2.6 Discussion

2.6.1 Secrets of the martial arts

I began training martial arts at age 14 and continued until my early 20s, competing and regularly ranking at regional and national tournaments. During this time I became aware that some practitioners, most of them masters, possessed a level of ability beyond my understanding. I became obsessed with understanding their techniques: What was it they did that I could not?

In many older Kung Fu movies the protagonist discovers an ancient scroll with a secret technique, a technique that allows him to defeat all comers, and though I poured over texts searching for such secrets, I found none.

[15] I have "failed" literally thousands of times during my martial arts studies. If I were graded on my progress, I would have received a long string of failing grades. Yet neither I nor anyone else gave this much thought, because it was seen as natural and even necessary.

By my early 30s I realized that I looked in the wrong place. It is true that there are martial-art secrets that were, traditionally, taught only to the so-called *indoor students* (students who would stay with the master). This was during a time when being, say, a caravan guard or a royal bodyguard were lucrative yet coveted positions. Consequently, martial arts knowledge was a desirable, well-guarded commodity.

But what was the nature of these secrets? As Bernstein hints, there were no secret *techniques*—"any movement can be taught by demonstration" (p. 187). The secrets were in the *practice*. What the masters kept hidden from their outdoor pupils were not the techniques, but the training practices that made those techniques effective.

2.6.2 Closing remarks

Like most mathematics education researchers, I want education to focus on conceptual understanding. I am, however, also deeply concerned with what I perceive as a general lack of discourse on how practices and concepts shape each other.

In order to make sense of procedures, I wanted to understand routine practices in general. Yet practice in mathematics brings up images of worksheet after worksheet of what students might call "busy work." This led me here, to what we have discussed so far in this chapter.

I offer that teachers guide students' enskilment by means of certain embodied practices—rehearsable disciplinary routines—which I called *enactive artifacts*. My claim is that enactive artifacts are routines through the practice of which students acquire disciplinary understanding. As Nikolai Bernstein argued, teachers can teach *outer* aspects of knowing—hands go there, elbows here—but *inner* aspects must be experientially felt. This need for aspects of knowing to be *felt* is why no one becomes a skilled athlete through attentive listening alone.

I have tried to pick case studies illustrating this process. Students are told or shown the enactive artifact yet never *told* what it is that they are supposed to achieve through practice, precisely because that pedagogical goal is only achieved through practice.

Even the comparatively simple Pop Up practice on the sand served to highlight for the student certain aspects of surfing that are simply too nuanced and context-dependent to communicate otherwise, such as the effect of pushing on a particular area of the surfboard, or the difference between pressing the board on sand versus water.

Thus, even as they make use of enactive artifacts to cope with some problem situation, working with and through enactive artifacts disciplines students' perception (Goodwin, 1994; Stevens & Hall, 1998)—that is, shows them what it means to encounter the world as an expert might.

This is even more evident in the case of training for rootedness. In the case study of taiji push hands, the enactive artifact provides explicit opportunities for the teacher to highlight certain aspects of interaction as significant (that is, signified). But these aspects must be differentiated by the student from what he was doing previously. That is, the student must be open to having a *novel* experience, a discovery. For this reason practice of enactive artifacts ought to be viewed as an open exploration of possibilities.

To put it differently—the central idea I draw from Bernstein's notion of repetition-without-repetition and my studies of enactive artifact "in the wild" is that the point of repetitive practice, of routines, is often about more than "merely" improving the routine itself. In other words, some pedagogical functions of the form are not necessarily evident in the form itself. Instead, routines are practiced because they orient or attune the practitioner towards certain previously inaccessible or unnoticed features of the world. This attunement is a felt, bodily sensation enabling greater context-sensitivity as skilled responding improves over time (see Dreyfus & Dreyfus, 1999). These sensitivities are of the contextual and interactive rather than general and symbolic sort and take the form of what *we can do with the world*.

In other words, enactive artifacts are a guide, pointing to something. And my claim is that this sort of practice is necessary in developing rich disciplinary understanding.

Finally, I wish to again emphasize that I am not pining for business-as-usual procedural drills. Here I refer to Rory Miller, author of *Meditations on Violence: A Comparison of Martial Arts Training & Real World Violence* (2008), an expert on controlling physical violence and well-regarded self-defense teacher. In one of his training courses, *Joint Locks*, Miller endorses the common belief that joint locks are extremely difficult to apply under pressure. Yet, he argues, the culprit is not any inherent difficulty of applying joint locks, but the *teaching methods* commonly used. He recommends using a "principles-based approach" to teaching. Here is what he had to say:

> Some of you have been studying joint locking arts for years and years and years. And I know people who have studied those arts and can't apply them under stress. [Yet] [t]hey aren't hard. We complicate them….If you go with a Rolodex of techniques that have to look picture perfect in your head, and you walk into a fight hoping to God that one of those opportunities that you happen to have practiced with is going to be the one to pop up, you are counting on luck as your primary strategy.

I find Miller's notion of "counting on luck as your primary strategy" to be a rather apt description of the sort of learning that comes about from rote drill practice in mathematics classrooms. Rather than teaching "a Rolodex of techniques," Miller employs the sort of training I have outlined here: Students pursue routines as opportunities for exploration, with not the aim of "getting it down cold" but "getting a sense of things."

In short, practice ought to be viewed as an opportunity for insights that cannot be told and must be felt. And yet, though no one can force discovery, it is not an accident, either, but born of guided exploration facilitated by the enactive artifacts and the teachers who oversee the process.

Chapter 3: Practicing Mathematics

> Learning processes are marked by a succession of changes in perspective which should be provoked and reinforced by those who are expected to guide them.
>
> (Freudenthal, 1991, as cited in Streefland, 1993, p. 133)

> But such shifts in meaning happen only when previously tacit norms are liberated—for example, the capacity to recognize what "sounds nice."
>
> (Bamberger & Schön, 1983, p. 37)

In domains such as surfing and martial arts, novices are taught *enactive artifacts*, or disciplinary routines that, through repeated use, bring forth valuable aspects of the discipline, aspects that cannot be *directly* taught (say, by demonstrating or by verbal instruction). The previous chapter explored how this pedagogical approach lives in some explicitly embodied disciplines, such as surfing and martial arts. The following conjecture emerged: recurring experiences of consistent sensorimotor patterns in explicitly embodied disciplines give rise to understandings we might call conceptual.

Consequently, students across disciplines may better understand concepts through engaging with interactive learning environments latently bearing the targeted conceptual systems rather than exclusively engaging with text-based quasi-realistic word problems. A design principle could be articulated as follows: students should first develop new dynamical physical coordinations and then signify these coordinations mathematically. The learning environment (of course, including teachers) should be sensitive to and facilitate changes in students' attention from physical routine to mathematical representation.

To illustrate this principle, imagine a simple activity: grouping seven red and eight green beads into a single pile. Being mathematically informed, we might be inclined to model this situation like so: $7 + 8 = 15$. This, we say, is a mathematical model of the situation. Yet what of the grouping action that was just performed? Mental in this case, but even if it were physical? This routine, or we might say, this enactive artifact, is an embodied model of addition. Granted, to a person who has not yet learned the meaning of addition, this is "just something one does"—that is, these actions *per se* would not convey mathematical meaning. And yet these actions bear the *potential* of building mathematical meaning. This potential is put to work when students begin making sense of their physical models via mathematical representations, and negotiate their perspective back and forth across the two (see also Bartolini Bussi & Mariotti, 2008).

Specifically, in this chapter I present and discuss educational technology in which students practice raising their hands in a continuous motion at a designated ratio even before any mathematics content is introduced and even before students themselves are aware that there is such a thing as "a designated ratio" present in their practice. Later students are provided with mathematical frames of reference and guided to discuss their actions. Navigating back and forth between physical action and mathematical representations, they are able to generate rich, interconnected observations that are grounded in their new way of moving yet informed by how the discipline signifies these events (for example, negotiating additive and multiplicative constructions of proportional dynamics).

As in the previous chapters, during data analysis emphasis is placed on the emergent nature of understanding. I first outline general findings. Then, through microgenetic analysis across three case studies, I detail how guided practice of enactive artifacts stands to brings forth for the student valuable aspects of the discipline. Specifically, during close analysis, it appears that students' Eureka! moments are best understood not as intended outcomes of preplanned routes but as noticing a felt difference between old an new ways of interacting. As such, this Chapter concerns itself with the unanticipated, opportunistic, and serendipitous nature of learning through interaction and reflection.

3.1 Overview of the Mathematical Imagery Trainer for Proportion

The Mathematical Imagery Trainer project[16] (PI: Dor Abrahamson) began with the following assertion: To the extent that mathematical knowledge is grounded in motor action schemes, constructivist instruction should attend to motor action knowledge—its nature, construction, and interaction with enactive, semiotic, and epistemic means in the learning environment. One model for such instruction, *embodied design* (Abrahamson 2009, 2012, 2014; Abrahamson & Lindgren, 2014; Abrahamson & Trninic, 2012), is to design technologically enabled fields of promoted action that elicit existing motor schemes yet challenge the learner to adapt and articulate new schemes and ultimately signify them within the discipline's semiotic system. Embodied design is thus a framework that seeks to promote grounded learning by creating situations in which learners can be guided to negotiate tacit and disciplinary ways of perceiving and acting. In turn, the hands-on nature of embodied-design learning activities typically renders users' implicit mental actions physically explicit and thus accessible for instruction as well as non-invasive investigation.

Our[17] experimental design was driven by a general conjecture that some mathematical concepts are difficult to learn due to a resource constraint of mundane life. Namely, everyday being does not occasion opportunities to embody and rehearse the particular dynamical schemes that would form requisite cognitive substrate for meaningfully appropriating the target concepts' disciplinary analysis of situated phenomena (Abrahamson & Howison, 2008, 2010a, 2010b; Reinholz, Trninic, & Abrahamson, 2010; Trninic, Reinholz, & Abrahamson, 2010). Specifically, we conjectured that students' canonically *incorrect* solutions for rational-number problems—"additive" solutions (e.g., "2/3 = (2 + 2)/(3 + 2) = 4/5")—may indicate a lack of multimodal kinesthetic–visual action images with which to model and solve situations bearing proportional relations (c.f., Pirie & Kieren, 1994).

In response to the design problem articulated above, we engineered, implemented, and analyzed an embodied-interaction computer-supported inquiry activity for learners to discover, rehearse,

[16] This chapter draws on numerous published and unpublished texts by the Embodied Design Research Laboratory (2008-2015).
[17] As a reminder, I frequently use the plural we (us, our) to refer to the majority of work done in the Embodied Design Research Laboratory. This reflects the collaborative nature of the work. Yet, I also note individual contributions when they are not my own.

and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the *Mathematical Imagery Trainer for Proportion* device (MIT-P; see Figure 1, below).

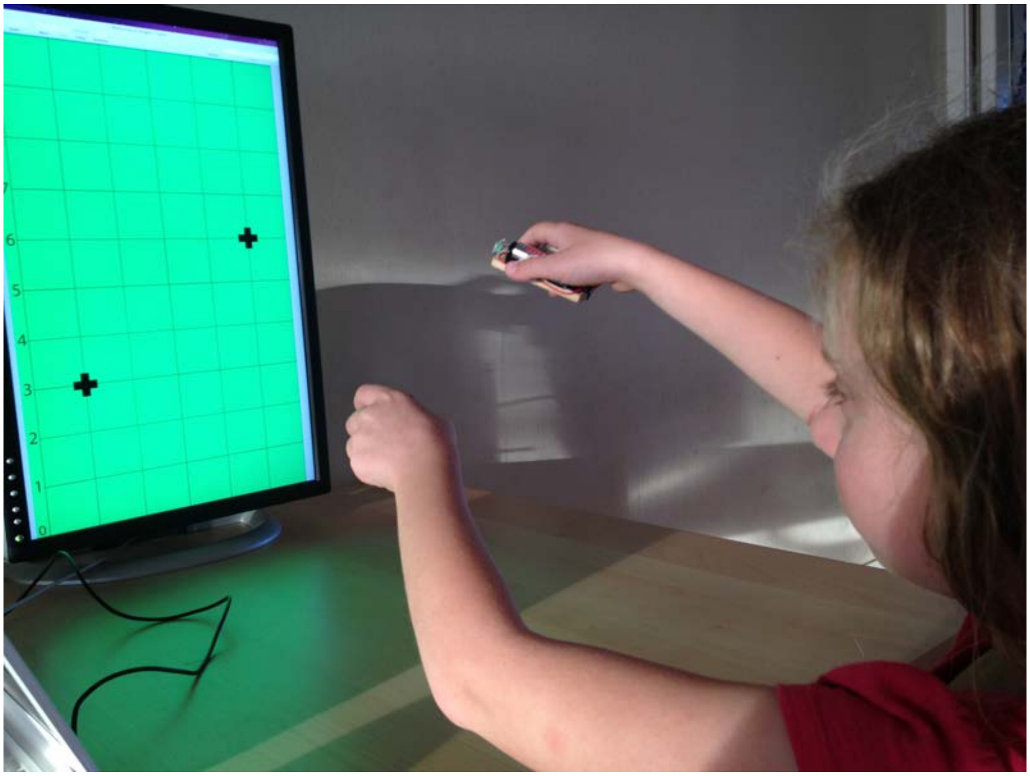
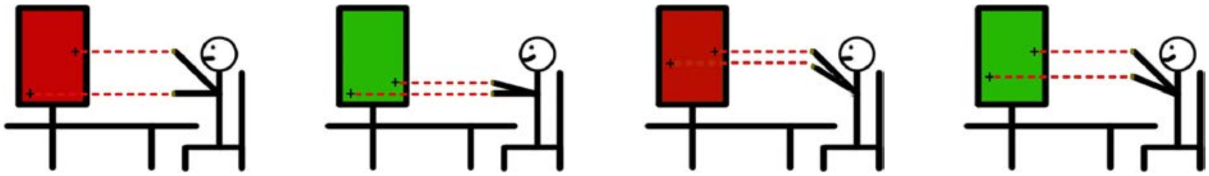


Figure 1. A student works with the Mathematical Imagery Trainer for Proportion. Unknown to her, the device is set at a 1:2 ratio. Note that her hand positions correspond to plus-shaped black markers on the screen. She is remote-holding the markers at 3 and 6 units above the baseline, and so the screen is green ("correct"). Otherwise, the screen turns red ("incorrect"). She first learned to move both hands simultaneously, keeping the screen green. Now she is modeling her new embodied skill mathematically, using the grid and numerals as frames of reference. MIT-P thus fosters a quasi-naturalistic, pedagogically oriented developmental sequence from sensorimotor coordination to mathematical articulation of curricular topics.

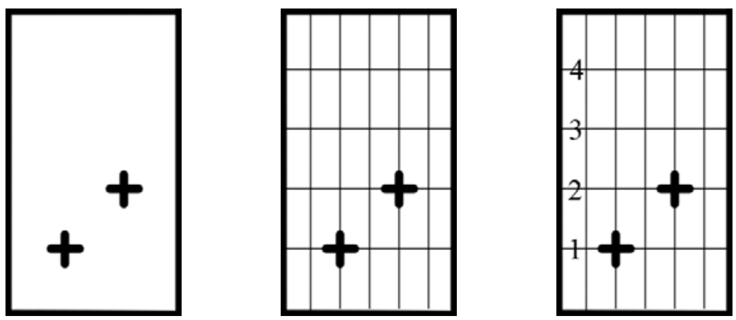
The MIT-P measures the heights of the users' hands above the desk. When these heights (e.g., 10 cm and 20 cm) relate in accord with the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. If the user then raises her hands in front of the display at an appropriate rate, the screen will remain green; otherwise, such as if she maintains a fixed distance between her hands while moving them up, the screen will turn red (see Figure 2, below; watch https://www.youtube.com/watch?v=n9xVC76PlWc for a video demonstration of this technology in action). Study participants were tasked first to make the screen green and then, once they had done so, to maintain a green screen even as they moved their hands. I address the technical aspects in the following sections (also see Howison, Trninic, Reinholz, & Abrahamson, 2011).



a. b. c. d.

Figure 2. The Mathematical Imagery Trainer for Proportion (MIT-P) set at a 1:2 ratio, so that the target sensory stimulus (green background) is activated only when the right hand is twice as high along the monitor as the left. This figure encapsulates participants' paradigmatic interaction sequence: (a) the student first positions the hands incorrectly (red feedback); (b) stumbles on a correct position (green); (c) raises hands maintaining a fixed interval between them (red); and (d) eventually "corrects" position (green). Note the difference yet similarity between 2b and 2d.

At first, the condition for green was set as a 1:2 ratio, and no feedback other than the background color was given alongside markers that "mirrored" the location of participants' hands (see Figure 3a, below). Next, a grid was overlaid on the display monitor to help learners plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions (see Figure 3b). In time, the numerical labels "1, 2, 3,…" were overlaid on the grid's vertical axis on the left of the screen to help learners construct further meanings by more readily recruiting arithmetic knowledge and skills and more efficiently distributing the problem-solving task (see Figure 3c).



a. b. c.

Figure 3. MIT-P display configuration schematics: (a) markers; (b) a grid; and (c) numerals along the y-axis of the grid. These schematics are not drawn to scale, and the actual device enables flexible calibrations of the grid, numerals, and target ratio. A few students originally began with no markers (blank screen), yet we ultimately found this condition unproductive.

Participants of ages 9 through 12 years old appropriated these artifacts as strategic or discursive means of accomplishing their goals. Yet, so doing, they found themselves attending to and engaging certain other embedded affordances in these artifacts that they had not initially noticed. Consequently, their actions (corresponding to "make the screen green") were modified as students saw the situation anew and, moreover, as they acknowledged their emergent strategies as enabling advantageous interaction.

My group proposed to characterize this two-step guided re-invention process as: (a) *hooking*—engaging an artifact as an enabling, enactive, enhancing, evaluative, or explanatory means of effecting and elaborating a current strategy; and (b) *shifting*—tacitly reconfiguring current strategy in response to the hooked artifact's emergent affordances that are disclosed only through actively engaging the artifact.

I offer that this hook-and-shift—opportunistic, unexpected, serendipitous learning phenomenon observed in our technologically-enabled learning environment—is fundamentally akin to the forms of learning described in the previous chapter. Indeed, the design of MIT-P was in part inspired by Karate Kid's "wax-on wax-off" motion (see Chapter 2). In both cases, the enactive artifact is a means to develop meaning, not a pedagogical end-goal in and of itself. The enactive artifact in this case is the "make the screen green" motion, where hands move up and down at a proportional rate (say, 1:2 ratio). As students practice this motion across a variety of contexts and situations enabled by our technology and our interview protocol, they *find themselves* incorporating, literally, novel aspects of the environment into their doing and thinking.

3.2 Methodology

3.2.1 Design-based research

The investigative approach undertaken was that of design-based research, in which theory and design co-develop iteratively (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Design-based research (DBR) is best understood not a methodology per se but a rich disciplinary context within which we carry out investigations of learning and teaching. It begins with a conjecture as to how learning could be better, and yet current learning environments are unsuitable for addressing the conjecture. Therefore, we design and evaluate a novel learning environment expressing this conjecture.

Furthermore, because it is impossible to consider ahead of time all the consequences of our actions, the process of design and its implementation provide opportunities to generate additional data-driven conjectures about learning that are then incorporated into the theoretical framework driving the design. In turn, this framework drives the next iteration of design, and so on. This type of reflective practice (Schön, 1983), while vital in all aspects of scientific inquiry, is foregrounded in DBR. In particular, the frequency and amount of observations emerging from DBR, many of which are unexpected, make it ideal when dealing with a novel research space—for example, embodied cognition.

3.2.2 Technology

Since its initial design, the Mathematical Imagery Trainer has been implemented in a variety of media, including mechanical devices (Abrahamson & Howison, 2008), remote-sensing technology (Howison et al., 2010), laboratory studies (see also Rick, 2012; Petrick & Martin, 2011), and touchscreen tablet application (both laboratory and classroom studies, see Negrete, Lee, & Abrahamson, 2013; Shayan, Abrahamson, Bakker, Duijzer, & van der Schaaf, 2015). My dissertation evolved in the context of the earlier versions, and those are the ones under scrutiny here.

The MIT-P leverages the high-resolution infrared camera available in the inexpensive Nintendo Wii remote to perform motion tracking of students' hands, similar to that described by Lee (2009). The light source is provided via an array of 84 infrared (940nm) LEDs (light-emitting diodes) aligned with the camera. In turn, 3M 3000X high-gain reflective tape attached to tennis balls can be effectively tracked at distances as great as 12 feet (see Figure 4, below). Lee reports that the camera has a 100 Hz refresh rate and a 45 degree field-of-view. In practice, our engineer[18] found the field of view to be slightly more restricted, requiring that we place the camera and LED assembly 10 feet from the student to reliably capture a 3 foot window of arm movement. Camera was oriented on its side to achieve greater resolution (1,024 versus 768 pixels) along the vertical axis. The Wii remote used is a standard Bluetooth device, with several open-source libraries available to access it through Java or .NET.

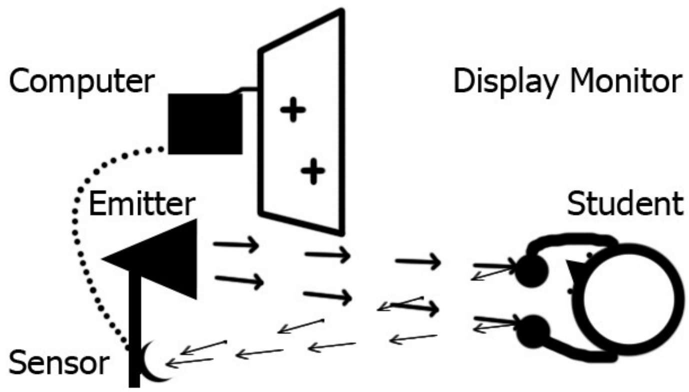


Figure 4. The Mathematical Imagery Trainer: top view of the system featuring an earlier MIT version, in which students held tennis balls with reflective tape. Newer versions (see Figure 1) had students hold IR emitters, removing the need for the emitter pictured above.

The accompanying Java-based software, called WiiKinemathics2, presents students with a visual representation on a large display in the form of two crosshair (+) symbols (markers). The orientation of the 22″ LED display (rotated 90 degrees and aligned to table height) and the responsiveness of the markers is carefully calibrated so as to continuously position each marker at a height that is near to the actual physical height of the students' hand above the desk. This feature is an attempt to enhance the embodied experience of the virtual, remote manipulation.

Occasionally detection of the reflective balls was too sensitive to the rotation caused by students' natural arm movement as they lifted the balls. Natural upward arm motion pivots the arms about the shoulder and consequently moves the main reflector attached to the balls upward, away from the axis of the camera and LED array. Our solution was to replace the LED array and reflective balls with battery-powered, hand-held IR (infrared) emitters that the students point directly at the Wii camera (compare Figure 1 with Figure 4). With LEDs repurposed from generic TV remote controls, these emitters have a wide enough angle of operation to robustly capture students' hand motion. Our software uses the WiiRemoteJ Java library, available

[18] Mark Howison was the engineer who—working with other members of the EDRL research group—turned Dor Abrahamson's original idea into a material reality.

from http://code.google.com/p/bochovj/wiki/WiiRemoteJ; additionally, a .NET library called WiimoteLib is available from http://wiimotelib.codeplex.com/.

3.2.3 Participants

The empirical data presented and analyzed in this paper were collected at a private K-8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students). In addition, we collaborated with the school principal, the head of general studies, and five mathematics teachers. Within each grade level (4th, 5th, and 6th), we grouped the pool of volunteering students according to three achievement levels as reported by their teachers (high/middle/low). Across these performance groups, we selected roughly equal numbers of students, balancing for gender, for a total of 24 students. One student was disqualified from the study. We preferred working with students whom the teachers had indicated as typically more disposed to communicate their thoughts. These more verbose students were distributed almost uniformly across gender and achievement level. Whereas this bias in our selection of participants, as well as the by-and-large upper-middle-class demographics of the school, limits the generality of conclusions coming from this study, we reasoned that this initial stage of research required dense real-time verbal feedback from the students as they interacted with the designed learning tools.

3.2.4 Data collection

Empirical data were gathered by conducting interviews using the semi-structured clinical technique championed by Jean Piaget, the founder of modern cognitive-developmental research (see also Ericsson & Simon, 1984, on veridicality of retrospective verbal reports; and diSessa, 2007, on legitimacy of drawing inference from verbal reports). We spread the implementation of the interviews thinly (no more than two per day), such that from day to day we would be able to introduce changes to the materials, activities, and protocol in light of the emergence and refinement of theoretical constructs. These rapid-prototyping changes—all motivated by the goal of optimizing the pedagogical quality and empirical utility of our subsequent interviews—were based on fieldnotes, preliminary analyses, verbal transcriptions, minutes from our team’s daily debriefings, and collaborative wiki postings. Thus, both the interview protocol and the interactive affordances of the instructional materials evolved as we progressed through the pool of participants. We gradually incorporated into the protocol any activities and prompts that arose during interviews and that, in debriefing, we evaluated as eliciting “researchable moments” from the participants. These were moments in which unexpected behavior from a participant suggested new theoretical constructs that we wished to test in subsequent interviews.

A note of validity and varying protocol. There is an inherent trade-off of control for exploration in design-based research (also see Ginsburg, 1997, on varying the interview protocol). Because we modified our protocol and even our materials over the course of the study, we cannot compare among or draw conclusions for all participants. The scientific merit of case studies emerging from design-based research is often disputed on the grounds of generalizability: regardless of how rich, systematic, and extensive they may be, our studies present only a minor collection of data points sampled from a larger universe of potential cases. Drawing on Robert Yin’s (2009) work, I offer an alternative view. Yin distinguishes two types of generalization. Statistical generalization is the (rightfully) acknowledged standard approach where researchers sample individuals from a population (people, classrooms, cultural groups) in order to generalize to the larger universe. Yet when dealing with context-dependent activities, such as learning, in a particular community, we encounter a difficulty with statistical generalization. After all, the community under investigation is merely one of myriad human communities.

Acknowledging the utility of statistical generalization, Yin argues for a different kind of generalization, one that he terms “analytic.” In analytical generalization, the focus is not generalization to the larger population from which a sample was drawn. The target is theory, and the endeavor is to refine theoretical constructs as their utility is explored and corroborated. In our work, we have found analytic generalizability well-suited to design-based research, which is itself characterized by its reciprocal iterations of design and theory. With this approach to generalization, theoretical considerations inform the design, and the design, in turn, is used to further develop theory. From such a perspective, this research contributes to characterizing the larger universe not through statistical inference but by refining our theoretical understanding. As Yin emphasizes, and particularly relevant to our work, analytic generalizability is a natural approach when phenomena and “context” have fuzzy boundaries, which is to say, whenever one studies learning in a rich setting.

One remaining issue concerns the validity of determining causal relationships between observable physical actions and “invisible” neural activity. In other words, even in a comparative study, how would we ever know that moving in some specific way (and not in any other way) is what led to observed forms of reasoning? This is a legitimate concern, yet one difficult to untangle. The studies reported in my dissertation were not designed to establish this sort of validity per se. Indeed, it is difficult to find any studies which have attempted to do this sort of work. That said, I wish to mention a paper by Mitch Nathan et al. (2014). The authors report a study involving undergraduate students (n = 120) asked to perform either a grounding action or a non-grounding action in the context of generating mathematical proofs. Grounding actions were actions relevant to the task; in turn, non-grounding actions were actions not relevant to the task (e.g., simply tapping on the board). As predicted, grounding actions proved pedagogically superior, suggesting that it isn’t *any* action, but a *specific* sort of grounded action that results in learning gains of the sort found in our study.

3.2.5 A “typical” interview

Interviews took place in a quiet room within the school facility. Students participated either individually or paired with a classmate (duration: mean 70 min.; SD 20 min.). Only six students, our final participants, were interviewed as dyads (i.e., 3 pairs). For the students interviewed in pairs, each student controlled one of the two hand-tracking devices. The interviewer guided the participant through a sequence of activities by first explaining each activity and then monitoring the participant’s performance and providing formative comments so as to ensure that the task was clear. The first activity was introduced with the simple instruction, “Make the screen green” (see Appendix A for the finalized interview protocol).

Once the participants found an initial “green” position, the interviewer asked them to “find green somewhere else.” If participants responded by “locking” the distance between their hands in a fixed interval and moving them up or down, the screen turned red. Only once they relaxed this fixed distance between their hands and attempted to adjust it appropriately would they strike green again. Participants might also at this point identify and articulate a rule to the effect that, "The higher you go on the screen, the greater the distance should be between your hands."

Following some variation of the above articulations, the interviews would introduce a grid (refer to Figure 3b, above). The grid bears the capacity to shift participants' attention so as to re-construe each marker's location as its height above the base line—a height that can be quantified in terms of discrete units (e.g., 1 and 2 units, respectively). In practice this tended to foster a "snap to grid" strategy utilizing the grid's inherent discrete-quantity relations. For example, a recursive rule for transitioning from one green spot to the next: "For every 1 box left marker goes up, raise right marker up by 2 boxes." Note that this hand-to-hand relation is also a covariation, just as the height-to-distance relation, above, was a covariation. Yet here the covariation shifts from continuous–qualitative descriptors ("higher," "greater") to discrete–quantitative values ("one," "two"). Thus, though both covariations refer to the same hand motions, the meanings and planning of this physical enactment have evolved. The latter covariation is closer to normative mathematical practice and discourse for proportional equivalence, and in line with our pedagogical goals.

In the final mode, numerals appeared to the left of the grid (refer to Figure 3c, above), potentially alleviating a need for counting the grid boxes. Specific "green" numeral pairs, such as "3" and "6" in the case of 1:2, may evoke basic arithmetic operations and "facts," so that students recognize that the right hand should always be double (the height of) the left hand. The interview ended with an informal conversation, in which the interviewer explained the objectives of the study, to help participants situate the activities within their school curriculum and everyday experiences. At the conclusion, the interviewer answered any questions participants had, such as about our technology.

If students appeared "stuck" at any point during the interview, the tutor reminded the participants what they themselves had said and done earlier in the interview, at times highlighting the fact that the participants had used more than a single strategy. More specifically, the tutor oftentimes did one of the following:

- Recounted two strategies that the participant had previously articulated (3 cases)
- Presented two strategies as potentially different/similar/related (7 cases)
- Suggested to explore whether one strategy could be used to describe another (1 case)
- Presented one strategy and then asked whether there were another (1 case)
- Spoke about an earlier strategy in a way that highlighted a new feature (1 case)

3.2.6 Data analysis

As mentioned above, we spread the implementation of the interviews thinly (no more than two per day), such that from day to day we would be able to introduce changes to the materials, activities, and protocol in light of the emergence and refinement of theoretical constructs. These rapid-prototyping changes were based on fieldnotes, preliminary analyses of multimodal utterance, minutes from our team's daily debriefings, and collaborative, editable online postings on a shared wiki. Thus, both the interview protocol and the interactive affordances of the instructional materials evolved as we progressed through the pool of participants. We gradually incorporated into the protocol any activities and prompts that arose during interviews and that, during our debriefing, we evaluated as eliciting "researchable moments" from the participants. These were moments in which unexpected behavior from a participant suggested new theoretical constructs that we wished to test in subsequent interviews. This could be considered a formative version of data analysis, one that iteratively feeds back into ongoing interviews.

In addition to such preliminary analysis undertaken during the project's implementation phase, we engaged in more intensive retrospective analysis, using a technique I will now outline (also refer to Appendix B for a sample coding scheme).

My group's primary approach to making sense of the video data collected during the implementation of the MIT-P is *microgenetic analysis*. Generally speaking, microgenetic analysis is an intensive investigation of a relatively brief period (often less than a minute) of rapidly changing competence, with the aim of making sense of processes underlying this change. Why undertake this type of intensive investigation, rather than, say, pre- and post-test analysis? One answer is provided by Siegler:

> If learning followed a straight line, [microgenetic analysis] would be unnecessary. Yet, cognitive change involves regression as well as progression, odd transitional states that are present only briefly but that are crucial for the changes to occur…and many other surprising features. Simply put, the only way to find out how children learn is to study them closely while they are learning. (2006, p. 468)

My own reasons for this time-intensive analysis involve not only commitment to investigating the embodied aspects of learning, but also commitment to vigilance against forming plausible-sounding and rational yet historically faulty narratives of the learning process. This statement bears some explaining. First, intensive interrogation of video data is necessary to detect nuances that may go otherwise unnoticed. What may have been ignored as a series of "inconsequential" gestures is instead evaluated as potentially vital to the gesturer's understanding. Transcriptions, even highly detailed ones that include bodily movement, cannot capture in vivo activity with sufficient fidelity for explorative analysis. Thus I value an investigative approach where one human observes another's activity, typically over and over again.

Second, there is a constant lure to interpret learning process as *teleological*, as though the child reasons rationally toward a logical conclusion. This is a form of historical revisionism, that is, reading onto the beginning of a micro-process something that emerged only at its end; ascribing to a child an understanding that was not present at a particular moment. This type of revisionism is liable to ignore aspects of activity that do not fit the researcher's "tidy" narrative, such as bodily movement at odds with verbal utterances. Microgenetic analysis serves as a strong guard against historical revisionism and confounding our expert observations with those of the students we study (see also Trninic, Reinholz, & Abrahamson, 2010, on the challenge of negotiating expert and novice perspectives in design research).

In the final analysis, making inferences on the basis of videotaped performance data is a "highly subjective and interpretive enterprise" (Schoenfeld, Smith, & Arcavi, 1991, p. 70). To this end, the EDRL research group approaches microgenetic analysis in a collaborative yet competitive spirit: each member of the research team puts forth an interpretation of recorded events. We approached rater reliability in the following manner. An interpretation is considered viable if all members of the research team are convinced by it. This includes members witnessing it for the first time as a video recording. Interpretations that fail to gather sufficient support are not discarded but instead recorded as such. Through this process, only the "strongest" interpretations survive, and none are admitted without a consensus of at least three core members of the group (including Dor Abrahamson). Yet we encourage and welcome challenging interpretations, because they force us to embody the observed learner's actions. We may even physically act out what we observe, thus acquiring unexpected and unanticipated perspectives. Through personal mimetic reconstruction of recorded multimodal activity we may arrive at a completely unexpected insight, for example, that perhaps our subject lowered her arms not as an exploration action but simply because she was physically fatigued from holding them up in the air for too long! While intensive in time and effort, we have found microgenetic analysis invaluable in our investigations of embodiment. The high frequency of observations and emergent conjectures make it an excellent companion to design-based research.

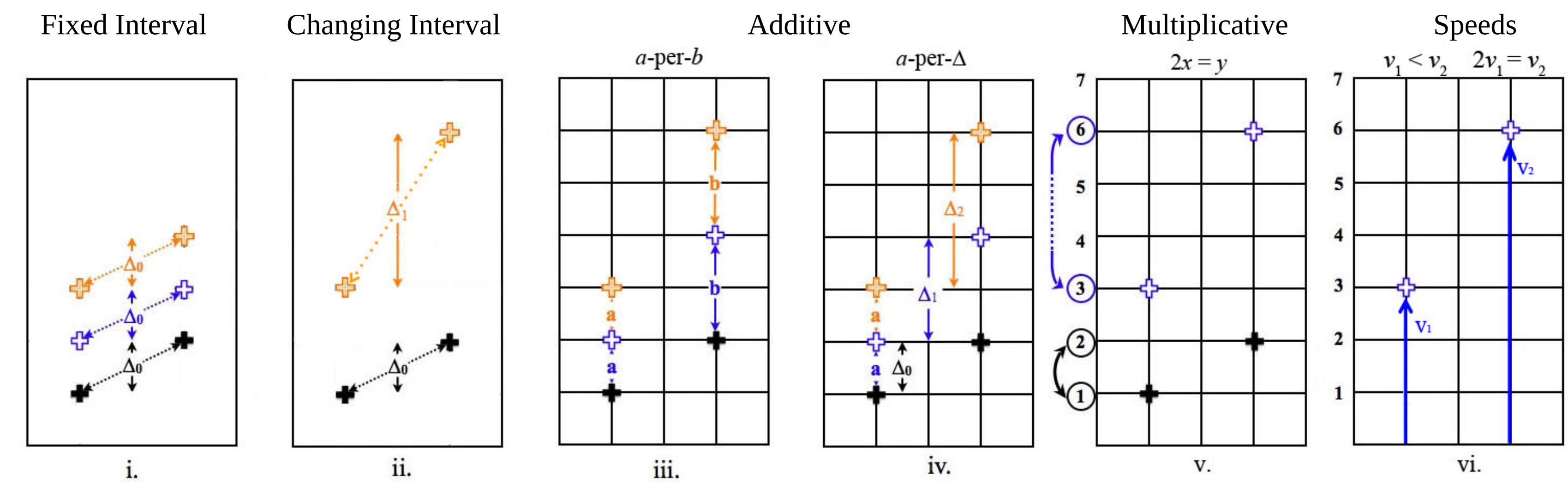
3.3 General Findings

3.3.1 Solution strategies as changes in perspective

Figure 5 (originally used in Abrahamson, Negrete, Lee, Gutiérrez, 2014), below, offers a schematic representation of the main solution strategies observed for the MIT-P problem across all the participants. Whereas these strategies are phenomenologically distinct from the learner's perspective, they are functionally equivalent. In other words, while the subjective experience of performing varies across the strategies, they yet bear similar effects and could be derived from each other through a chain of inferences by an expert.

Initially, the participants attempted to effect a green screen by moving their arms about, with some students waving one arm at a time, some students waving both arms uniformly up and down, some cautiously exploring only at the bottom of the screen, and others exploring the entire vertical extent. Eventually, all students chanced upon a "green" position of the hands. Prompted, they next sought and found another such position. Concurrent with these first successes, the students realized they should pay attention not to each hand's individual location along the screen but to the relation between the hands.

All participants articulated a strategy relating the cursors' position (how high the pair is elevated) and interval (the distance between them, hence "Δ"). Initially hypothesizing that Δ should remain constant ("Fixed Interval"; see Fig. 5i), ultimately they inferred that Δ should vary with elevation ("Changing Interval," see Fig. 5ii). This inference was of central interest to us, because it could potentially ground the meaning of proportional equivalence (e.g., 1:2 = 2:4), in which within-ratio differences vary across the equation (c.f. 1:2 = 2:3). Table 1, below, provides a breakdown of the strategy occurrence frequencies according to grade level for the last eighteen participants (i.e., once the protocol was relatively stabilized); the primary purpose of this table is to give the reader a sense of how likely (or not) each strategy was to surface during any given interview.



Legend: Lm = left-hand marker; Rm = right-hand marker; Δ = magnitude of interval between cursors (vertical and diagonal variants); v = velocity

Figure 5. Student generated solution strategies for making the screen green (the case of a 1:2 ratio): (i) Fixed Interval—maintaining Δ constant regardless of Rm-and-LM elevation (incorrect solution); (ii) Changing Interval—modifying Δ correlative to Rm-and-LM elevation; Additive, either (iii) Co-Iterated Composite Units—both Lm and Rm either ascend or descend at respective constant values *a* and *b* (*a*-per-*b*), or (iv) Lm rises by *a* (usually 1), Rm by 1 box more than the previous Δ; (v) Multiplicative—relocating to a next "green" position as a function of the height of only one of the cursors (given Lm at *x* and Rm at *y*, $2x = y; x = ½ y$), e.g., determining Lm *y*-axis value, then doubling to find Rm, or determining Rm value, then halving for Lm; and (vi) Speeds—LM and Rm ascend/descend at different constant velocities ($v_1 < v_2$) or Rm velocity is double Lm velocity ($2v_1 = v_{2;}\ v_1 = ½\ v_2$).

Table 1.
Frequency of Solution Strategy According to Grade Level for Grades 4 (n=3), 5 (n=11), and 6 (n=4)

Grade	Fixed Interval	Changing Interval	*a*-per-*b*	*a*-per-Δ	Multiplicative	Speeds
4	1	1	2	–	3	2
5	2	10	6	6	9	6
6	2	3	5	1	5	5
Totals	5	14	13	7	17	13

Elaboration on above Figure 5:

i. *Fixed Interval*. Initially, students articulated a strategy relating the hands' elevation above the baseline and Δ. In particular, once students had found an initial "green spot" and were prompted to find another, they all believed that Δ should remain constant as they move. A typical statement a student made was, "I think it's just, like, you stay the same distance apart"—a fixed-interval strategy. However, students eventually inferred from the feedback that this strategy does not constitute a valid solution.

ii. *Changing Interval*. Students inferred that Δ should vary correlative with the pair's height above the baseline. They stated, for example, "The higher you go, the bigger the distance".

Overlaying a grid onto the screen caused students to revisualize the cursors' positional properties. Incorporating this new frame of reference into their operatory schemas, they shifted their attention from Δ per se to construing two topical cursor locations. These revisualizations were concomitant with determining a strategy for recursively relocating the hands/markers, as follows.

iii. *a-per-b*. Building on Cobb and Steffe (1998), we name this proto-ratio strategy "co-iterated composite units". This strategy involves moving the hands/cursors by their respective constant rates. Students state, for example, "For every one I go up on the left, I go up two on the right". This manipulation may be enacted sequentially so as to facilitate accurate execution.

iv. *a-per-*Δ. Another recursive strategy we observed is, more accurately, "a-per-(Δ+/-1)". Here, too, the left hand paces iteratively by a constant (composite) unit, yet the right hand moves in relation to the previous interval. For example, students stated that "the distance grows by one every time". By necessity, this strategy requires sequential rather than simultaneous hand motions, because the new left-hand location must be established prior to determining the new right-hand location relative to it.

Overlaying numerals onto the grid caused students both to construe cursor locations as heights above the baseline and to recruit their arithmetical knowledge. As a result, they yet again changed their strategy, as follows.

v. *Multiplicative* (either $2x = y$ or $x = ½ y$). This strategy is non-recursive, that is, one need not attend to previous pair locations so as to determine new locations: Given one cursor's numerical position index (e.g., Lm at "2") the other cursor's "green" position is determined computationally (e.g., by doubling to "4"). Using multiplication, one student stated, "[you] double the number that the left one is on, and you put the right one on that number"); using self-adding, another student explained, "One plus one is two, two plus two is four".

vi. *Speeds*. Finally, some students described a strategy whereby the left and right hands move simultaneously, up or down, each with a different respective constant velocity. Some analogized their hands to moving vehicles, whereby each hand moves at a constant speed, v1 and v2 respectively, stating that the "right hand moves faster than the left," or that "Like this [LM] one's going 20, and this one's [RM] going 50. And they have to keep on going…" ($v1 < v2$) . Yet other students noted that Rm moves at double the speed of Lm (2v1 = v2) or vice versa (v1 = ½ v2)

Having presented some general observations across all students, next follows an illustrative case study.

3.3.2 A case study: Shani builds a meaning of proportion

What follows is an annotated transcription of a selected excerpt from our videography of Shani, a 5th grade female student ranked by her teachers as low-achieving. We will witness a succession of mathematical insights, as in Figure 5, that Shani achieves as she uses the mathematical frames of reference to articulate action routines co-enacted with the instructor. These shifts in perspective—from embodied and spontaneous to symbolic and disciplined—are where Shani builds a meaning of proportion.

In this and the following excerpts, I will be making use of the following terminology and annotations. In their right and left hand, a student would hold a right-tracker device Rd and left-tracker device Ld, respectively. The location of Rd corresponded to the position of the right marker Rm on the computer screen, similarly for Ld and the left marker Lm. Ellipses are used for noticeable pauses in utterances and // is used when one interlocutor interrupts another.

Twenty-seven minutes into the activity, the grid and its y-axis numerals are overlaid on the screen. The head researcher (DA) takes ho of the left-hand controller, and Shani operates the right-hand controller.

DA:	What should happen if I'm all the way down here? [placing Lm on lowest line on the grid, Line 1 out of 10. Shani slowly and incrementally moves her marker up towards 2. The markers are now positioned at 1 and 2, respectively, and the screen turns green.]
Shani:	So that's two.
DA:	One, and two.
Shani:	So basically like… um… if you put, like, either one at a point you'd be able to find a green.
DA:	What if this one is at two? [DA places his marker on 2. Shani quickly moves her marker up to 4.]

Shani realizes that every left-hand position can be matched by a right-hand position, and generalizes a strategy for transitioning iteratively from each such "green" ordered pair to the next.

Shani:	Oh, and they are getting further apart as it goes up. Like, the last… [She moves her marker down to 2 as DA moves his down to 1.]

Moving back and forth between these locations (see Figures 6a and 6b, below) leads Shani to attend to the vertical distance between the markers that should increases as the markers rise, in order to maintain a green screen (see Figure 4, Strategy ii).

a. b.

Figure 6. A change in perspective: (a) Shani believes she noticed a difference between 1-and-2 (not pictured) and 2-and-4 (pictured); (b) With her free hand (see circle), she gestures downwards, inviting DA to return to the previous green. This confirms her hypothesis that the markers are "getting further apart as it goes up." Her use of "it" also indicates that the space between the markers is the new focus of attention, rather than individual locations of markers. "It" changed even as the screen stayed green.

Having confirmed her observation, Shani continues:

Shani: Yeah, here it's one and two, but here [DA and Shani move their respective markers up simultaneously] it's four.
DA: What do you think it will be if I move it up to three?
Shani: Probably [moves her marker quickly upwards] six.
DA: [moves Lm up to 4]
Shani: [moving Rm upward] So this would be eight, probably.

Shani then states that the bottom, left-hand marker is "going up by one box but the top one is going up by two" (see Figure 5, Strategy iii), so the space between the marker is increasing when moving upward (see Figure 5, Strategy ii). She thus coordinates two mathematically complementary visualizations of the "green" bimanual dynamical enactment.

DA: What else can you say about those numbers? One and two, two and four…
Shani: One and two, two and four, three and six… hey wait. Oh! It's… [fidgets and becomes animated] it's all doubles. The bottom number times two is the top number [gestures at markers on the screen]…

Shani has discovered a multiplicative relation between the constituents (see Figure 5, Strategy v).

Shani: Cause this one [points at Rd] is always going up by two and this one [points at Ld] is going up by one… which would mean *that*. [long pause]
DA: Which would mean that *what*?
Shani: That um this one [right side] is always double this [left]!

Shani thus finally coordinated additive and multiplicative visualizations of the situated proportional dynamics (see Figure 5, Strategies iii & v above), a major and enduring challenge of implementing proportions curriculum (see Fuson & Abrahamson, 2005).

In this excerpt, which spanned only four minutes of guided exploration, Shani alternated rapidly between enacting effective action routines and modeling those routines mathematically with the available frames of reference. Despite being labeled “low-achieving,” Shani sustained her attention on this activity for almost an hour, making appropriate and mathematically significant connections between a variety of multiplicative and additive features of her actions.

The excerpt exemplified my view of productive engagement in mathematical reasoning. I believe that the continuous engaged focus of student and teacher on meaningful actions in a shared interactive field, rather than on writing solution procedures on paper, may have enabled Shani to infer, articulate, and connect many and different ways of acting and modeling related to curricular objectives for multiplicative concepts (Trninic & Abrahamson, under review). Analysis indicates that Shani’s case is representative of this “typical” learning experience with the MIT-P.

Note that, as evident in Figure 6 and proceeding events, Shani’s insights were not merely associated with a change in perspective, but driven by it. Close analysis suggests that Shani’s Eureka! moments reported above are better understood as experientially differentiating new ways of interacting, oftentimes in response to the interviewer’s suggestions that this is something she ought to consider (an invitation to explore this space, if you will). The next two case studies further illustrate this phenomenon of learning as serendipitous discovery.

3.4 Learning as Serendipitous Discovery: Hooks & Shifts

3.4.1 Overview of Hooks & Shifts

In this section we consider the unexpected and emergent nature of students’ insights.

In earlier texts (see Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011), the following scenario was used to outline what *hooks and shifts* might look like in an educational setting.

> Consider an apprentice carpenter who, tasked by his Master to drive a screw into a solid wood plank, elects to apply a hammer. He sedulously pounds the screw with great might but minor success, occasionally striking his thumb. The Master carpenter, alarmed to witness this travesty, hastily proffers the apprentice a screwdriver. However, being a radical-constructivist Master carpenter, she merely places the screwdriver within the apprentice’s visual field. The apprentice responds to the cue: he lifts the screwdriver, inserts its tip into the screw-head groove, lifts the hammer again and… pounds the screwdriver’s handle butt, a larger and thus more convenient and safer surface. So doing, though, his clenched fist that holds the screwdriver in place inadvertently rotates it and, with it, the screw. Ah, observes the apprentice, this is a better way of handling things. He lays down the hammer and thereafter applies the screwdriver masterfully. (p. 56)

Thus, by virtue of engaging artifacts introduced into a problem space—namely symbolic artifacts that an instructor layered onto a computer-based microworld and presented as possibly helpful for the students—the students' problem-solving strategies transformed in conceptually important ways in line with our didactical objectives. It is this unanticipated transition I found intriguing and attempted to understand (see also Abrahamson & Trninic, 2015, on characterizing transitions).

In the following sections the plan is to contribute a proposal for how students "bootstrap" towards higher levels of mathematical reasoning by virtue of engaging and incorporating artifacts into their routines (cf. Hall, 2001; Neuman, 2001). Once students engage artifacts that are framed or recognized as bearing problem-solving utility, contextually emergent affordances of these artifacts modify students' naïve actions. Eventually, students *find themselves* employing new, potentially more sophisticated forms.

The vague colloquial idiom "find themselves" is used deliberately, because—I conjecture—students experience conceptual change inadvertently and often realize vital aspects of their discovery only after they have adjusted their schema to incorporate emergent features in the environment.

Students initially recognize the new artifact either as an auspicious means of enhancing their control over the interaction space or as a discursive means of explaining and evaluating their strategy, in possible accord with the instructor's pragmatic prompt (i.e. they "hook"). Yet as they engage the artifact, embedded meanings of its features present themselves as more powerful operative–discursive grips on the interactive situation, so that the original strategy becomes modified (i.e. they "shift").

Recall that students initially worked with markers, then markers and a grid, and finally numerals were added to the grid's y-axis (see Figure 3). For this reason, the subsections below are presented in the following order: (3.3.2) Grid and (3.3.3) Numerals.

3.4.2 Thinking with, through, and about the grid

Following the pair of markers, the Cartesian grid is the second symbolic artifact layered onto the computer display. Introducing the grid onto the display implicitly catalyzed many participants to reconfigure their green-making strategies into pedagogically desirable forms. This section demonstrates how this strategic reconfiguration process can be explicated through the analytic lens of our hook-and-shift construct. In particular, I present and analyze video data from a paired-student interview to argue that the dyad collectively hooked to, and then shifted by using the grid.

Eden and Uri, two Grade 6 male participants, were selected for a paired interview on the basis of compatible mathematical achievement (both were identified by their teachers as "high achievers"). Their interview was conducted by an apprentice researcher (DT, author), and the lead researcher (DA) occasionally intervened. Eden and Uri were seated side by side in front of the remote-action sensor system and computer display and each operated one of the two tracker devices.

Hooking to the grid

Prior to the introduction of the grid, Eden and Uri had been working together for nearly 18 minutes. So doing, they identified two spatial dimensions—height and distance—as relevant to making the screen green and had articulated two theorems-in-action (Vergnaud, 1983, 2009) with regard to each of these dimensions: (a) Rm should be higher than Lm; and (b) the vertical distance between Rm and Lm is non-arbitrary. Both students had thus articulated cognitive content with respect to interaction prior to the introduction of the grid and apparently a requisite factor for hooking it.

However, Eden and Uri disagreed as to whether this vertical distance should change or remain constant as the markers move. Whereas both Uri and Eden observed different distances between the Rm and Lm at certain green locations, Uri interpreted this difference as a systemic principle for making green, while Eden attributed it to an HCI issue, as though the physical manipulation were inaccurate (Eden, apparently an avid video-game designer, referred to this error as the "human factor"). Uri articulated a covariant principle relating distance and height, explaining that "it has to get, like, farther away, the higher up we are" and that "the lower you are, the less distance apart it has to be"—a changing-distance theorem-in-action. Eden, however, courteously responded with, "Well I'm not sure if it matters if you're lower or higher, but I think it's just, like, you stay the same distance apart"—a fixed-distance theorem-in-action.

Thus, Uri and Eden's collaborative hands-on problem solving enabled them each to notice and explicitly articulate a relation between the markers' height and distance (both students bore cognitive content with respect to the interaction). Yet whereas Uri concluded from their empirical data that the distance should vary, Eden concluded from the same data that it should not (each student had validation for their subjective content, yet note that at this point Eden bore a correct dimension, incorrect value, i.e. he attended to the distance but judged it to be constant).

The students' disagreement bore practical implications, because the dyad was co-operating the two devices—each student depended on the other to enact a green-making theorem-in-action, yet their respective theorems were mutually exclusive. Consequently, the students' success within this collaboration became contingent on whether or not they could rule between their incompatible fixed-distance and changing-distance theorems-in-action. At the same time, they were apparently under-equipped to arbitrate within the continuous space. Namely, when the grid was subsequently introduced (see below), they recognized its potential for ruling between the theorems—they "hooked" the grid largely for its discursive, argumentation, and arbitration affordances. Specifically, the grid served these boys to quantify the distance between the markers and ultimately determine that this distance should in fact change between green spots, as Uri had believed and Eden soon concurred.

The excerpt below begins immediately after DT had layered the grid onto the screen. In passing, note how both students immediately recognized the grid's mathematical function, i.e., both demonstrated fluency. That is, we can assume that both Uri and Eden are graph fluent, because they immediately identify the object as “Grid” and orient to it as parsing the working space into enumerable quotas of spatial extension, which was not the case for all study participants.

Eden:	<19:36> Grid.
Uri:	Yeah. [grabs Rd, lifts it, and remote-places Rm on the 1st-from-the-bottom gridline (hence “Rm up to 1-line”). Simultaneously, Eden, too, brings Lm up to 1-line. On the way up, between 0-line and 1-line, the screen flashes green for a moment but then turns red. Eden lowers Lm back down, holds it at .5 units. The screen turns green.] Oh so you can like show where… Let’s see, so [Rm up from 1-line to 2-line]// if you’re on here…
Eden:	//maybe it has to be two… [Lm up to 1-line (see Figure 7, below)] an entire box apart.
Uri:	[Rm up to 3-line] If I go here…
Eden:	[Lm up to 2-line; screen goes red (see Figure 8, below)] Then maybe you should raise it [Uri raises Rm to just below 4-line; screen flashes green]. So maybe the higher you go, the more boxes it is apart.
Uri:	Let’s just say like I’m here [Rm down to 2-line], then he has to be one box under me…
Eden:	[Lm down to 1-line; screen goes green] And then the higher he goes//
Uri:	//and when I go here [Rm up to 3-line], he has to be like in the middle [Eden moves Lm up to 1.5 units; screen goes green]
Eden:	So the higher//
Uri:	//And here [Rm up to 4-line, while Eden moves Lm up to 2-line] he has to be like two boxes under me.
Eden:	So like the higher it goes, the more space there has to be between each. [both Eden and Uri place their tracker devices on the table]

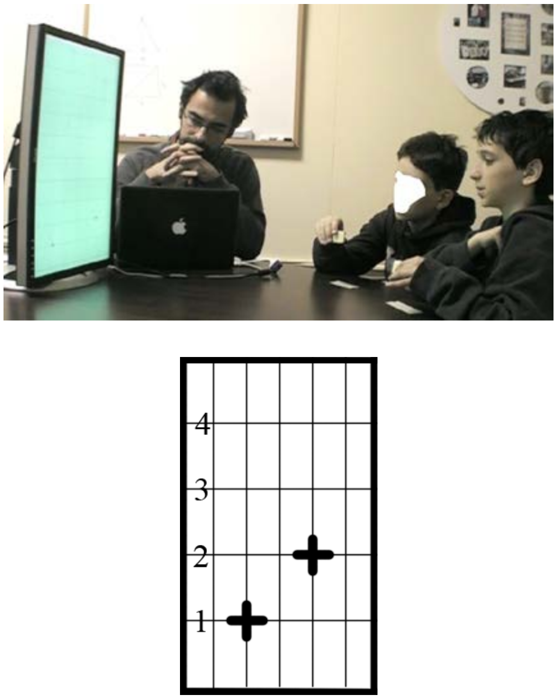


Figure 7 (above). After the introduction of the grid, Uri (middle) and Eden (far right) find green with Rm at 2-line and Lm at 1-line, respectively. Noticing the distance between the Rm and Lm, Eden predicts that the fixed-distance subtends “an entire box.” The diagram directly above this caption is a partial schematic recreation of the screen (actually, the *y*-axis ran to 10).

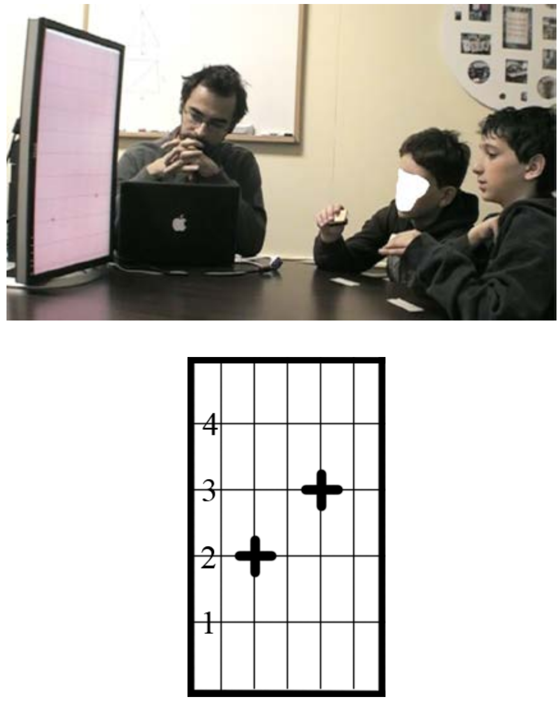


Figure 8 (above). Immediately, Uri and Eden reposition Rm and Lm to 3-line and 2-line, respectively. The screen turns red. Upon noticing that the fixed-distance theory does not obtain, Eden says, “Then maybe you should raise it. So maybe the higher you go, the more boxes it is apart.” This diagram, too, was recreated for clarity.

Thus, it appears that both Eden and Uri immediately appropriated the grid in view of its affordances to arbitrate among their conflicted theorems, however they differed with respect to the nature of their discovery, and this difference can be related to their idiosyncratic beliefs prior to the introduction of the grid. Namely, Uri had articulated a changing covariant relation between height and distance, so for him the grid afforded reiterating and quantifying this qualitative principle. Specifically, the grid enabled Uri to reformulate his continuous qualifier “get farther away” as the discrete quantifiers “one box” and then “two boxes.” Eden, who had acknowledged the in-principle possibility of a changing-distance rule yet maintained a fixed-distance rule, soon changed his mind and articulated a changing-distance hypothesis (“the higher you go, the more boxes it is apart”). However, Eden ends with a qualitative statement about “space,” which suggests that Eden construed the grid as a means not of quantifying the “higher–bigger” conjecture but of evaluating whether or not this conjecture even obtained. Thus, Eden and Uri both hooked to the same artifact, yet they utilized the collaborative inquiry activity it enabled for different purposes.

This episode demonstrates a common-sense view that an artifact’s subjective utility is contingent on the individual’s goals. Yet the episode also suggests that a dyad can engage in physically co-

enacting collaborative inquiry even as they develop and hold different theorems-in-action (compare to Sebanz & Knoblich, 2009, who suggest otherwise). Finally, the capacity of learners' to engage in deep reflection over collaborative manipulation suggests that embodied reasoning can be distributed intersubjectively, with perception of vicarious action acting as proxy for action, as long as perception is monitoring vicarious action against the enactment of a particular theorem-in-action.
In the following excerpt, we continue at a point where the dyad initiates further inquiry.

As we shall see, the dyad's exploration will shift them from the now-consensual "higher–bigger" strategy toward a proto-ratio a-per-b strategy. Both strategies can be viewed as expressing covariation—"the more x, the more y"—that is enacted as coordinated bimanual operations embodied and monitored in the functionally extended spatial medium of the computer interface. However, the former strategy is continuous–qualitative, whereas the latter is discrete–quantitative, so that adopting and articulating the latter strategy is a pedagogically desirable outcome. The students' shift was apparently contingent on their consensus over covariation rather than fixed-distance as their theorem-in-action. Namely, some of our study participants, who identified the correct dimension (i.e., distance) but inferred an incorrect value for this dimension (i.e., constant), did not experience a shift or experienced difficulty in shifting.

Shifting with the grid

Having reached consensus, Eden and Uri elaborated their explanation. At this point, they had instrumentalized the grid to quantify the distance between the hands. This new conceptualization of space is soon to engender the semi-spontaneous emergence of a new mathematical form. In the transcription that follows we will observe that the students shift with the grid from a continuous–qualitative strategy to a discrete–quantitative strategy. In particular, the students are about to change the object of their co-manipulation from: (a) the distance between the hands/markers; to (b) each hand/marker's location independent of the other one. Following this excerpt, I offer an interpretation of media and interaction factors inherent in the shift.

Uri:	<20:19> I think, like, uhm, when I go up to here [points to 2-line], he has to be one. Then when I go up//
Eden:	Like for every… for every box he goes up, I have to move, go down //
Uri:	//You have to go up half// //a box.
Eden:	//Yeah//

The dyad's coordinated production of green tacitly modulated from pre-grid simultaneous motions, in which, ideally, the distance constantly changes and green coloration is maintained throughout, to with-grid sequential motions, in which each hand separately ratchets up to its respective designated destination and green is effected after a brief red interim, once the second ratcheted motion is completed. Imperceptibly, the dyad thus shifted from their "the higher, the bigger" continuous–qualitative strategy to an a-per-b discrete–quantitative strategy.

We wish to highlight several properties of discourse that contributed to the shift beyond each child's strategic perceptuomotor interactions: (a) the sequentializing (linear) constraint of the speech modality, which introduces order into originally simultaneous actions; (b) the indexing or deictic affordance of the new symbolic artifact, which enables unambiguous reference to particular physical locations germane to successful enactment of strategy subgoals; and (c) turn-taking norms of conversation about distributed actions, which suggest splitting the description into respective complements.

Eden and Uri thus co-discovered that in order to maintain green, they should progress at intervals of 1/2 (Eden) and 1 (Uri) coordinated vertical units, by either both going up the screen or both going down. It is through this serendipitous discovery that their earlier observation, "the higher you go, the more boxes it is apart," a covariation between height and distance, transformed (shifted) into a covariation that foregrounds the independent actions of the left and right entities, "For every box he goes up—you have to go up half," a new strategy that is closer to normative forms for ratio (i.e., a-per-b). I wish to underscore that whereas the general x-per-y covariation form was maintained, its semantic–mathematical content was replaced (see Table 2, below).

Table 2.
Consistent "Covariation" Linguistic Structure Across Strategy Micro-Shifts

Mathematical Properties	"The more *x*,…"	"the more *y*."
Continuous–Qualitative:	"The higher you go,…"	"the bigger the distance"
Discrete–Qualitative:	"The higher you go,…"	"the more boxes it is apart"
Discrete–Quantitative:	"For every box he goes up,…"	"you have to go up half"

In addition to explicating our hook-and-shift construct, our analysis of the case has demonstrated that collaborative mathematical learning processes are impacted by nuances of personal/interpersonal framing to the extent of dissociation between a dyad's mechanical and epistemic actions. Namely, whereas the two dyad members collaborated on using a single symbolic artifact (the grid), their joint experiment simultaneously enacted an exploration of two different hypotheses ("same distance" vs. "different distance"). Uri was quite comfortable from the very onset with the higher–bigger principle, so he did not need arbitration but refinement, whereas Eden, who challenged Uri with a same-difference theorem, needed resolution. Once a common ground had been established, the dyad was able to continue mathematizing the mystery artifact–phenomenon and jointly articulate a new mathematical form, which the researchers recognized as pedagogically desirable.

3.4.3 Thinking with, through, and about numerals

Following the pair of markers and the Cartesian grid, the y-axis numerals are the third symbolic artifact layered by the interviewer onto the computer display, in accord with the interview protocol. This section presents a case of a student who hooks and then shifts by using the numerals. Our case-analysis participant, Siena, is a 6th-grade student identified by her teachers as low achieving. She, too, was interviewed by an apprentice researcher (DT), with the lead researcher (DA) occasionally intervening.

About 20 minutes into the interview, the grid was introduced. Siena immediately responded that the grid would "make it easier to say where it is," gesturing the second "it" to mean the hand locations effecting green. As such, Siena hooked to the grid as enabling her better to explain her strategy content. However, she did not go on to use the grid so as to “say” or otherwise demonstrate green locations. In order to probe her statement, DT held the Rd so that the Rm fell precisely on a gridline and asked Siena to predict where the Lm should be so as to effect a green screen. Initially, Siena did not lift the Ld but instead communicated her predictions for the Lm’s location by pointing with a finger to one of the horizontal lines on the screen. Siena was seated too far away from the screen so as to literally place her finger on the particular line she was referring to, and so the interviewers, who were positioned to her sides, could not know unequivocally which line she was referring to. A need for repair action thus emerged in the conversation. The following excerpt begins shortly before the introduction of numerals onto the screen.

But first, a brief clarification should help the reader make sense of Siena’s otherwise abstruse statement, below. The computer display monitor was rotated 900 from its normal “landscape” orientation to a “portrait” orientation, so as to accommodate the vertical interaction space required by our design. Consequently, the silver “DELL” logo, which is usually located directly below the screen at the center of the framing panel, was located halfway up the left-side frame panel and oriented downward. Siena will be using features of this logo to refer to a particular location on the screen.

DT:	<22:15> So how about for here? [places Rm on 8-line]
Siena:	I think… [lowers right hand toward the Ld that is lying on the desk, lifts it slightly, but then lets go; hand rises, index drawn out] Uhhm, I think… it would be [right index points toward the left-side panel of the screen, then glides horizontally to the right along the 5-line to an empty space in the Lm column, then back to the left-side panel (see Figure 9, below)] right at the E on the DELL.
DA:	Here? [places right index finger upon the “E,” such that the finger is pointing horizontally across the screen to the right, along the 5-line that Siena had indexed (see Figure 10, below, noting DA’s index finger)]

Figure 9. DT holds Rm at 8, and Siena is asked to predict the position of Lm. She utters, “Uhhm, I think… it would be right at the E on the DELL.” Her right index finger points to “E” then sweeps back-and-

Figure 10. Immediately following Siena’s gesture, DA places left index finger upon the “E” (see middle left), such that the finger is pointing horizontally across the screen to the right, along 5-line.

forth horizontally along 5-line.

Siena: Yeah.
DA: Let's see. [moves finger away from screen]
Siena: [lifts Lm toward the height of the 5-line. On the way up, the screen flashes green at 4-line; she hesitates very briefly then continues upward to 5-line. The screen turns red] Oops! Never mind. Guess it was down here [lowers Lm to 4-line, where she had briefly effected green on the way up]. On the line below.
DA: On the line below, uh-huh…mmm.
DT: Okay, so, maybe instead of having to, you know, point to every line, it would be easier if we had… names for them? [chuckles]

DT's utterance, which might be analyzed grammatically as an interrogative probe, as though DT is seeking information, in fact serves pragmatically as more than a mere question. Namely, DT is implicitly communicating to Siena that it is in the interest of the collective activity that Siena now assume agency in elaborating on the symbolic artifact, and in particular by interpolating appellations for the gridlines, such that the lines can be referred to unambiguously. As such, DT, who is about to introduce the numerals onto the screen, is framing the numerals' designed function in advance, so that when they appear, they will be immediately construed as serving a particular goal, a discursive goal of regulating the conversation by repairing the ambiguity of referents. DA will now follow up on DT's pragmatic framing, as though it has been established that it is in the general interest of the interlocutors to label the referents. Note also that the new symbolic artifacts, the numerals, will appear immediately after their intended function has been established. As such, Siena has been primed to frame the numerals as serving a particular function. Priming is necessary and critical, because even if students have established the prospective affordances of a symbolic artifact in anticipation of its appearance and in accord with the design, still they might forget this framing by the time it appears.

DA: What would be good ways of naming those lines?
Siena: A, B, C, D… [Following this utterance, she performs three descending chopping gestures with her right hand, her palm facing downward, marking that the top gridline, at shoulder height, should be named "A," the line below it—"B," etc.]
DA: We could do that. We chose //
Siena // …or numbers.

Note that Siena is fluent in basic representational strategies that avail of common symbol strings, such as "A, B, C, …" or "1, 2, 3, …" Though this particular fluency may appear trivial for a 6th-grade student, the dimension of fluency becomes more pertinent when students are expected to use symbolic artifacts that they have not mastered sufficiently, such as when some 4th-grade students behold a Cartesian grid.

DA: Or numbers. We chose numbers. [DT operates the console, and the y-axis numerals appear on the screen]

In analyzing the above excerpt, we find it helpful to orient ourselves with the following question: Why does Siena suggest to label the horizontal gridlines as A, B, C, D, …descending—rather than 1, 2, 3, 4, … ascending?

To adequately answer this question, we must trace Siena's activity leading to the "A, B, C, D" utterance. Specifically, recall that when DT placed the Rm on a particular gridline and challenged Siena to find its green Lm counterpart, Siena responded by pointing toward the screen so as to indicate her suggestion for its location. Yet pointing became cumbersome, because her distance from the screen prevented unambiguous deictic indexing. Thus a local discursive goal emerged for Siena to better index the location—the specific grid-line—she was gazing toward. The "E" in "DELL"—a contextually salient perceptual landmark located in the appropriate vertical position, if off to the left of the intended marker position—occurred to Siena as a direct pragmatic means of inviting the interviewers to co-attend with her (see Figure 11, below).

Figure 11. An example of using fortuitously available features of the environment to resolve the ambiguity of a speech referent. One interviewer (DT) had placed Rm on 8-line. Siena estimated that Lm should be "right at the E on the DELL." The other interviewer (DA) points accordingly along the 5-line. (Scene recreated by the author for this text.)

It thus appears reasonable to assume that Siena did not, at that point, assign any mathematical/quantitative meaning to the grid. Moreover, her initial evaluation that the grid would help her "say where it is" notwithstanding, Siena struggled to utilize the grid as a means of indexing the markers' locations. Indeed, it appears she valued the discursive utility of the grid at best as equal to the alphabetical characters in the "DELL" logo—that is, purely as spatial placeholders. Moreover, she viewed the "DELL" as a superior index, compared to the grid, possibly because the stylized, tilted "E" is a unique landmark on the monitor, whereas the many gridlines are indistinguishable save for their serial location. That is, enumerating the lines did not apparently occur to Siena as a viable means of distinguishing among these many lines. Whereas a graph-fluent person would very likely count the gridlines from the bottom and up toward the line in question, to Siena the gridlines per se did not afford any means of remote disambiguation that would support the interaction flow.

Siena, then, scanned the environment for a means of indexing a particular horizontal line on the screen. Seen from this perspective, her suggestion to label the lines from top to bottom is a fine

solution to her localized discursive goal of unambiguously indexing the lines. Siena thus anticipates the numerals' contextual indexing affordance and can therefore hook the numerals, once they actually appear on the screen.
We continue at the point where the numerals have appeared on the screen. Again a brief clarification is due. The y-axis numeral "8" marks the gridline where DT had been holding the Rm just earlier, when Siena had guessed that the Lm should be at the "E" so as to effect green. The "E" is adjacent to, and roughly at the height of the "5."

DT:	See those? [sweeps hand downward toward the numerals on the screen]
Siena:	Mmhm.
DT:	So, let's kind of do the same thing//
DA:	//Even before you do that—so if, if Dragan [DT] puts it [Rm] up at eight [8-line], where do you think... the other one [Lm] should be?
Siena:	Five.
DA:	Let's try. [DT lifts Rm to 8-line and holds it at that location]
Siena:	[raises Lm to 5-line] Oops. Ah! I always do it wrong. [lowers Lm to about 4-line] I'm always wrong... I always won't find it … Wait!! Wait, wait… Go to ten. [Still holding Ld, with the Lm suspended at 4-line, she points her right index finger up toward the "10" gridline. DT lifts Rm to the "10" line and holds it at that location; Siena lifts the Lm to the "5" line] Oh! [glances at DA] It's always half!

With her utterance "It's always half" Siena first enunciated the multiplicative constant that relates all Lm and Rm locations co-effecting green under the 1:2 setting. Siena's reasoning consisted essentially of evaluating the contextual utility of applying an emergent affordance toward the solution of an unresolved problem.

Some sociocultural theorists (e.g., Newman, Griffin, & Cole, 1989) might thus view this episode as validating the hypothesis that naïve and expert views are cognitively incompatible—that Siena's learning experience is marked by a clear break from before to after she engaged the new semiotic potential of the cultural artifact. Whereas I agree with the judgment that the process of learning ought to be viewed as strongly framed by its sociocultural context, I would disagree with an assessment that Siena's *personal* experience is disconnected. Namely, whereas an expert may identify logical discontinuity in Siena's reasoning, we must be vigilant against positing this discontinuity inside her personal experience. Simply stated, no third-person account of a dance captures the dancer's first-person experience. Furthermore, for the student to perceive the alleged discontinuity, she herself would have to be an expert! Thus the danger we run here, to paraphrase Marx, is mistaking our modeling of the world for a world of models. I posit instead that a personal connection exists and is phenomenological rather than logical: it emerges reflexively in the sequencing of Siena's interactions with the artifact as expressed in her utterance sequence. That is, I suggest that Siena's sense of connectedness is grounded in spatial–temporal continuity of intentional, embodied activity. Her shift to a new mode of thinking cannot be explained as a logical continuation of indexing but rather as a recognition that emerged in the activity of doing and telling (cf. Roth & Thom, 2009).

Lastly, I underscore that, in cases such as this, it is deceptively easy to commit historical revisionism and claim that the student interacts with the numerals because she sees them as enhancing her reasoning in the manner they would enhance the reasoning of an expert. Precisely the opposite happens here: Siena recognizes the expert utility of the numerals because she interacts with them in the first place (see Wertsch, 1979). Specifically, what we might call "progressive mathematization" (Freudenthal, 1986) emerges through the student's active interaction with the numerals: mathematization can be caused by rather than be the cause of interaction. We thus agree with Prawat (1999) that the so-called learning paradox is such only inasmuch as we conceptualize learning as a purely logical-deductive process—a topic we shall return to in Chapter 4.

3.5 Closing Remarks

Using the Mathematical Imagery Trainer for Proportion, students first learn to move in a new way and describe their action routine in qualitative language. Next, they use mathematical frames of reference to describe their actions quantitatively, effectively developing a mathematical model grounded in their embodied scheme. So doing, students link conceptual features both among compatible ways of moving and among commensurate mathematical models.

More specifically, I outlined what I consider the core finding presented here, namely cases of learning that can be characterized as unanticipated, opportunistic, serendipitous. For example, Shani is focused on making the screen green across various marker locations when she notices that "it" (the distance between the markers) appears to be changing from location to location—an important qualitative observation of proportion at work. As far as the two boys, Uri and Eden, are concerned, their shift to *a*-per-*b* additive strategy for making green was "merely" the result of using the grid to better communicate their strategies to each other. And—in what is perhaps the most striking case—Siena stumbles upon the multiplicative strategy not because she wanted numbers to do *arithmetic* with, but because she desired numerals as *labels*. In each case the student(s) did not plan out their discovery ahead of time, or even aim in the general direction, but *found themselves* moving and seeing in new ways, a difference which they recognized to be in some way advantageous.

I drew on Wittgenstein in Chapter 1, and here it is useful to return to *Philosophical Investigations* (1953) once again, to elaborate on what I mean by the unanticipated nature of insight in learning:

> An intention is embedded in its situation, in human customs and institutions. If the technique of the game of chess did not exist, I could not intend to play a game of chess. In so far as I do intend the construction of a sentence in advance, that is made possible by the fact that I can speak the language in question.
> After all, one can only say something if one has learned to talk. Therefore in order to want to say something one must also have mastered a language; and yet it is clear that one can want to speak without speaking. Just as one can want to dance without dancing. (337-338)

Here Wittgenstein is alerting us to the notion, prevalent throughout this text, that we cannot intend to do something we do not already know. I *want to* speak in French; but I cannot *intend to* tell you about how my day went in French because I do not know how.

Similarly, I cannot intend to learn something I do not know, I can only want it. Some aspects of learning are, therefore, unexpected, occurring when students recognize in their changing actions and perspectives novel yet situationally advantageous ways of moving and seeing.

In the final chapter, I draw on both MIT-P findings reported in Chapter 2 and findings on enactive artifacts in Chapter 3 to make sense of the original question posed in this dissertation: Can what we call conceptual understanding emerge from embodied interaction? If so, how does this process come about, and what does it tell us about the procedural/conceptual divide in mathematics education?

Chapter 4: Practice Makes Practice

Practicing should be experiment, not drill.
(Artur Schnabel, as cited in Bamberger, 2013, p. 3)

4.1 Overview

Can conceptual understanding of mathematics emerge from embodied interaction? This question, which oriented my dissertation work, was posed in the context of issues regarding the roles of procedures and concepts in mathematics education. Overall, my work can be characterized as an attempt to understanding the consequences of enactivism for the would-be procedural/conceptual divide in educational research of mathematical cognition and instruction.

The issue might be stated like this: We ought to acknowledge either that we are Platonists or that conceptual understanding comes, at least in part, from *something we do*. I wondered what, if any, role procedural practices play in this process. (After all, quite often we find ourselves working out some routine or other.) Gradually, this investigation turned from procedures to routines and then to practice, broadly conceived, and the roles of all these activities in the development of disciplinary competence.

In much of mathematics education research, there appears to be an unspoken consensus to view the routine practice of procedures as, in some substantial sense, unrelated to conceptual understanding. A recent, comprehensive meta-survey of literature (Crooks & Alibali, 2014; see Chapter 1 for an overview) reports that even if this opinion is not explicitly, *overtly* professed by education researchers, it is nonetheless the reality of how many of us do our work.

I have argued that this account appears at odds with theories of embodied cognition which posit that our mental landscape is necessarily vested in to the physical one. In these frameworks, particularly the so-called radical strain of enactivism I advance in this text, conceptual understanding is theorized to come from *and incorporate* actions we take in the world, including procedural actions (such as long division, multiplication algorithm, and so on).

It is worth a brief detour, here, to acknowledge that not even the most ardent Cartesian denies that *some* aspects of our cognition depend on our bodies and the ways we move. For example, nobody doubts that our prehensile hands allow us to encountering the world in a way different from dolphins. These differences are readily explained by differences in embodiment. This, however, is a weak notion of embodiment, one that can be made consistent with disembodied views of the mind. This text is about something more radical: cognition as not secluded or elevated from perception and action but rather embedded in, distributed across, and inseparable from these corporeal processes.

In this account, our ostensibly unique human mental capacities are not due to modal symbolic–propositional processing capabilities, but due to the fact that we can incorporate our bodies and even the world into our thinking—that we literally think and problem solve not merely “in our heads” but by interacting with the world, whether this interaction is real or simulated. The world mediates our thinking because we think with it. We might even say that this interaction *is* our mind:

> Tools are not simply external markers of a distinctive human mental architecture. Rather, they actively and meaningfully participate in the process by which hominin brains and bodies make up their sapient minds. (Malafouris, 2012, p. 230)

But more on these important issues later (next section, in fact). For now I only offer that if we take these frameworks seriously—and a growing consensus in cognitive sciences is that we should—we find that the conceptual versus procedural dichotomy requires another look.

Many debates in math educational policy, at least in the United States, have taken the form of either:

(1) foremost, students ought to understand the mathematics they are learning (e.g., answering: “What is a slope?”), or
(2) foremost, students ought to learn how to do the mathematics (e.g., answering: “How do you calculate a slope?”).

One claim put forth here is that we cannot have one without the other.

As noted in Chapter 1, analysis of the literature yields *zero results* indicating an individual who simultaneously *understands* mathematics yet has no ability to do *procedural* mathematics.[19] Alternatively, when we say a student has procedural knowledge but lacks understanding, we mean more precisely that he lacks the type of understanding *we* are interested in. To illustrate this point—when I teach, I find that *all* students have an understanding of what they are doing, even if most of them were “merely” following rules they hoped led to a positive final grade. Assessment-driven understanding, literally. Not the type of understanding we hope for, I think.

In Chapter 1 we also explored the perspective that meaning springs forth from doing. Thus, what a thing means for us is what we *use it for* and *do with it*. When students do endless repetitions of FOIL—a mnemomic for $(a + b)(c + d)=ac + ad + bc + bd$—only for the sake of getting better at FOIL-ing, mathematics will mean to them something different than if their practice consisted of, say, repetitions of binomial multiplication in the context of investigating binomial probability.

And this is a crucial point I hope to explore. What is it we want students go learn? How to FOIL, or something more than that?

The issue with FOIL is that it is a procedure that doesn’t go anywhere. In fact, it seems to be worse than that, because it seems to lead to *incorrect* conceptions just fine (Koban & Sisneros-Thiry, 2015). FOIL seems to be at its core an admission to our students that we are primarily interested in teaching them how to obtain correct answers during a test.

I am strongly against using FOIL in classrooms. But what is to be the solution? Students must *do* something unless we believe that they can learn polynomial multiplication purely through listening to the teacher talk about it. We will come back to this, but what I want to suggest for now is that FOIL and similar procedural practices aren’t bad *because they are procedural*

[19] With the same caveat mentioned in Chapter 1.

practices; instead, I argue that such practices are bad *because they are bad procedural practices* (also done out of context). At least, they are bad procedural practices insofar as we want students to develop the types of mathematical understandings we value.

Like most mathematics education researchers, I want education to focus on conceptual understanding. I am, however, also deeply concerned that a focus on conceptual understanding without an equal focus on practice is akin to throwing the proverbial baby out with the bathwater. Not all procedures, and not all practices, are created equal—this is true. Some procedural labor is menial and perhaps students would be better off without it at all. But surely the solution should include better procedures, better practices, rather than throwing them all out?

This interest in practice led me to a cross-disciplinary journey. The rhetorical gambit I have employed throughout this work has been to step away from the world of mathematics education and immerse myself in the investigations of other pedagogical domains, typically ones with overtly embodied modes of practice. My rationale was arguably similar to that found in a recent paper by Saxe and collaborators "*Regardless of the domain,* teachers engage children with the equivalent of historically elaborated scientific concepts and procedures" (Saxe, de Kirby, Bona, Le, & Schneider, 2015, p. 41, emphasis mine).

In this process, I was particularly interested in pedagogical practices that have been validated historically; in other words, practices that have passed the test of time by remaining relatively stable over decades or even centuries. When discussing the explicitly embodied disciplines in Chapter 2, namely martial arts and surfing, I offered that teachers guide students' enskilment by means of certain embodied practices—rehearsable disciplinary routines—which I called *enactive artifacts*. The short of it is that enactive artifacts are routines through the practice of which students acquire disciplinary understanding. As the Soviet neurophysiologist Nikolai Bernstein (1996) argued, teachers can teach *outer* aspects of knowing—hands go there, elbows here—but *inner* aspects must be experientially felt. This is why no one becomes a skilled athlete through attentive listening alone. Practice of enactive artifacts is a *repetition-without-repetition*, the purpose of which is not in performing some would-be perfect routine but precisely in developing the repertory of agile fixes to emerging contingencies. Bernstein writes:

> Repetitive solutions of a problem are necessary because, in natural conditions, external conditions never repeat themselves and the course of the movement is never ideally reproduced. Consequently, it is necessary to *gain experience relevant to all various modifications of a task*, primarily, to all the impressions that underlie the sensory corrections of a movement. (p. 176)

Thus Bernstein's notion of repetition-without-repetition speaks of repetitive routines that are necessary to gain experience relevant to "all the impressions that underlie the sensory corrections of a movement."

The central idea I draw from Bernstein's notion of repetition-without-repetition and my studies of enactive artifact "in the wild" is that the point of repetitive practice, of routines, is often about more than "merely" improving the routine itself. Routines are rather practiced because they attune the practitioner towards certain previously inaccessible or unnoticed features of the world (which includes their motor movements and mental landscape). This attunement is a felt, bodily sensation enabling greater context-sensitivity as skilled responding improves over time (see Dreyfus & Dreyfus, 1999). These sensitivities take the form of what *we can do with the world*. Overall, this perspective is *far* removed from the conceptions we have of drills as boring or mindless—here, routines reward the attentive and open mind with secrets of the discipline.

Chapter 3 continued the investigation of enactive artifacts, this time in the context of a mathematics education design-based research study. If cognition is grounded in bodily experience, learning environments can be made more effective if they tap into everyday bodily knowledge. Yet, we noted that

> In much of everyday activity, meanings are tacit, contextual schematized orientations toward obtaining goals under given circumstances….STEM disciplines, however, concretize, parse, analyze, and quantify these naturalistic interactions. To understand STEM content, students must reconcile their unmediated perceptions and actions with the mediated structures of disciplinary practice. (Abrahamson & Lindgren, 2014, p. 360)

Accordingly, Chapter 3 explored the reconciliation between spontaneous and scientific ways of encountering the world. Here I introduced the notion of *hooks and shifts*, or moments wherein learners recognize something new in and through their actions. The short of it is that, through *guided* repetitions of a routine—repetition-without-repetition—learners *find themselves doing something new* even as they adapt to changing circumstances (e.g., the introduction of a novel artifact into the routine).

Additionally, this chapter shored up some theoretical support for applying theories of embodiment equally to martial arts and mathematics and, in turn, for using findings in martial arts pedagogy to inspire design for mathematics education.

This brings us to the present chapter. In terms of rhetorical structure, our eventual goal is returning to the questions of conceptual emergence in the context of mathematics education. However, the journey will prove valuable in and of itself, perhaps more than the destination. This journey is about exploring novel cross-disciplinary ideas of what it means to learn and to know, and what those accounts have to say about conceptual understanding. Along the way, I will continue drawing on my own research as well as research ranging from archeology to neuroscience, from philosophy to kinesiology.

The first step will be to contextualize this work in a broader context. Admittedly I have been doing this all along; however, so far this dissertation has been about enactive artifacts, with a focus on *enaction*—and thus embodied cognition. Yet enactive artifacts are also artifacts. Thus, in the next section, I will re-contextualize my work in the broader context of recent scholarship concerning the role of artifacts in thinking. Building on Vygotsky's legacy, this line of scholarship argues that artifacts not only mediate thinking, but "actively and meaningfully participate in the process by which hominin brains and bodies make up their sapient minds" (Malafouris, 2012, p. 230).

Once I present the case that we literally think with artifacts, I will return to my work, continuing the case that (enactive) artifacts also change our minds and asking how this process occurs. I emphasize that routines can shift our actions and thinking in ways unanticipated and unexpected; furthermore, some of these shifts can be gradual, while others abrupt. Ultimately I argue that, rather than focusing on concepts as something that should or even can be taught directly, we ought to focus on the sorts of practices that result in particular forms of reasoning we consider conceptual. These types of practices I characterize as explorative. Along this path, I will argue that this *unanticipated* nature of interaction is useful in making sense of the classical learning paradox.

Respecting these two observations—that artifacts play an active and meaningful part in our thinking, and that our interactions often shift our thoughts in unanticipated ways—I offer the notion of an "instrumented field of promoted action" as an alternative to characterizing mathematical activities as merely procedural or conceptual.

At the end, I recap some major landmarks encountered along the way and offer closing commentary.

4.2 The Role of Artifacts in Thinking and Learning

In his landmark *From Social Interaction to Higher Psychological Processes* (1979), James Wertsch provides empirical evidence that young children learn through scaffolded interactions and so come to understand the adult's "definition of situation." The empirical context was mother-child interaction in the joint solution of the cargo truck puzzle. Wertsch found that "One of the major ways that [situational coherence] is created for the child is by carrying out the behaviors specified by the adult *and then* building a coherent account of the relationships among speech, definition of situation, and behavior" (p. 20, emphasis mine). I wish to emphasize this *opportunistic* rather than premeditated nature of insight, a theme we will pursue throughout this chapter. It is only *after* engaging in the activity that the child understood it as the adult does.

If we choose to take this opportunistic aspect of learning seriously, we may have to lessen our intellectual grip on the primacy and importance of individual intention. To illustrate what I mean, consider that the children in Wertsch's study could not have *intended* to understand a situation they did not understand ahead of time.

In fact, this is one variation of the learning paradox (see, e.g., Prawat, 1999). The argument goes like this: If children already know what they were going to learn, no learning takes place; If children do not know what they are to learn, how do they go about learning it? If we are not Platonists, we can assume that children did not, in fact, already know what they were going to learn. What about the second assertion? As Wertsch writes, children in his study learned through doing—they *found themselves* knowing something they did not anticipate ahead of time. We could say that they didn't intend to go there, yet, through interaction and guidance (other-regulation) *recognized* their arrival.

So learners *find themselves* doing something new. If they find themselves doing something new, how did they get there? Across the examples presented there, it seems futile to suggest that they *intended* to, because if they intended to walk down a particular path, there'd be no Eureka! moment of insight after all. (Unless, of course, we suppose that they must have forgotten about their earlier decision to walk down this particular path—but that won't do.)

As we lessen the spotlight on *individual* intention, we are able to cast a light on other elements involved in intelligent activity. This section explores contemporary studies concerning the role of artifacts and the ways in which artifacts do our intending for us.

Concept formation, according to Piaget (Piaget & Inhelder, 1969) emerges from sensorimotor interaction with objects. For Vygotsky, concept formation is best understood as utilizing artifacts "to master one's own mental operations" (Vygotsky, 1987, p. 131). Recent theories of radical embodiment may be seen as an extension of these perspectives. Here, artifacts—including routine ways of using our body—do not only support, mediate, or master, but *constitute* thinking; that is, what is outside the head may not necessarily be outside the mind (Abrahamson & Trninic, 2015; Chemero, 2009; Hutto & Myin, 2013; Hutchins, 1995; Malafouris, 2010, 2013; Nemirovsky, Kelton, & Rhodehamel, 2013).

Naturally, it is true that artifacts, and the outside world in general, lack the specialized neurons commonly found in our heads—nobody questions that. However, nobody is arguing for an extended *brain* theory. Instead, we advance the position that what we call a *mind* need not be contained under our skull or even under our skin. I will begin to advance this position by considering Einstein's (1916) articulation of the theory of relativity, borrowed from educational psychologist Mitchell Nathan (2012; due to my lack of training in physics, I will follow Nathan's account closely). According to the traditional historical account, Einstein's process of scientific discovery was both intuitive and abstract, drawing profoundly on the prevailing scientific theories of electromagnetism and inertial frames of reference (Stachel, 1982). However, as the circumstances of Einstein's contributions come to light, the account of his discovery is being revised, with a greater appreciation for the role artifacts—especially novel technological artifacts of his time—played in Einstein's thinking.

4.2.1 The artificiality of Einstein's genius

As Nathan (2012) writes, the foundation of Einstein's groundbreaking theory was to reframe the question of *simultaneity*, or what it meant for two events to happen at the same time even when they were not located at the same place. Again, the traditional account has the solitary theoretician, Albert Einstein, working in the confines of the Swiss patent office and conceptualizing Isaac Newton's widely accepted laws of the universe. By this account, Einstein's discovery was "exceedingly abstract, couched deeply in theory and inscribed in the language and notation of formal mathematics and physics" (p. 126). However, an analysis of Einstein's everyday life suggests something else altogether.

Einstein was puzzled by the question of how clocks at different places could accurately mark simultaneous events. In Newtonian physics—according to the postulate of Absolute Time—each place could be calibrated to a single common time standard. This postulated a "master" clock that sent a signal to secondary clocks to denote the simultaneity of an event. It occurred to Einstein that, because the secondary clocks could not all be equidistant, and because light travels at a finite speed, clocks closer to the master clock would receive the signal and record the event before the others. Einstein deemed this implausible; in turn, he offered a *Gedankenexperiment* (or thought experiment) to investigate idealized time-keeping procedures. Yet even in this simulated world, with idealized clocks and cables,

> Einstein reasoned that if one took into account the time for the signal to travel from a central clock to the different locations of each secondary clock along these cables (with each cable distance divided by the speed of light), then the time recorded by a secondary clock would be independent of its proximity to the master clock. (p. 126)

This, in turn, led Einstein to realize that there was no need for a master clock at all, because times could be determined relative to clocks' distances from each other.

In thinking through this newly proposed model of relative time, Einstein came upon yet another *Gedankenexperiment*, "this one addressing the relative experiences of observers on trains headed toward or away from a lightning strike, as compared to a stationary observer positioned along the railway embankment" (p. 126; also see Einstein, 1916).

The position I wish to consider here is that Einstein's references to trains and systems of synchronized clocks were not mere illustrations of concepts (also see Galison, 2003). They were, instead, integral to Einstein's thinking on relativity. Calibrating clocks and coordinating time were practical concerns in Einstein's time, largely due to the spread of train travel across Europe. As Nathan notes, during his time as a patent officer, "Einstein was exposed to applications proposing electro-mechanical ways to coordinate clocks and reliably schedule trains" (p. 127). The technological artifacts of his day, such as clocks and trains, were productive things to think with. Indeed, it was this interaction with existing technological artifacts that enabled a major advancement of formal theories of physics. Interaction was the source, not the application, of conceptual understanding.

According to Nathan, examples of this sort are abundant and span the range of scientific study. Drawing on Davis Baird's *Thing Knowledge* (2004), he provides another illustrative account, this time of Michael Faraday's contributions to our basic understanding of the physical world.

Despite minimal formal education, "Faraday constructed a device that produced rotary motion by adjusting current that varied a magnetic field in synchrony with the rotor, effectively pulling the rotor forward throughout the cycle" (p. 127). In this groundbreaking *invention*, Faraday created the electromagnetic motor. Nathan writes:

> In so doing, Faraday revealed fundamental knowledge about all forms of electromagnetic phenomena, including light. He also identified important aspects of the conservation of energy, as electricity was converted to mechanical motion. Notably, he accomplished all this without deriving his design from formal theories or equations….To disseminate his scientific work, Faraday actually shipped prebuilt versions of the motor the way scholars today share reprints and digital files of their scientific papers. (p. 127)

Faraday's breakthrough was borne from his passion for tinkering, an intimate bond with the materials and tools of his practice. The formalized scientific theories and mathematical formulae *trailed his discovery by years*.

The point is that working with technology and artifacts can and—contrary to stock beliefs—often does precede the development of formal, scientific theory (see also Cajas, 2001; Meli, 2006). Yet,

> Despite evidence of the power of material invention to advance scientific theory, we generally accept [the view] that technological advancements are born from theory, and to be legitimate they must be derived from formal knowledge represented in symbolic and specialized notation. (Nathan, 2012, p. 127)

The next challenge lies in articulating an account of artifact use that acknowledges this sort of *material knowing*, or thinking with, through, and about things. So doing, we continue to advance the position that a *mind* need not be contained under our skull or even under our skin; furthermore, it may be appropriate to argue, at times, that the environment does our *intending* for us. These are the moments when we find our thinking shift in an unanticipated direction, without ever intending to do so.

4.2.2 Expanding the mind

Cognitive scientist David Kirsh (2013) provides an elegant argument for why a reconceptualization in thinking about thinking with tools is needed.

The argument echoes a famous line usually associated with Marshall McLuhan: "We shape our tools and thereafter our tools shape us." Kirsh begins by considering the well-known case of a blind man using a walking stick. In cases of familiar tool-use, the neural representation of one's body schema changes as they recalibrate their body perimeter to absorb the end point of the tool (Làdavas, 2002). In short, tools change the way we encounter, engage and interact with the world. They change our brains. They change our minds.

Kirsh goes on to argue:

> If a tool can at times be [neurally] absorbed into the body then why limit the cognitive to the boundaries of the skin? Why not admit that humans, and perhaps some higher animals too, may actually think with objects that are separate from their bodies, assuming the two, creature and object, are coupled appropriately? If tools can be thought with, why not admit an even stronger version of the hypothesis: that if an object is cognitively gripped in the right way then it can be incorporated into our thinking process even if it is not neurally absorbed? Handling an object, for example, may be part of a thinking process, if we move it around in a way that lets us appreciate an idea from a new point of view. Model-based reasoning, literally. (p. 3-3)

I find the argument convincing. Besides defining the mind to be so *a priori* because that's "what we've always done," what reason have we to limit the notion of thinking to inside our heads?

This question expresses itself even more forcefully once we note that much of our cognitive life depends on internal simulation of events. Here Kirsh quips, "if internal simulation counts as thinking why not also count external simulation as thinking" (p. 27)? If rotating an object "in the head" is thinking, why is physically rotating the same object not thinking?

Another noted proponent for this view is Lambros Malafouris (2010, 2012, 2013), whose landmark analysis of knapping—a Paleolithic form of stone-tool production—advanced forth the field of neuroarcheology, or archeology of the mind. The following line is representative of his perspective: "We need to abandon our common representational/internalist assumptions, and recognize knapping as an *act of thought*; that is a cognitive act" (2010). He extends this argument, positing that not only is thinking distributed, but also the *intentional* act itself. Because Malafouris' findings (independently) parallel and, I offer, corroborate some of my own conjectures concerning the learning process (as argued in this dissertation), I will now spend some time unpacking the arguments found in *Knapping Intentions and the Marks of the Mental* (2010) and elaborated in *How Things Shape the Mind* (2013).

The above-quoted statement on "knapping as an *act of thought*" is based on Malafouris' extended analysis of *knapping* (see Roux & Bril, 2005), a relatively simple fracturing process practiced for more than 2.5 million years. Knapping involves striking flakes off a stone core, resulting in a edge or a point used for meat preparation, hunting, and a variety of other activities suitable for such implements. For many archeologists, this process and its products represent one of the defining characteristics of the genus Homo, the descent of "man the toolmaker" (see Ambrose, 2001).

We will be considering the biface Acheulean handaxe in particular (see Figure 1, below). Acheulean refers to the industry of stone tool manufacture characterized by distinctive oval and pear-shaped handaxes, and *biface* comes from the fact that the archetypical model is a generally bifacial Lithic (stone) flake with an almond-shaped morphology.

Figure 1. A genuine Amygdaloidal (almond-shaped) handaxe—roughly eroded—from a site in the Valladolid province of Spain[20]. Note the evident symmetry, despite erosion.

As evident in the accompanying images, these handaxes can be characterized as symmetrical. Malafouris considers and rejects the prominent theories of what the handaxe symmetry implies about the human mind. These theories specifically concern the so-called "handaxe dilemma." Namely, are we justified in attributing some kind of symmetry concept to the knapper, or is the property of symmetry simply a part of our modern perceptual apparatus and was in no way intended by the knapper?

The prominent accounts of the "handaxe dilemma" can be characterized as implicitly identifying knapping with some sort of "causal and intentional transaction between mind and the world" (Searle, 1983, p. 88). Seen as such a form of causal and intentional transaction, knapping can be described as a sequential process composed of two essential parts: (1) an intentional state in the mind of the knapper that (2) causes an external movement into the outside world. In other words, the knapper's intentional states are seen as prior intentions, which are presumably formed inside the knapper's head *in advance of the action itself*. Thus, in this account the intention is an internal representational state that temporally and ontologically precedes and causes the agent's movement which, then, as an external physical act, produces the handaxe. In Malafouris' words, the traditional account of symmetry can be summarized as, "whatever the precise adaptive reason (e.g. functional, social, sexual, aesthetic or symbolic) behind the symmetry of the handaxe, it is the product of conscious intention" (p. 16). As means of an example, he considers the classical exposition of the handaxe enigma by Thomas Wynn (1995), which, in short, *explicitly* identifies in the symmetry of handaxes the intentional execution of a preconceived mental plan.

Malafouris' solution to the handaxe enigma involves reframing the central question.

> …[T]he key issue underlying the handaxe enigma is not about whether humans in the Stone Age were producing one sort of intentional states rather than another….The key issue, rather concerns….how and when humans became aware of the intentional character of their actions and of the actions of others. (2010, p. 17)

Based on a fine-grained study of the knapping process (an approach I understand as analogous to the microgenetic analysis in our own discipline), Malafouris offers an alternative account:

> The topography of the knapping activity and the accurate aiming of a powerful blow is neither pre-planned nor recollected; it is embodied and therefore needs, instead, to be *discovered* in action (p. 18).

Thus, the directed action of stone knapping does not simply execute but rather "brings forth" the knapper's intention: "The decision about where to place the next blow, and how much force to use, is not taken by the knapper in isolation; it is not even processed internally" (p. 17).

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[20] "Bifaz amigdaloide" by José-Manuel Benito Álvarez (España). Locutus Borg. Own work. Licensed under CC BY-SA 2.5 via Wikimedia Commons http://commons.wikimedia.org/wiki/File:Bifaz_amigdaloide.jpg#/media/File:Bifaz_amigdaloide.jpg

According to this account, the flaking intention is constituted, at least partially, by the stone itself—a radical claim, yet likely to resonate with anyone who intimately works with materials. The tools and materials we use have their own demands, and oftentimes we do our best work when we simply pay attention to these demands. This interaction is captured in Figure 2, below.

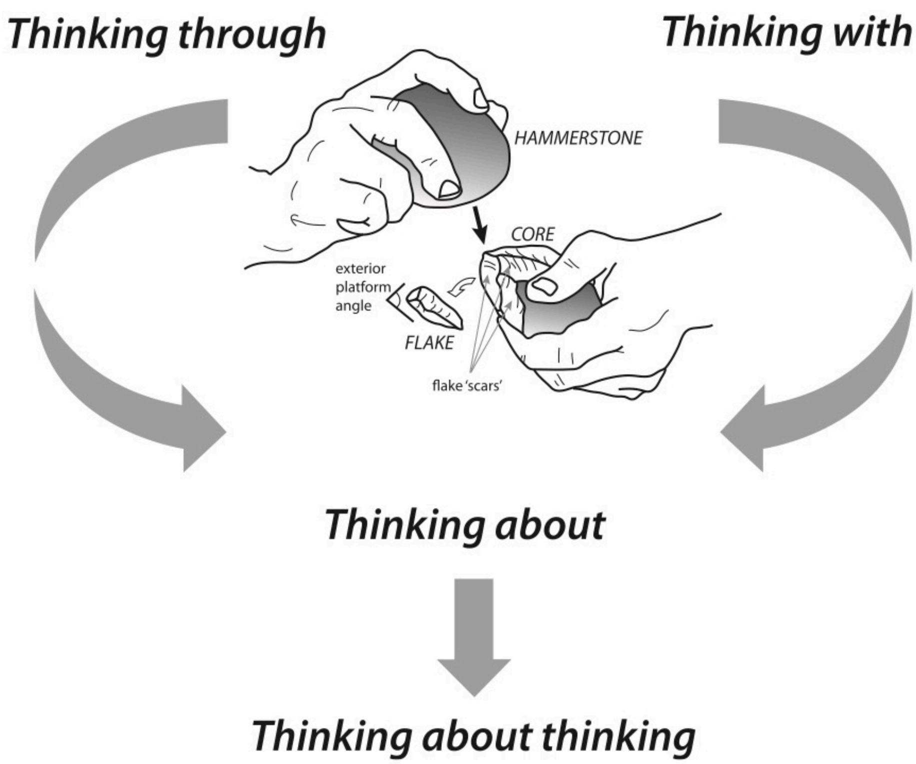

Figure 2. According to Malafouris, "the knapper first thinks through, with and about the stone (as for example in the case of Oldowan tool-making) before developing a meta-perspective that enables thinking about thinking (for instance as evidenced in the case of elaborate late Acheulean technologies and the manufacture of composite tools)" (p. 18).
(figure redrawn from Malafouris, 2010)

In turn, this material intentionally leads him to conclude that

> not only is the tool often smarter than the toolmaker but it can also come to possess….a mind of its own. In other words, my claim is that the knapper had to learn first 'how' to make a symmetric handaxe before she/he could identify what symmetry as a parameter of human design abilities was….[I]ntention no longer comes before action but it is *in the action*; the activity and the intentional state are now inseparable. (pp. 18-19; also see Abrahamson, Trninic, Gutiérrez, Huth, & Lee, 2011, for comparable analysis of learning via technology-enabled enactive artifacts)

This section began with a quote about us shaping tools, and our tools shaping us—a quote sometimes attributed to McLuhan. I believe McLuhan spoke of just this sort of breakdown of conceptual barriers between us and artifacts, wherein intention can be said to originate from the things we made. We become *servomechanisms* of our technologies (McLuhan, 1964, p. 46).

Malafouris' conclusion, above, matches my own thinking on the proper course of education. He concludes that the knapper first had to learn how to make a symmetric handaxe in order to comprehend symmetry as a parameter of human design ability. When I ask my taiji teacher about his frequent admonition to "never separate, never collide," he is more likely to physically demonstrate this to me, on me, than to talk about it. As he told me early on in my training, "You want to figure everything by understanding it intellectually. That won't help here. Do the work for at least one hundred hours and you'll start to understand." And, indeed, after hundreds of hours I felt in my body what he said in words: never separate, never collide. Same words, but now I began to understand what he meant by them.

In mathematics too, I have not infrequently found myself understanding some particular *definition* or a *proof* without feeling like I understood what it means. Invariably, once I used this definition or theorem over a period of some weeks, it would have a meaning! Here, too, I recall an experienced (topology) teacher telling us to "give it a few weeks time, it'll make sense." And he was right.

One major claim here is that, in modeling intelligent behavior and learning, intention need not be confined to the inside of our heads. Knappers did not intend to create symmetry any more than learners today *intend* to understand any specific principle of taiji or mathematics. The French phenomenologist Maurice Merleau–Ponty writes:

> In so far as I have hands, feet, a body, I sustain around me intentions which are not dependent upon my decisions and which affect my surroundings in a way which I do not choose. These intentions are general….they originate from other than myself, and I am not surprised to find them in all psycho-physical subjects organized as I am. (1962, p. 440)

When I speak of students *finding themselves*, this is what I mean. This also echoes an account of learning I have elsewhere labeled *Action Before Concept* (e.g., Trninic & Abrahamson, 2012), yet a better term might be Action *Becomes* Concept. What these accounts ultimately point to is that one way to resolve the learning paradox and the handaxe enigma begins with letting go of our notions of intentionality and agency as strictly confined in the individual. This is the negative outcome. The positive outcome is that we gain the world to think with.

4.3 Learning Environments that Move Us and Change Our Minds

So this is the position with which we move forward: interaction with artifacts is an opportunistic process. Even as we change them, artifacts change our minds. And this precisely is my central argument about enactive artifacts. Mastering these routines, they master us.

Another way to state this is to say that disciplinary practices, by which I mean forms of engagement organized via artifacts, bring forth the world to the learner. Practices *enact* some aspect of the world. This does not mean that the practice literally creates something new in the physical world, but that something is there for the student that wasn't there before, like symmetry that the knapper found in the stone handaxe millions of years ago.

Everyone is familiar with some variation of *the law of the instrument*: "Give a small boy a hammer, and he will find that everything he encounters needs pounding" (Kaplan, 1964, p. 28). My favorite variation of this aphorism: "If all you have is a hammer, *everything* looks like a nail." But—and here's the crux— we could also turn it around: "If you've never used a hammer, *nothing* looks like a nail!" This is what I mean when I say that enactive artifacts evoke aspects of the world. Small metal spikes with a broadened flat heads might exist in a world without hammers, but they wouldn't be nails.

Another way to say this is that the practice is often more powerful than the individual: it is not wrong to say that the practice is a tribal system, and the young who join become part of the tribe, learn to see the world as the tribe does.

As a reminder, Chapter 2 investigated the pedagogical practices found in two overtly embodied disciplines, surfing and martial arts, and coined and introduced a novel construct: *enactive artifacts*—rehearseable routines that serve as resources for encountering and coping with particular problem situations in the world (see Trninic & Abrahamson, 2012; also see Abrahamson 2006 on the meaning of *situations* in situated cognition).

In Chapter 3, I provided a descriptive account of how artifacts change our minds. First, learners appropriate artifacts, including enactive artifacts (which show us how to use material artifacts), as helpful equipment. In other words, artifacts draw learners in, appearing useful for some present purpose. Yet in the course of utilizing artifacts, they *find themselves* enacting and availing of embedded affordances in these artifacts as supporting the development of a more sophisticated strategy.

Through enactive artifacts (which often make use of material artifacts), the practitioner is able to encounter and solve previously intractable, impossible, or even invisible situations in the world. I offer that pedagogical interactions involving enactive artifacts do not so much involve any sort of *knowledge transmission* from teacher to student, but rather *the method for acquiring understanding*. Enactive artifacts are the vehicle of this method: We might even say that the teacher teaches the student how to pursue understanding.

Enactive artifacts live in multiple worlds, connecting spontaneous and disciplinary ways of encountering the world. On the one hand, they must be something the learner can *do*, however approximately. Yet, at the same, they expand the horizon of what the learner can do; they enable a new sort of movement. And yet again, they are *also* practiced because *the actual act of practice* offers insight beyond the routine practiced (such as finding symmetry in a stone). This insight is *in the action* and *through* it, inseparable from it, as Malafouris might say.

A mathematical example can be used to illustrate this point. The division algorithm is a series of steps I was taught at some point in my mathematical career (likely in 3rd grade). If I go through each step carefully, I can find the desired quotient. At a much later point in my mathematical career, I carefully investigated why the division algorithm works, a process which led to, for example, an appreciation of place value in our system of notation. Yet—and here is the crucial point—I can *also* use the division algorithm to investigate certain features of rational numbers, namely that every repeating or terminating decimal is a rational number and vice versa. If you asked me why this is so—I mean, why every repeating or terminating decimal is a rational number (and vice versa)—it is likely I would incorporate the division algorithm (or some variation of it) in my answer. Moreover, if I didn't know the answer immediately, I would resort to using the division algorithm in order to make sense of it (cf. Pirie & Kieren, 1994). I offer this example as a case of a mere procedure becoming a conceptual performance. Indeed, *this is how I convinced myself years ago that terminating decimals are rational numbers and vice versa*. My understanding of this mathematical principle is not merely stored "in my head" but enacted—literally brought forth—through the things I do, including the procedure of long division.

Let me again note that one of the central reasons I felt compelled to investigate routines is that I found talk about mathematical procedures conspicuously absent in most discussions on mathematical meaning. Yet as even this simple thought experiment involving long division demonstrates, a sense of understanding can be found in and through procedures.

This, then, is our pedagogical conundrum. Teachers want students to grasp the meaning—the spirit, not the letter—of the disciplines. Alas, understanding cannot be handed over like a set of keys. What I have suggested is that practices, routines are an effective and powerful way of solving this educational dilemma—a notion I will pursue for the remainder of this chapter.

In other words, I cannot teach you how *I* understand the world, but I can show you how *you* might get here. In short, enactive artifacts may provide learners with not only new ways of moving and performing, but *through this*, learners become receptive to discipline-valued insight (literally internal sight, from Middle English), which we also call understanding.

Our engagement with artifacts can take us places we never anticipated, yet find invaluable. "Walking a mile in someone's shoes," but literally. This account is supported by Tim Ingold, a British social anthropologist, who writes, "Practitioners, I contend, are wanderers, wayfarers, whose skill lies in their ability to find the grain of the world's becoming and to follow its course while bending it to their evolving purpose" (2010, p. 92). This single sentence, I think, beautifully captures the pedagogical aim of enactive artifacts. Routines, which *can* be taught directly, restrict our movements and direct our attention, so that we may find the grain of the world and learn to follow it. What is this grain? Ingold gives an example:

> Consider, for example, the operation of splitting timber with an axe. The practised woodsman brings down the axe so that its blade enters the grain and follows a line already incorporated into the timber through its previous history of growth, when it was part of a living tree. (p. 92)

Here, the grain is literal: at the moment of impact, the grain of the wood is aligned with the blade of the axe. This can be seen as a metaphor for a larger scheme. Our skill as practitioners is in aligning our capacity to follow the grain of the word, as determined by our practice. The grain directs us even as we bend it to our "evolving purpose." All disciplines can be said to follow a "grain"—whether natural or man-made. To teach is to guide students towards this grain.

This raises a question: Is looking for a grain like groping in the dark? What motivates and drives this process? Merleau–Ponty provides an answer: we are constantly trying to get a "maximum grip" on our situation. This expression comes from observing that when grasping something, we tend to grab it in such a way as to get the best grip on it. Similarly, when we are looking at something, we tend, without thinking about it, to find the best distance for taking in both the thing as a whole and its different parts (see Dreyfus & Dreyfus, 1999, for an extended discussion). This human drive towards a maximum grip is the underlying motivation to locate and follow the grain. Note that it is not something we intend to do: we no more intend to locate and follow the grain than we intend to establish a firm yet comfortable grip on the hammer handle. Rather, we strive for a maximum grip with our senses, for it feels *more comfortable*—it reduces the sensations of tension we experience when something doesn't quite feel right. A proper grip on the situation simply "feels right," something we all recognize yet rarely reflect on.

If we take seriously the notion that we think with our bodies and with artifacts, and that practices can shift our thinking in ways we could not have anticipated or intended—What would it look like, from this perspective, to characterize and design learning environments for "conceptual understanding"?

4.3.1 Instrumented Fields of Promoted Mathematical Action

Edward Reed, an ecological psychologist, and Blandine Bril, a social anthropologist, describe indigenous practices that apparently foster infant development of culturally valued physical capabilities. For example, mothers in remote Sub-Saharan villages were observed to enact shared routines of handling their infants so that they learn to move in new ways. Society thus intervenes in shaping infant development by creating circumstances—*fields of promoted action*—that encourage the building and exercising of particular motor capacities required for effective participation in cultural activities (Reed & Bril, 1996). Dor Abrahamson and I (2015) have extended this pedagogical construct to include artifacts. Thus, *instrumented* fields of promoted action.

Instrumented fields of promoted action utilize enactive artifacts (routines), material artifacts (tools and symbols), and discursive forms as a means of guiding and shifting learner's actions and thus understanding. Ultimately, however, it is the learner who engages these artifacts, developing a sense of what this situation is about. In this sense, instrumented fields of promoted action are a designer's interpretation of Wertsch's account of learning (Wertsch, 1979): learning through interaction.

In instrumented fields of promoted action, learners find themselves modifying and shifting their actions to get a maximum grip on the situation. These transitions towards a maximum grip can be either smooth or abrupt (Abrahamson & Trninic, 2015).

To illustrate, let us briefly consider some transitions across the disciplines. In surfing (Chapter 2), learners work with a material artifact, the surfing board, and an enactive artifact, the pop-up motion. As they practice on sand, their execution of the enactive artifact becomes smoother, more unified. They might notice minor details, such as which hand locations lead to more explosive pop-ups. This noticing occurs *after* the fact. So the practitioner first places her hands at a novel location, notices that her pop up is now more explosive, and recognizes this as advantageous. Later, surfers will struggle with executing the pop-up motion in the water. Again, with practice, their execution will become smoother, more unified. They might notice, for example, that in the water they need not push down as hard on the board to create physical space needed for the pop up. They might start noticing which places on the board are optimal for balance—something not evident when practicing on sand—and adjust accordingly. The reason we call this transition smooth is because it involves only minor adjustments of one's motor coordination (e.g., placing one's foot here instead of there).

Abrupt transitions are more drastic, and often experienced as surprise, joy, or similar expression of a Eureka! moment. In the taiji example (Chapter 2), the student works with a partner through an enactive artifact, the two-hand push hands form. One of the primary goals of this practice is for the student to redirect rather than to resist opposing force. Because resisting force with force is a natural tendency for most adults (if someone pushes us, we tense up and resist), discontinuing this tendency is a moment of abrupt transition requiring a drastic *shift* in interaction. In the example provided in Chapter 2, the student experiences a Eureka! moment only after the teacher demonstrates the very same principle repeatedly and provides multiple opportunities for the student to do so himself. In contrast to smooth transitions, which require only minor adjustments in motor coordination, abrupt transitions are more likely to require significant scaffolding (other-regulation), including the introduction of novel artifacts.

In the case of MIT-P (Chapter 3), we can observe both smooth and abrupt transitions. As students practice making the screen green, their hands constantly adjust to whatever position produces this goal. In general, this sort of smooth transition is indicative of refining a practice we are familiar with, yet which requires us to be responsive to its changing demands (as in the case of changing contexts).

On the other hand, the introduction of novel artifacts such as the grid and numerals directed students to approach "making the screen green" from an entirely new perspective. For example, in contrast to their previous continuous vertical movements, students moved in discrete units, focused entirely on the lines on the screen and ignoring the spaces between. Or, as in the case of Shani in Chapter 3, students use artifacts in enacting completely novel ways of "making the screen green" such as "[left marker] is always half [the height of right marker]."

Thus smooth and abrupt transitions are thus characterizations of how artifacts change learner's actions, depending on how drastic of a change the transition imposes on the learner's action.

Earlier I offered the artifacts alter some aspect of the world for the learner: something shifts for the learner, and something is there that wasn't before. The continuation of this argument is that the things we call "conceptual understandings" are outcomes of organized forms of engagement involving various transitions of the sort just discussed.

Of course, if we are to return to conceptual understanding as a construct, we ought to recall that educational researchers do not agree on what conceptual understanding actually is (Crooks & Alibali, 2014). For example, even the influential *Adding It Up* (NRC, 2001) defines conceptual understanding as "the comprehension of mathematical concepts, operations, and relations," (p. 118) which merely replaces conceptual understanding with "comprehension."

Still, we should be able to agree that conceptual understanding forms a continuum: we never understand everything all at once—generally speaking, our learning progresses piecemeal. As Wertch informed us, learners engage situations *all the time* without understanding them. Thus it often happens that a practitioner *literally operates some instantiation of a concept* before being aware that this is what he is doing. It is, I posit, only once these moments of awareness occur that we announce insight or understanding. In other words, I suggest that what we call moments of "insight" or "understanding" is thinking-about-doing and thinking-about-thinking that happens *after the fact* (Figure 3, compare to Malafouris' account of knapping in Figure 2, above; also see Bamberger & Schön, 1983). I would even extrapolate this to conjecture that perhaps "conceptual understanding" is nothing more (and nothing less) than the accumulation of such moments of serendipitous insight acquired through practice.

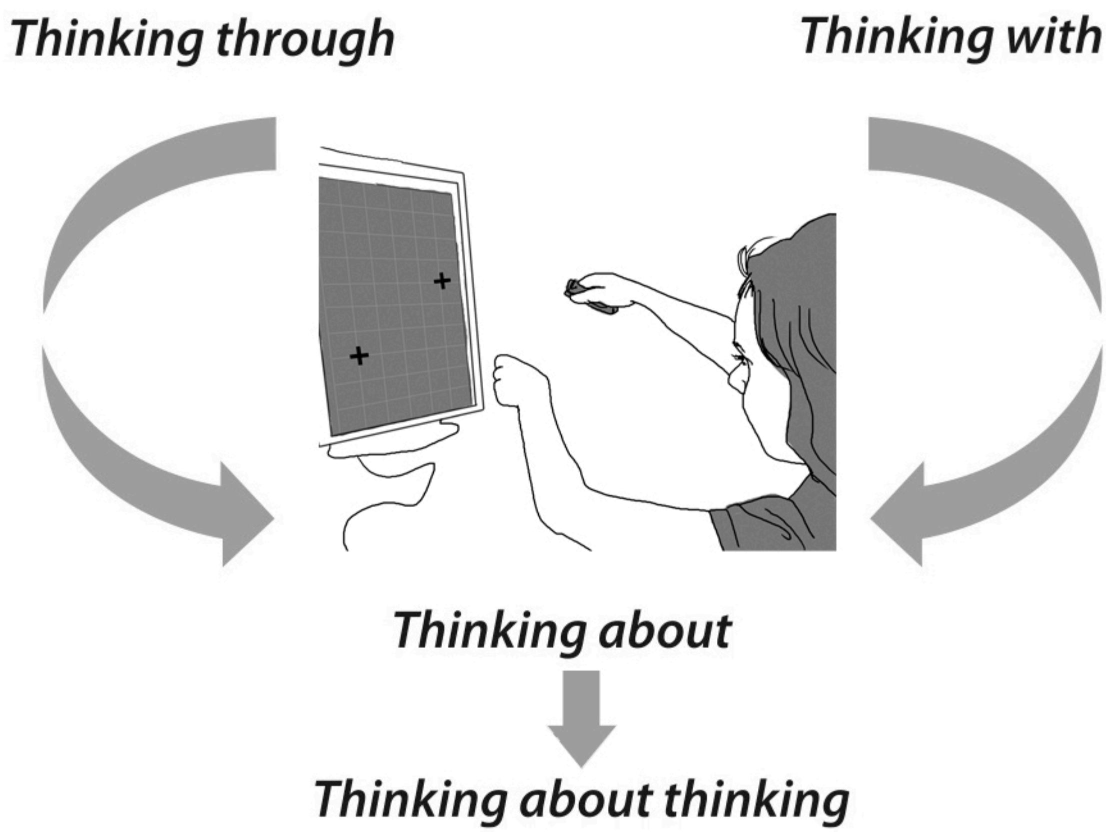

Figure 3. In the case of the MIT-P, the student is asked to first think with, through, and about "making the screen green" enactive artifact (that is, moving hands up and down proportionally so as to effect green) and then to progressively incorporate mathematical artifacts, such as a grid and numerals, into her actions and thinking. Incorporation of novel artifacts in this field of promoted action is, to the child, just another case of executing the enactive artifact. Yet so doing, she finds herself moving in new, unexpected ways to reestablish the maximum grip. Thinking about these moves, and thinking about *that* thinking, takes the form we call "mathematical

understanding." Yet these understandings (e.g., connecting "the right moves faster" with "1-per-2" with "the right is twice as high") are grounded in an enactive practice.

I hope it strikes the reader as banally obvious: we *always* have the Eureka! moment *after* we figure something out. Rarely do we say, "Eureka, I'm about to discover it!" Yet it is equally common for us to employ historical revisionism when narrating these moments of understanding, and say: "I figured it out—that's why I did it," instead of putting things in their proper chronological order: "I did it—that's why I figured it out."

Moments of understanding are the awareness that a *transition* has transpired. Ginsburg (2010) writes: "Learning itself is not conscious. The integration process itself is not conscious. Nevertheless the process depends on conscious processes in feeling and detecting changes. The consequence is *felt as difference*" (p. 185, emphasis mine). This notion—that unconscious, oftentimes subtle interactions drive adaptations to behavior, and that consciousness plays a *post facto* appraisal role in making sense of these changes—is crucial to my thesis. Specifically, I posit, *what we call conceptual understanding emerges from guided interaction through a felt sense first noticed as difference*.

This means that understanding cannot be forced. It is a gradual process, psychologically driven by our desire for a maximum grip, and characterized by transitions whose value is recognized yet unanticipated. As Gendlin writes:

> …the words must come. If they don't come, we cannot make them. We have to wait.… We recognize this kind of coming: it is characteristic of all bodily comings. It is how sleep comes, and tears, the appetites….You can feign anger, but to have it, it must come to you.… So also, the muse cannot be forced, or invented. She must come. *We can make ourselves receptive*, but we cannot control her coming… (1999, p. 105, emphasis mine)

We can't make a realization happen by willing it: it happens when we find ourselves following the grain of the world while bending it to our purpose in novel ways. This is why we *find ourselves* doing something new.

How do we extend this to educational design? In instrumented fields of promoted action, students experience transitions they do not anticipate, yet transitions they are driven towards in their search for a maximum grip on the situation. Both smooth and abrupt transitions are integral steps towards understanding. To illustrate this point, let us return to knapping. If smooth transition is knapper's gradual improvement of the flaking process, abrupt transition occurs when symmetry emerges from this process. Importantly, what constitutes conceptual understanding in this promoted field of action isn't "symmetry" as a mathematical form, but symmetry as a design ability. "Conceptual understanding" always emerges within some particular instrumented field of promoted actions. Symmetry was the outcome of a routine, not the goal.

My recommendation is to step away from our field's fascination with "conceptual understanding" altogether. Not permanently!—but to gain some distance and reevaluate how we conceptualize conceptual understanding. I offer that we cannot *make* students develop conceptual understandings anymore than we can *make* ourselves do the same. What we can do is design instrumented fields of promoted actions that make students receptive to particular understandings. And we can emphasize and develop routines and practices likely to result in students' understanding and reduce the occurrences of those unlikely to.

4.4 Against FOIL

Not all routines and procedures are pedagogically equal, and an account attempting to lionize the educational use of routines, or procedural practices, should acknowledge this fact. The purpose of this section—a brief interlude—is to emphasize by means of an example the sorts of routine practices I do *not* consider pedagogically useful. It would be foolish to spend a dissertation lionizing the pedagogical importance of routine practices without also acknowledging that all is not well with the current state of procedural practice as it occurs in mathematics education (for a convincing account, if one is needed, see Schoenfeld, 1988).

Specifically, I aim to illustrate that as long as we are not *exclusively* concerned with a view of mathematics learning as obtaining the correct answer, the FOIL method is a poor choice of a routine. I argue that FOIL isn't pedagogically deleterious merely because it's a procedure, it's pedagogically deleterious because it stands little chance of making students receptive to any mathematical insight. Quite the opposite.

It may come as little surprise that FOIL has been empirically shown to be pedagogically deleterious. A recently study finds that the FOIL method is, at best, problematic as an educational tool (Koban & Sisneros-Thiry, 2015). Specifically, the empirical finding ($n = 252$) is that while most first year undergraduates are able to use the FOIL method to appropriately multiply two binomials, less than half were able to extend this method to general polynomial multiplication or provide explanation for why FOIL works at all.

What is FOIL? FOIL, or the FOIL method, is a mnemonic for multiplying two binomials. It is not a mathematical rule, at least not in the sense that a professional mathematician would use. It is a shortcut. FOIL stands for: First, Outer, Inner, Last. So:

$$(a + b)(c + d) = ac \text{ (First)} + ad \text{ (Outer)} + bc \text{ (Inner)} + bd \text{ (Last)}.$$

Typically it is assumed that the learner will memorize this mnemonic, memorize that the four terms—First, Outer, Inner, Last—corresponding to each letter are to be added, and know how to multiply and add these terms. The result, it is hoped, is an appropriate binomial product. It teaches students a relatively quick way of multiplying binomials.

Mathematically, the most obvious problem with FOIL is that if one learns First, Outer, Inner, Last as a necessary series of steps, *it becomes impossible to apply this routine to other polynomial multiplication*. How to FOIL $(a + b + c)(d + e + f)$? In other words, FOIL is a rule that only works for the special case of *bi*nomial multiplication, and then must be forgotten for more general cases of polynomial multiplication. When students learn FOIL, they *also* learn this—that mathematics consists of narrow rules that must be memorized.

Perhaps this would be an acceptable pedagogical outcome if FOIL provided us with an invaluable service. Yet it does nothing of the sort. To see this, we can compare FOIL method to directly applying the distributive property, a process that works for all polynomial multiplication. This "long" form would look like

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd.$$

In other words, all FOIL does is allow us to cut out a *single step* in a computation that students *ought* to be able to do in the first place. And even then, a Bay Area teacher confides:

> FOIL is one of the many things I am trying to un-teach my students, because it leads to only finding two products (The "F"irst times the "O"utside! The "I"nside times the "L"ast!) instead of 4, and it doesn't aid when multiplying anything except two binomials. (personal communication, 2015)

It would seem that FOIL doesn't do much for students *even as a mnemonic*.

Earlier, I suggested that enactive artifacts are practiced not only for what they enable us to do, but because they can make us more aware and thus receptive to certain discipline-valued insights or discoveries. For example, working with the long division algorithm can contribute to insightful observations concerning rational (and irrational) numbers.

What does FOIL do beyond—and even this is debatable—enabling students to multiply binomials? This is an important question in evaluating routine practices, and while it emerged in my work, other scholars have already asked as much.

As just one example, Director of Mathematics for Strategic Education Research Partnership (SERP) Phil Daro emphasizes that when evaluating mathematical practices we ought to ask questions like: What kind of mathematics does a student learn from solving this problem? I will also note that the SERP website lists the following quote, attributed to Daro, as one of its mottos: "Correct answers are essential... but they're part of the process, they're not the product. The product is the math the kids walk away with in their heads" ("Daro Talks," 2015; of course, we'll ignore the part mentioning knowledge "in their heads").

To answer this question—what FOIL provides beyond the ability to multiply polynomials—I will share an illustrative anecdote. A colleague, now a professor at a Midwestern university, raved about his favorite mathematics teacher, insisting that this teacher made math both meaningful and accessible. The short of it is that this teacher had students practice FOIL on aluminum foil with oil paints: this way they would always remember the mnemonic. Now, when my colleague said that this teacher made the mathematics meaningful, he wasn't being sarcastic. Instead, what mathematics meant to him—and, one can assume, to many of his classmates—was a host of seemingly arbitrary rules that must be memorized. Consequently, even *tricks to memorize a mnemonic* were welcome and appreciated. Naturally, my colleague—and one can assume many of his classmates as well—believed that FOIL is a mathematical *rule*, similar to rules we follow to determine that 5 + 2(2) = 9 and not 14.

Thus what FOIL conveys to students is that what is most important is producing the answer by following a rigid rule (Koban & Sisneros-Thiry, 2015; also see Jiménez-Aleixandre, Bugallo Rodriguez, & Duschl, 2000, on "doing the lesson").

If one *had to* teach the FOIL method, I would recommend not relying on First-Outer-Inner-Last but on virtually or physically drawing connections between each element being multiplied (see Figure 4, below). At least there the student's attention could be drawn to the fact that what is important is that each element in (a + b) be multiplied to each element in (b + c) rather than worry about what is first, outer, inner, or last—a worry about something which *isn't* mathematically important.

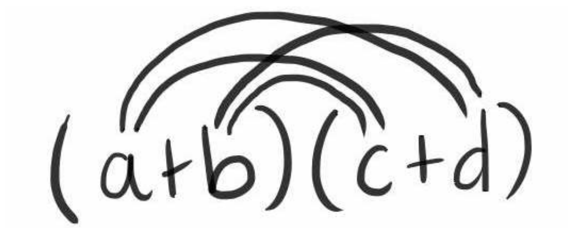

Figure 4. A student-generated example of a FOIL-like alternative to FOIL. Each link "stands for" multiplication, with the rule is that each element in (a + b) links to each element in (c + d). Unlike FOIL, which focuses on specific combinations characterized as First, Outer, Inner, Last, this approach is generalizable in the sense that any polynomial multiplication could be so represented. This approach even affords potentially interesting questions, such as, "How many links are there between any two polynomials?" Regardless, this approach, like FOIL, is merely a "helpful" rule on top of an actual mathematical rule; however, unlike FOIL, it directs attention to an underlying mathematical structure.

Yet I would suggest that the FOIL method be removed from the curriculum entirely. FOIL does not help students understand mathematics—it merely creates yet another layer that must, in turn, be understood. In contrast, something like the area model of polynomial multiplication at least stands a chance of providing students with some mathematical meaning to what they are doing (see Figure 5, below).

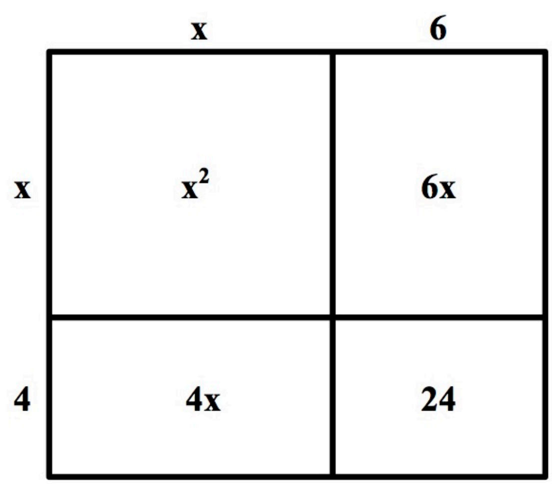

Figure 5. An area model representation for (x + 4)(x + 6). Note that this particular procedure does not immediately yield the product. However, the spatial representation provides justification as to why there are four elements in the product (x^2, 4x, 6x, 24). Contrast to FOIL, where there are four elements for the seemingly arbitrary reason that there are four letters in FOIL.

And finally, FOIL need not be in the curriculum because FOIL is exactly the sort of shorthand procedure that students frequently discover on their own. After all, a variation of FOIL is what I use, and many other non-US schooled mathematicians do as well, despite never being taught to do so. Freudenthal (1971) stated it was criminal to teach students something they could (easily) discover on their own—and I suggest that FOIL is an ideal candidate for this indictment.

Being cautious, I wish to note that the FOIL method, as an educational practice, *can* be utilized more or less effectively. One could even attempt to teach FOIL in the context of meaningful work. And it is even possible to attempt extending the FOIL method to general polynomial multiplication (this, however, would not yield a similar shortcut). But why use FOIL at all? What can students learn from it other than how to FOIL? A felt sense of *what* does it enable the student to develop? This is what I mean when I say that not all routines, or procedures, are created equal.

In short, my main criticism of FOIL as a procedure can be summarized in a question: What is the advantage of practicing FOIL? The answer seems to be: getting better at FOIL-ing, an activity no mathematician values. Thus FOIL is *not* a desirable educational practice not because it is procedural but because it is unlikely to lead to any sort of mathematical understanding we might find valuable.

I hope that this detour shows that we are *not* coming around full circle. In other words, though it advocates the importance of practice, this dissertation is *not* advocating for business-as-usual worksheet drills, of the endlessly-apply-FOIL sort. Quite the contrary.

4.5 Closing Remarks

Can conceptual understanding emerge from embodied interaction? I began my dissertation work by orienting myself with this question. In turn, my work became an exploration of what it means to take seriously the claims emerging from radical embodied cognition regarding the roles of bodies and artifacts in the learning of disciplinary subject matter content. This text has been my attempt at taking the reader along on the journey. Here I will revisit a few landmarks we encountered along the way and offer some final commentary.

Motivated by the prospect of alternative accounts to the traditional conceptual/procedural dichotomy, I have investigated the role of routines and practice in the learning of the disciplines. As such, this is a dissertation about practices, routines, procedures—in general, about labor, and what comes of it.

At the beginning, I identified two central issues with the dichotomy of procedures and concepts. The first was that there was little, if any, agreement on how to conceptualize conceptual understanding. The second was that any clear-cut division of doing and thinking should be viewed with suspicion. Yet there is a more pernicious third issue with this dichotomy. It consists of looking at a child and declaring, “She can do the procedural work, but she lacks conceptual understanding. *Therefore, let us teach her concepts*.” In other words, this approach takes the *description* of child’s current knowledge and, in the same breath, offers a *prescription* of how a child should be educated. And, most dangerously, it potentially denigrates labor as something beneath “conceptual knowledge.” What if concepts cannot be taught directly? What if conceptual knowledge emerges through practice, through labor?

While I make no claim that procedures are necessarily required for everything we call conceptual understanding, I offer that they have been incorrectly ignored and even maligned in the literature. And I offer that if we conceive of procedures more generally as disciplinary routines, their cultural and cognitive significance becomes more readily apparent.

After decades of research linking drills (“getting it down cold”) to students’ inability to make personal sense of mathematics (e.g., see Carpenter, Lindquist, Matthews, & Silver, 1983; Erlwanger, 1973; Schoenfeld, 1988), it is reasonable to be suspect of any claims lionizing procedural routines. It is true that not all procedures are pedagogically effective (as argued earlier, FOIL seems particularly ill-suited), and it is true that teachers ought to create contexts of meaningful work for their students practice. However, what this dissertation points to is that perhaps we have been practicing the wrong routines—there may be routines better suited for guiding students towards developing mathematical understanding, such as those utilized in the Mathematical Imagery Trainer for Proportion (Chapter 3 of this dissertation). In this sense, eschewing routines and procedural practice is throwing the proverbial baby out with the bathwater. There are alternatives worth trying.

Namely, it is generally accepted that there are spontaneous ways and scientific (or rather *disciplined*, my preferred translation of Vygotsky’s construct) ways of encountering the world. I offer that instrumented fields of promoted action—a “middle path” of sorts—help bridge these two (consider Figure 6, below).

Figure 6. Education as creating a field of promoted actions. Two women teaching a child to walk (Rembrandt; image used by Michael Cole in Abrahamson & Wilensky, 2007, to illustrate the complexity of articulating pedagogical activity).

Whereas "reform" schools of thought believe that students should begin from concrete situations and then progressively formalize their understanding towards normative abstract representations, "reactionary" perspectives hold that students should embark from symbolical representations and then apply their formal strategies to situated contexts (see Nathan, 2012 for a discussion of these perspectives). The work outlined here puts forth a *third* position, between the reform and reactionary approaches (see Figure 7, below). Namely, it is here proposed that mathematical concepts should be grounded in those felt sensations that live between the "concrete" and "abstract."[21] Thus relevant routines and procedural practice can be used to develop the sensorimotor schemas that bridge scientific ways of thinking to our existing, spontaneous know-how (Abrahamson & Trninic, 2015).

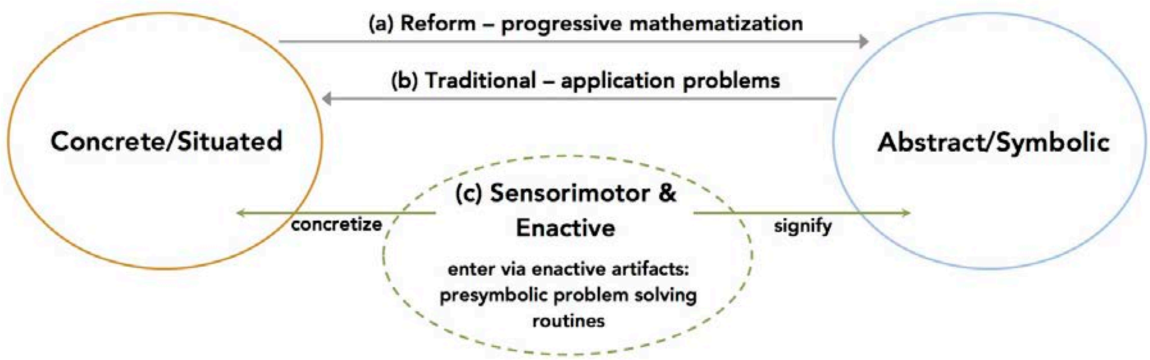

Figure 7. Three approaches to mathematics education, understood as fostering a meaningful relation between concrete/situated and abstract/symbolic. Reform approaches (a) advocate students begin from concrete situations and then progressively formalize their understanding towards abstract representations. Traditional approaches (b) advocate students begin with symbolic representations and then practice applying their formal strategies to situated contexts. The third path (c), represented in this dissertation, advocates that students ground their learning in felt sense mediating between particular situations and representations. So doing, they find themselves reconciling spontaneous and disciplinary ways of acting and thinking.

What are some warrants for the applicability of this approach? Admittedly, this dissertation was written in the spirit of a generative case study rather than confirmative experimental experiment (Clement, 2000), and this sort of work does not, in and of itself, warrant strong prescriptive claims. That said, I have the following arguments to make on this account. In the MIT-P data, we can see forms of reasoning that surpass what we ordinarily see in schools—in particular, the opportunities to spontaneously link multiple notions of proportion (see Figure 5 and Shani's

[21] This insight and its articulation in the form of a concise diagram are the culmination of seven years of concerned efforts at the Embodied Design Research Laboratory. Going further back, the roots of this can be seen in Abrahamson (2006). Moving forward, this structure is now driving our design of the next-generation Mathematical Imagery Trainer for Proportion that involves a gesturing avatar. A general version of this diagram was first shared outside of the laboratory in an accepted grant proposal submitted to the National Science Foundation to procure summer funding for our undergraduate team member, Dana Rosen. That document was prepared by multiple team members, as is our practice. Notwithstanding, I am adapting the diagram here with her permission.

excerpt in Chapter 2, see Abrahamson, Lee, Negrete, & Gutiérrez, 2014; Abrahamson, Trninic, Gutiérrez, 2012).

And while I will not go into detail here, Carmen Petrick (2012; also Petrick & Martin, 2011) has demonstrated statistically significant conceptual gains for MIT-P students (n = 128) as compared to a control group. Furthermore, eye-tracking research by collaborating labs (see Shayan, Abrahamson, Bakker, Duijzer, & van der Schaaf, 2015) continue to corroborate our original findings. All things considered, I am convinced that this approach is worth pursuing further.

I have outlined the notion of instrumented fields of promoted action as a framework for thinking about teaching and learning that does away with the procedural/conceptual dichotomy. The introduction of an artifact (e.g., a Cartesian grid) in these fields of promoted action is, to the child, just another case of executing the enactive artifact ("finding the green"). For the educator, however, this is a designed event that engages yet transforms the enactive artifact. The child, in order to maintain a maximum grip on the situation, now does something *different*, a transition that may be either smooth or abrupt. These transitions are *not* the result of the student's understandings, their *post facto* recognition *is* the understanding. Here we escape the learning paradox: children aren't consciously intending or aiming for some particular insights (they cannot aim at something they cannot see), they are simply orienting themselves to the situation in whichever way reduces the embodied tension they experience. Then, shifts happen.

Along the way, I have aligned myself with scholars arguing that the mind itself is best understood as *literally* found in the interaction rather than in the head. So too is intention not only what an agent does, but the things we do direct us to places we never anticipated.

If it seems like I have played the radical throughout this chapter, I offer the following in my defense: Everything I uttered here, others pointed to already. For example, I draw on sociocultural theories (Vygotsky, 1987; Wertsch, 1979) and accounts of embodiment because I see the two as highly complementary. Vygotsky directed us to the importance of artifacts, and radical accounts of embodiment (e.g., Malafouris, 2013) extend these arguments in novel and, I find, productive directions. Nor am I the only scholar exploring the interplay of sociocultural and embodied accounts of cognition (see, e.g., Geoff et al., 2015; Roth, 2009).

My findings, while generally aligned with the push in contemporary mathematics education research towards meaning and understanding (see, e.g., Thompson, 2013) nonetheless deviate in the sense that I focus on, and advocate for, routines or procedures and their role in the development of understanding. But when I assert that knowledge emerges in and through students' labor, I follow in the footsteps of scholars who recognized the reflective nature of practice (Bamberger & Schön, 1983).

And, almost a hundred years ago, John Dewey (1916) knew that thinking, or knowledge getting, is far from being the armchair thing it is often supposed to be.

One notion I have found particularly difficult to communicate is what I call the *opportunistic* nature of learning: that learners *find themselves* doing something new (and not merely something random, but something valued in the discipline). Allow me to recreate a recent discussion I observed about this very topic. After hearing a conjecture that understanding happens after the fact,

Scholar A, “But this phenomenon of finding yourself seems so *completely commonplace* in everyday life.”
Why even bother studying something so patently obvious?
Scholar B, “But it is also *completely absent* in the literature.”

And this, I think, is exactly the issue. This phenomenon is largely absent in the literature. The idea that learning is opportunistic does not belong to me, nor is it novel:

> [I]t is not the case that the child first carries out the task because she/he shares the adult's definition of situation. It is precisely the reverse: she/he comes to share the adult's definition of situation because she/he carries out the task (through other-regulation). (Wertsch, 1979, p. 20)

Yet while we researchers may well acknowledge that we constantly find ourselves moving in novel and unexpected ways, we rarely if ever acknowledge or explore this in the literature. Why might this be an issue? If a physical therapist ignored gravity, a patently obvious aspect of our lives, she would struggle to make sense of her patients’ postural problems (see Feldenkrais, 1981). Like so, I offer that the learning paradox will remain a puzzle until we fully acknowledge the serendipitous and opportunistic aspects of learning; likewise, teachers will continue to tell their students secrets, wondering why no one understands.

I hope for a future where teachers and education designers embrace this phenomenon. Teachers would focus on providing practices and guidance and react to students’ shifts. Students would make themselves receptive to transitional moments: they would be less afraid of failure, yet also less satisfied with success, unafraid and hungry to continually explore anew. And once a student experienced a shift, or found herself somewhere anew, the teacher would be there, ready to unobtrusively facilitate the next step of their journey. Later, students would say, “Oh, we did it” and reflect with pride on the labor they undertook and understood as necessary in pursuit of knowledge.

Chozan Shissai, a 17th century Japanese Zen swordsman, had this to say about teaching.

> Question: What is the essence of teaching?
> Answer: The master first teaches form without wasting a word about its significance; he waits for the student to discover this himself. This is called drawing the bow but not shooting.
>
> Question: Why do so many schools miss this essence of teaching?
> Answer: In many schools, teachers and students alike are led away from the door through which I believe they must enter [away from practice]. Consequently, they grow enamored with the landscape along their paths, and many stay there and consider it right. Thus, one sees them making a great uproar over the most insignificant educational theories and arguing among themselves what is right and what is wrong. The landscapes along the

> way are merely appearances fashioned within the framework of the mind. As regards to this landscape, details could be discussed without end. (adapted from Ralston, 1989)

Shissai, considered one of the finest swordsman and teachers of his time, deeply valued practice. He echoed Miyamoto Musashi, another swordsman of renown: “The Way is in training.” I believe this is true for all practitioners, regardless of the discipline.

In the end, can conceptual understanding emerge from embodied interaction? The answer I have offered is that it does: for all involved, learning is moving in new ways.

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Appendix A—Complete Interview Protocol

What we say/do	Why we say/do it	Possible Responses	How to respond to those responses
Hi, we're from the University of California at Berkeley. First of all, thanks for agreeing to take part in this study. I hope you'll find it interesting. We'd like to get your help to understand how students think about our stuff. As part of our work, we design activities for learning. We use computers and all kinds of other technology. It's really important to meet talk with students as we develop our technology, to provide us with feedback. Today we're going to be asking you some questions and working with some activities, but remember there are no right or wrong answers. We just want to hear what you have to say about these things. We'll be glad to answer any questions you have now or later.	Introduce ourselves	My mother/uncle works at Cal. Do you know so-and-so?	Cool, Go Bears!
How old are you? When did you turn ___ (student's age)?	Document age	I'm 10. February 29.	Oh, so you're _ years and _ months!
You'll notice we're using a camera to video-tape this session. We use it because we don't want to miss anything you say. If at anytime you wish not to be recorded, please let us know and we'll turn off the camera.	Make students comfortable; Let them understand their rights as volunteer participants.	Will I be on TV?	Probably not, but lots of professors will see you, maybe in some far-away countries!
[*Show the first set of cards; lay them out, scattered across the table*] Here we have a set of cards in random order, and each card has two numbers. Can you find a set of four cards that keep growing, but that also tells a story (or pattern) with the numbers? Let's add the rule that the numbers have to increase/go up in a sequence. Optional questions: Can some other card go before/after/between them?	Are students attending to sequential/additive or multiplicative relations?	Going up by two... [2 4] [4 6] [6 8] [8 10]	I see, cool. How about this one, if we start with [2 5]? Which card can we put next, in a way that makes sense to you?
[*Remove set of number cards from the table. Show the set of balloon cards; scatter them across the table*] Can you tell what these are? Can you do the same with the balloons as the numbers? Let's say that the balloons have to be going up as well. What's happening? Optional questions: Can some other card go before/after/between them?	Make sure student recognizes objects as hot-air balloons, all floating up. Are students constructing sequences that are additive or multiplicative?	It's like a hot-air balloon race! The red balloon is always higher than the blue.	Is there another story here?

[*Collect all the cards. Introduce the MIT-P. Casually point out and explain all the elements of the MIT. Once the student is comfortable/familiar with the technology, begin activity with main prompt*.]	Setting up the MIT-P activity by introducing and naming situation elements (trackers, camera, monitor, laptop), interaction mode (lifting/lowering trackers), and task (making the screen green). These introductory communications establish the perceptual field, significant elements, mode of physical interaction, and available resources.	Is this a game?! Are we going to play Nintendo Wii?	Not quite. We hacked the Wii remote so that it works as a camera only.
[*Set the MIT-P to "Continuous Space" mode and the target ratio to 1:2*] Your job is to make the screen green.	Establish task objective.	H'ok… Wha?? [Student could raise one hand at a time, both at the same time, perhaps flip-flopping them wildly up and down, etc., eventually coming across a green spot.] There, it's green!	You can move both trackers at the same time.
Ok, you found a "green spot." You think you can find green somewhere else?	Elicit their intuitive/default operatory schemas.	Here's another (green spot)! There are different places that make green.	You're doing great, figuring out the system! How about lower down/higher up along the screen?
Do you think it's possible to move between green spots, keeping green? Optional question: Do you think it's possible to keep green, while moving your hands from the bottom all the way to the top?	Further elicit their intuitive/default operatory schemes, specifically those properties of the system that they perceive as "static" (e.g., each hand has a certain elevation) versus "dynamic" (e.g., the right hand always moves faster than the left).	No it's not possible.	Try moving your hands at the same time, but very slowly. That's it—you're doing great!
What have you figured out about this so far? How do you make it green?	Reflect upon, analyze, verbalize, and problematize	The right hand has to be higher than the left… it has to be a	How far away from each other do they need to be?

	their default operatory schemas. Guide them toward discovering the proportional operatory scheme.	certain distance apart. It doesn't matter where they are, so long as the distance of the height between them stays the same.	Is that the same rule for all green spots? Can you show me your rule again? Can you show me just with your hands without the trackers, just as if you were doing it?
[*Set the MIT-P to "Cursors" mode and keep the target ratio at 1:2*] I'm going to add something to the screen here… do you see those? Make the screen green.	Make sure participant recognizes the cursors on the screen as the manipulation objects for making green; establish common language and continue with task objective.	They're pluses! They look like little "t's".	Do they help you make green?
You think you can find green somewhere else?	Elicit their intuitive/default operatory schemes	You put the pluses here and here, and then you get the green!	How about lower down/higher up along the screen?
Do you think it's possible to move between green spots, keeping green? Optional question: Do you think it's possible to keep green, while moving your hands from the bottom all the way to the top?	Further elicit their intuitive/default operatory schemes, specifically those properties of the system that they perceive as "static" versus "dynamic." Rehearse their developing operatory schemes.	No, it's green only in specific spots.	[*Identify another green location lower along the monitor; practice making green continuously*] Try it! Try to move your hands up, while keeping green.
Do these cursors help you make green? What's the rule?	Reflect upon, analyze, verbalize, and problematize their default operatory schemes. Guide them toward discovering the proportional operatory scheme.	Same as before, the right hand has to be higher than the left and it has to be a certain distance apart.	How far away from each other do they need to be? Is that the same rule for all green spots? Can you show me your rule again? Can you show me just with your hands without the tackers, just as if you were doing it?
[*Set the MIT-P to "Grid" mode and keep the target ratio at 1:2*] I'm going to add something else to the screen here… do you see that?	Progressive mathematization; establish common language and	Looks like boxes. It's a grid.	Does having these boxes/this grid on the screen help you in any way? How?

Make the screen green.	continue with task objective.		
You think you can find green somewhere else?	Elicit their intuitive/default operatory schemes	You put them on this line and this line, and then you get the green!	How about lower down/higher up along the screen?
Do you think it's possible to move between grid lines, keeping green? Optional question: Do you think it's possible to keep green, while moving your hands from the bottom all the way to the top?	Further elicit their intuitive/default operatory schemes, specifically those properties of the system which they perceive as "static" versus "dynamic." Rehearse.	No, it has to remain on certain lines/boxes.	[*Identify off-grid location lower along the monitor*] What about here and here? Try it! Try to move your hands up, while keeping green.
Does this grid help you make green? What's the rule?	Guide them toward discovering and articulating the proportional operatory scheme.	The left hand moves up by one, and the right hand moves up by two.	Can you show me your rule again, just with your hands without the trackers, just as if you were doing it?
[*Set the MIT-P to "Grid+Numerals" mode and keep the target ratio at 1:2*] I'm going to add something to the screen here… do you see those? Make the screen green.	Progressive mathematization; establish common language and continue with task objective.	Numbers.	Do these numbers help you in any way? How?
You think you can find green somewhere else?	Elicit their intuitive/default operatory schemes	You put them on this number and this number, and then you get the green!	How about lower down/higher up along the screen?
Do you think it's possible to move between numbers, keeping green? Optional question: Do you think it's possible to keep green, while moving your hands from the bottom all the way to the top?	Rehearse their developing operatory schemas.	No, it has to remain on certain numbers, like 2 and 4.	[*Identify off-grid location lower along the monitor*] What about here and here? Try it! Try to move your hands up, while keeping green.
Do these numbers help you make green? What's the rule?	Guide them toward discovering and articulating the proportional operatory scheme.	2 and 4, 3 and 6, 4 and 8, 6 and 12. The right number is double the left number.	Can you show me your rule again, just with your hands without the trackers, just as if you were doing it?
Did you notice anything about the difference in heights between the two hands?	Focus attention on key feature.	The difference is getting bigger! The interval between 2 and 1 is	Is there a pattern to the difference?

		1, 4 and 2 is 2, 3 and 6 is 3. It grows by one each time.	
There seems to be different ways to make green. You said that the one on the right is double the left. But you also said earlier that when the left hand goes up by one the right hand goes up by two. Are these related? Just now you said the difference is getting bigger as well. Is that related to doubling or your “skipping” strategy?	Coordination prompt; organize visualizations and integrate them into a coherent conceptual system.	The right hand moves up two every time the left moves up one, so that makes the overall height of the right double the left. As the number on the right gets bigger, so does its “half.”	Oh I see. Which number is doubling? Which number gets bigger?
[*Keeping the Cursors, Grid, and Numerals on the display monitor, now change the target ratio to 1:3*] I’ve changed something in the system; let’s see if you can figure out what changed. Optional question: Let me know if you want me to remove/add something from/to the screen, if you think it will help you make green. [*Remove/add artifacts onto the screen per the participant’s request*]	Guide them to discover and articulate a different proportional operatory scheme. Continue through stages parallel to the case of 1:2 all the way through to coordinations.	My old rule doesn’t work! It’s triple! For every one I go up on the left, the right one has to go up 3. 1 times 3 is 3, 2 times 3 is 6, 3 times 3 is 9. The right one is three times the left one!	So what’s the rule? How can you be sure?
[*Keeping the Cursors, Grid, and Numerals on the display monitor, now change the target ratio to 2:3*] I’ve changed something else in the system; let’s see if you can figure it out. Optional question: Let me know if you want me to remove/add something from/to the screen, if you think it will help you make green. [*Remove/add artifacts onto the screen per the participants request*]	Guide them toward discovering and articulating a different proportional operatory scheme. As before, go through the same stages as per 1:2.	This one is hard! None of my old rules work! For every two I go up on the left, the right one has to go up three. 2 and 3, 4 and 6, 6 and 9, 8 and 12. The right one is one-and-a-half times the left one!	So what’s the rule? How can you be sure?
[*Keeping the settings the same, introduce the Driver onto the screen*] This is called the “driver,” and it makes green by entering heights numerically, instead of using your hands. This way, if we have some idea about how the system works, then we can just input numbers to test our theory, instead of having to do it by hand, which has sometimes caused	Guide them toward discovering connections between numerical computations and bimanual operatory schemes for effecting green.	Student might input incorrect values into the driver module, for example, 3:4, 4:5, 5:6. I’m not sure. I need to go back to	Ok let’s try that [*running the module with these numerical values (see on left) should yield a red screen*]. [*If they get stuck and are unsure what*

minor errors… the computer won't make such errors. Shall we try it? Let's start by entering the numbers we were just working with…so I'll enter a "2" here and a "3" here [input 2:3 into the first row of the module]. To fill out this table, what numbers do we need to input to make green? [*If they try to enter a decimal*] Sorry, the computer still has some "bugs" so it only accepts whole numbers at this time. Optional question: Do you remember pairs of numbers that worked earlier, when you were working with your hands?		working with my hands to figure this out! It should be 2:3 and then 4:6… and 6:9... and 8:12.	*numbers to enter*] Earlier you were saying that for every two you go up on the left, the right one has to go up three. Does that strategy work here? Do you want to try again with your hands?
What did you notice just now that made you want to fix it? Why did you think it was wrong? Optional question: How are you figuring out each pair?	Elicit numerical patterns students are noticing.	The left one is skipping/counting by two and the right one is by three. You can add [the pairs] 10 and 15, and 12 and 18.	Oh I see. Let's put in a more few rows of numbers…what should they be? [*Run the module*] Look at that—it's green all the way up! You got it right.
When you added these new numbers, how did you do that? How did you know they would work?	Ditto.	It worked for all the other ones. You just add two to the left and three to the right each time.	Ah, I see.
Did you notice anything about the difference between the two numbers?		The difference is getting bigger! The difference between 2 and 3 is 1, the difference between 4 and 6 is 2, the difference between 6 and 9 is 3. Each time the one on the left goes up two, the *difference* grows by one!	Is there a pattern to the difference?
[*Suggest changing the target ratio to student's choice*] How about using some numbers that we've never even tried before, like 3 and 4, and see what happens? Let's try that… so what we're going to do now is put in a pair of numbers we haven't yet worked with our hands at all, and see if we can take what know and make it always green. To fill out this table, starting with 3 and 4,		We need another pair with a difference of 1, because 4 minus 3 is 1. No wait! The difference has to increase each time. The left is skipping	They have to have the same difference each time—a difference of one? Is that what happened in the other settings? Remember that "skipping/counting" thing earlier? You

what numbers do we need to input to make green?		by 3 and the right is skipping by 4. Student enters 6:8, and 9:12.	said each column was skipping by…
With these new numbers, 3 and 4, did you notice anything about the difference between the two numbers?		The difference is getting bigger here, too! The difference between 3 and 4 is 1, the difference between 6 and 8 is 2, the difference between 9 and 12 is 3. Each time the one on the left goes up three, the *difference* grows by one!	Is there a pattern to the difference?
[*Remove the trackers from the table and turn off MIT-P; bring back the first set of cards; lay them out, scattered across the table*] So let's try the cards again. Same objective as before. Can you find a set of four cards that keep growing, but that also tells a story (or pattern) with the numbers? Optional questions: Can some other card go before/after/between them?		Student could might same set as before Going up by two… [2 4] [4 6] [6 8] [8 10] No wait, these are the same distances so it wouldn't work for making green! Fix to [2 4] [3 6] [4 8] [6 12]	Do you think that these cards are like the stuff we're doing before with the computer? Is it a little different or is it the same? If we ran these numbers, would we get green or something else?
[*Remove first set of cards from the table. Show the second set of cards; scatter them across the table*] What about with the balloons, can you do the same with these cards as the numbers? What's happening? Optional questions: Can some other card go before/after/between them?		The red balloon is going twice as fast as the blue balloon… the distance between them is growing.	Is there another story here?
[*Wrap up the interview*] Well, that's all we have for you today. Thank you very much. Did you find it interesting? Do you have any questions for us?	Closure. Address all remaining issues. Create connections to future classroom activities.		

Appendix B—Coding Schemes

Coding Scheme for Determining Participant Interaction Strategy

Video data were coded as following to determine which strategy, if any, participants engaged to solve the interaction problem. The researchers considered all student actions and utterances as potentially manifesting enactment, explanation, and/or evaluation of the following strategies.

Code	Description	Examples
Fixed Interval	Student's strategy is to maintain a constant spatial interval between hands/cursors. Student explains that the distance between their hands or trackers stays the same as the trackers go higher on the screen.	"It's, it's [the interval] about the same distance." "They [left and right trackers] always have to be the same distance apart to get this green." "They always have to be the same distance apart to get this green."
Changing Interval	Student strategy is to modify the spatial interval between hands in relation to the height of the hands/cursors above the baseline.	"Oh and they're [left and right trackers] getting farther apart as it goes up." "It's kind of but you have to stretch it out as you get [unintelligible] as you get higher." "You—you stretch out the space between them." "Oh! No, it [the interval] gets shorter if you go down more and then it [the interval] gets (to) longer if you go up."
a-per-*b*	Proto-ratio strategy corresponding to sequential hand motions, in which each hand separately moves up or down according to its respective quota. For example, some students expressed that the right-hand cursor goes up by 2 units for every 1 unit that the left-hand cursor goes up.	"Um, well the ex—well, the bottom one is going up by one box… but the top one is going up by two." "For every half I go up, he [points at St2] goes up one." "[T]he right hand always goes up 2, the left hand goes up 1."

a-per-Δ	Proto-ratio strategy that attends to the interval between the left- and right-hand cursors as it changes with respect to the height of the lower cursor. For example, some students noticed that as the left-hand cursor moves up by 1 unit, the interval between the right- and left cursors increases by 1 unit	St1: "I think that um//" St2: "//the distance has to grow//" St1: //The distance has to grow slowly… first one then two then three and slowly growing." "And to get here you have to go up 1 [unit distance] to get the green. Here you have to get 2 [unit distances] to get the green. That's 3 [unit distances] to get the green. That's 4 [unit distances] to get the green. "1 to 2 is one square apart, 2 to 4 is 2 squares apart, 3 to 6 is 3 squares apart."
Multiplicative	Student expresses the numerical location of one hand directly as a product of the other hand's numerical location. This strategy need not attend to previous pair locations or the interval between the locations.	"Hey wait. Um, oh, it's... It's all doubles. The bottom number, like times two is the top number." "Oh! It's like—if, it's half of the numbers. 3 is half of 6, and 4 is at half of 8, and 2 is half of fours. "Wait, wait, wait, but then if you put it at 9, it's at 4 and a half, and 4 and a half plus 4 and a half is 9, and 5 plus 5 is 10"
Speed	Students articulate the relationship between the two cursor actions in terms of their respective velocity, typically using words such as speed, slow, fast, pace.	"So this one should be... so my right should be moving faster." "My right hand goes up a little faster than my left hand and that way it goes it stays green almost all the time." "Right hand needs to go slower."

Coding Scheme for Determining Participant Strategy Coordination

Video data were coded as following to determine whether students engaged in coordinating among interaction strategies. The researchers considered all student actions and utterances as potentially manifesting enactment, explanation, and/or evaluation of a coordination.

Code	Description	Examples
Engaged in coordination	Student compared two strategies or more; attempted to connect them, to determine how they might be commensurate or inter-derivative. This category includes spontaneous coordination as well as coordinations in response to a prompt.	"Because they're each going...'cause this [right-hand tracker]...this one's always going up by two and this [left-hand tracker] one's going up by one, which would mean that... That, um, this one [right-hand tracker] is always double this [left-hand tracker]." "The right has to go faster, they are getting farther and farther apart, because the right jumps 2 numbers when the left jumps 1." "The right one is going half the number faster than the other one." "Well, this one, if it goes, it goes kind of twice as fast as this one does. So you're kind of going like this and it's like a little pace [motions with right hand], this one [motions with left hand] a little one, this one, a little one, this one, ..."
Asked but did not engage in coordination	Student could only articulate one strategy and did not express a relationship between two strategies.	"That way of thinking does not work for doubling" "I don't think my theory works for this."
Not asked	Student was not prompted to think about two strategies and did not spontaneously discuss two or more strategies.	No examples applicable.