Mathematics Learning as Perceptual Reconstruction: The Role of Semiotic Breakdown in Collaborative Problem Solving

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Submitted
May 7, 2018

In Partial Fulfillment of Master's of Art Degree in Education

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Acknowledgements

First, I want to thank Dor Abrahamson for not only being my reader but also pushing me in both my thinking and writing with his scrupulous feedback. This thesis builds on my work from the Paradigmatic Didactical Mathematical Problematic Situations (PDMPS) course taught and designed by Dor, and I am forever grateful for all that grew from PDMPS, which undoubtedly made me a better teacher. I also want to thank Jake Disson, my second reader, for his consistent feedback, support, and sincere commitment to the MACSME program, which has turned out to be more than what I imagined and would not be the same without him. I would also like to thank all members and visitors of the Embodied Design Research Laboratory (EDRL) who watched many videos of the 4-Views problem and provided incredible feedback without feigning interest. Particularly, I want to thank David DeLiema and Virginia Flood who encouraged me and opened my eyes to research related to my thesis.

My math comrades, Annemarie Newell and Emily Gilden, have been incredibly supportive and made being a math teacher the most exciting thing I can imagine doing. I thank them for their support and never letting joy slip out of focus. With that being said, I want to thank everyone in the MACSME cohort for making these two years not only academically rigorous and intellectually interesting, but enjoyable and fun. I've learned a lot from each of you and feel inspired to be in the company of such amazing teachers. My family, both near and far, has also supported me throughout my two years in MACSME. Thank you for always being interested and proud of what I do. Finally, I would like to thank Jordan, who was there during lows and highs, reminding me that life is dynamic and beautiful just the way it is.

Abstract

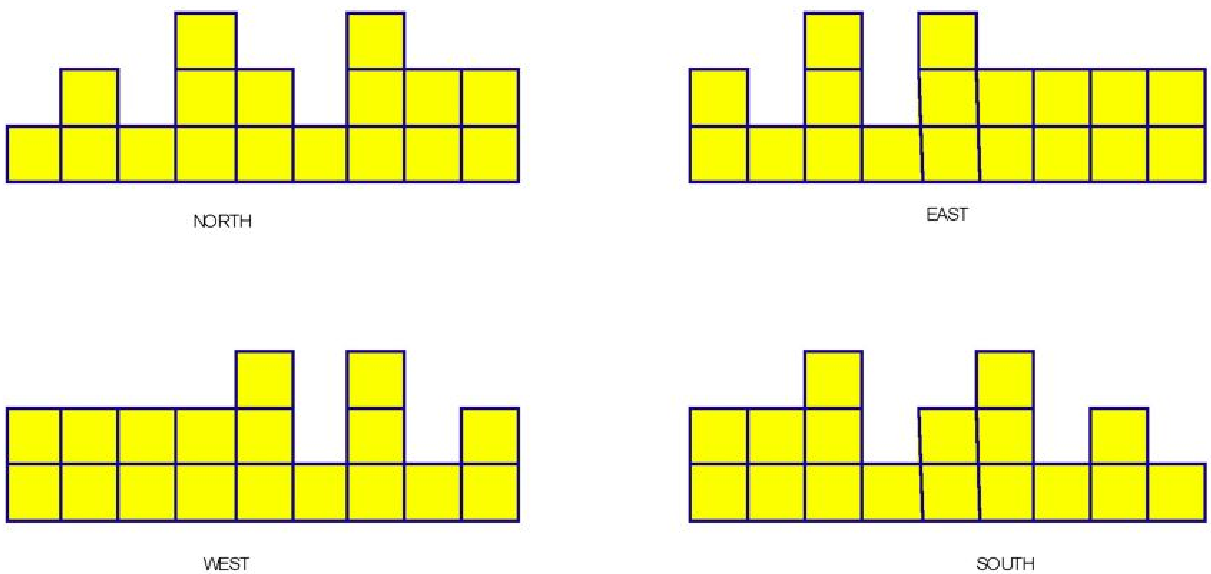
How do multiple participants coordinate their collaborative action so as to solve a novel mathematical problem situation? Operating within fractals of perspective, this paper is the culmination of my continued development in seeing how participants came to solve a particularly novel mathematical problem situation entitled 4-Views. From analysis of 3 different groups of 4 participants solving the 4-Views Problem, I posit that semiotic breakdown serves a discursive function for collaborative group work in mathematical problem solving, in which an individual invites other group members to recognize an impasse in coordinating their collective actions and thus revisit their implicit assumptions regarding to a mathematical sign. In the case of the 4-Views Problem, which provides participants with 2-dimensional diagrams of the North, East, South, and West views of an unknown construction and asks them ambiguously to "reconstruct" it in 3-dimensional space, the semiotic breakdown functions as the watershed moment through which multiple participants are able to reconstruct a new and collective way of seeing from their combined distributed perspectives. More specifically, the semiotic breakdown in 4-Views is sparked when an individual notices a designed impasse between the different 2-dimensional views, catapulting that person into uncertainty. When the breakdown is made public and accepted by the group, a collaborative negotiation process begins, in which participants reconstruct the meaning of the provided diagram (i.e., the views), and thus come to see the views not as 2-dimensional facades of a construction but as projections of a 3-dimensional construction that has depth. This realization resolves the designed impasse, leading the group to the solution of the problem: a 3-dimensional construction that satisfies the 4 views. Pedagogical implications include both the consideration of designing impasses which can

precipitate semiotic breakdowns into mathematical problem situations as well as how to support students who are working collaboratively within the epistemic state of uncertainty.

PROBLEM NAME: 4-VIEWS

Here are four different views of the same construction.
Can you reconstruct the construction?

Work in groups of four seated around a square table.



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Introduction

> *"Pardon me, my Lord," replied I; "but to my eye the appearance is as of an Irregular Figure whose inside is laid open to the view; in other words, methinks I see no Solid, but a Plane such as we infer in Flatland; only of an Irregularity which betokens some monstrous criminal, so that the very sight of it is painful to my eyes."*
>
> *"True," said the Sphere, "it appears to you a Plane, because you are not accustomed to light and shade and perspective; just as in Flatland a Hexagon would appear a Straight Line to one who has not the Art of Sight Recognition. But in reality it is a Solid, as you shall learn by the sense of Feeling."*
>
> ***-Flatland: A Romance of Many Dimensions, by Edwin Abbott Abbott***

I remember sitting in my classroom in Nashville, Tennessee, and thinking that there must be some other way that people think about mathematics education than the way that I was told to teach. Under the intense scrutiny of my alternative teaching licensure program, I was pressured for two years to teach students from low-income backgrounds how to efficiently carry out mathematical procedures to ensure their success on the state exam. I was told that this is what mathematics was - that, for example, I should be "proud" of my students when they carried out the correct sequence of calculator buttons to calculate standard deviation. I distinctly remember someone in my program saying "They love it! Even if they don't understand, it makes them feel smart to get the right answer." It doesn't escape me that these experiences were couched in deep-seated racist pedagogies and educational agendas.

This paper is emblematic of the "other ways of thinking about math" that I wondered about back in 2013. Since joining the Graduate School of Education at UC Berkeley, I have been continually astonished by the theory and applications of constructivist and constructionist epistemologies, in which knowledge is considered to be an active process where students construct new meanings from prior understanding, particularly through the authentic construction of a meaningful product. For me, taking such an epistemological stance has responded directly to the pedantic and oppressive teaching pedagogy put forth by my alternative licensure program; seeing my students as active agents who construct their own understanding has supported me in humanizing my students and garnering the patience needed to let them learn.

At the deep of my core, however, a question has lurked. If we construct our own meaning from our prior knowledge and experiences, how can we construct something new, as opposed to reconstruct what has already been? This question and its answer has major personal and academic implications. If it is impossible to avoid just reconstructing our past, how is it that we can break free of knowledge structures that oppress and bind us? Is it possible to imagine, and therefore construct something genuinely new? As I've come to see it, the 4-Views problem provides a modest example of how we can, indeed, construct new meanings - even hold two incongruous meanings simultaneously - through collaborative mathematical problem solving.

As I localized my thesis to understanding how students come to see in new ways, I became particularly interested in characterizing the moment in which this new construction happens. What precipitates the moment in which we are able to construct new ways of seeing? That is, what *makes* us reconstruct our understanding in the first place? In the case of collaborative problem solving, how do participants coordinate their collaborative action so as to solve a novel mathematical problem situation? Particularly, how do participants coordinate the collaborative reconstruction of a mathematical diagram when faced with a breakdown in their perception of the diagram?

To answer these questions, I analyze 3 different instances where participants experienced a breakdown in their perception of the 4-Views diagram (i.e., the yellow squares representing North, East, South, West) during collaborative problem solving. In each instance, an individual participant recognizes an impasse in the group's attempt to coordination of their collective actions, and thus revisits their implicit assumptions about the diagram. Accordingly, as participants struggle to make new sense of the diagram given the designed impasse that precipitates the breakdown, they attend to the inherent ambiguity of both the diagram and the prompt itself, which vaguely directs participants to "reconstruct" the construction. It is this moment that is of concern to this paper, in which the breakdown in their perception of the diagram causes participants to pivot towards a new perception of the diagram in which its latent ambiguity is untangled: that the 2-dimensional diagram accurately represents the 3-dimensional construction while obfuscating the construction's innate depth. That is, the 2-dimensional diagram is a projection.

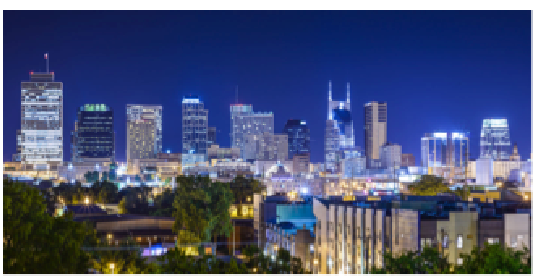
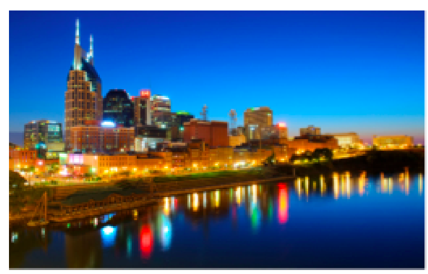
What is of interest to me is not just pinpointing the space-time location of this breakdown in the diagram in each of the 3 different group's collaborative problem solving process. Rather, as the breakdown in each group transpired quite differently, I analyze the dimensions along which there are similarities and differences across the case studies. While each group consisted of unique individuals who have varied life experiences and prior knowledge that they brought to the collaborative problem solving process, are there deeper patterns that illuminate the different ways in which participants collaboratively experience a breakdown in perception and thus share uncertainty? How do participants experience agency in revisiting their implicit assumptions and engage with mathematically ambiguous territory?

The pedagogical implications of this paper are that we take a closer look at how uncertainty, at times the outward expression of a breakdown in concept, is treated in the classroom. How do teachers respond to and make space for student uncertainty as their students encounter the ambiguous terrain of mathematics? With the case of collaborative problem solving, how do teachers effectively teach students to make this space for each other? As I transition out of graduate school and back into the teaching profession, these questions remain paramount to ensuring all students have access to productive breakdown moments, in which they can reconstruct their understanding of a mathematical artifact in line with normative disciplinary perception.

The 4-Views Problem: What's Going On Here?

For clarity purposes, it is necessary to briefly discuss the 4-Views problem and its designed features, namely the designed impasses that are crucial to this research project. To begin, what is in a view? This question drives the 4-Views Problem as participants struggle to determine what is discernible from the diagram (i.e., the yellow views). The act of viewing may seem simple: in our everyday lives we often obtain a view by observing a known object that exists 3-dimensionally and reconstructing it 2-dimensionally (i.e., the view), or observing a 2-dimensional figure (e.g., pixels on a television screen) that represents a familiar 3-dimensional object (e.g., the flat array of colors on the television represent a person). With the help of our imagination, we easily suspend disbelief and forgive the dimensional dissonance; we physically view a 2-dimensional figure, but see what we know. This suspended disbelief is precisely what gets in the way of painting, articulated by many artists as the need to "paint what you see, not what you know" (Disbelieve it is a bowl of fruit! It is really an assortment of geometric shapes.). It is in knowing the object that we casually include or leave out features when we are asked to describe it. For example, if giving directions to a building from the street, one might describe the 2-dimensional face of the building, rather than how it *really* looks from the street, with chimneys or other buildings sticking up from behind. Thus, the ease with which we shift between 2 and 3-dimensions depends on our knowledge of the object we see, as our knowledge permits us to make perceptual decisions. As Hutchins (2010) writes, “perception is something we do, not something that happens to us” (p. 428).

In this way, the 4-Views problem asks us to *do something*; that is, to take the views in the form of the diagram provided to us and build the 3-dimensional construction from which the views came. What is different compared to the previous examples, however, is that the 3-dimensional object is unknown to the viewer in the 4-views problem. For example, when using one 2-dimensional view to imagine what the 3-dimensional object might look like, we are often thrown off or proven wrong about our imagined object when confronted with other views of the same structure. For me, this point is salient as I remember that no matter how many times I had driven on the highway that encircles Nashville, Tennessee, I was constantly saying to myself, “Wow, this city is [bigger/smaller] than I thought it was.”

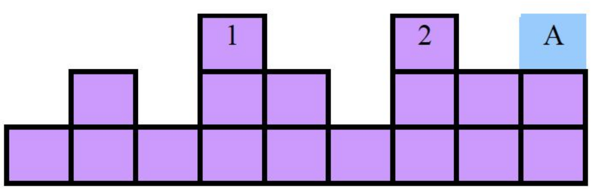
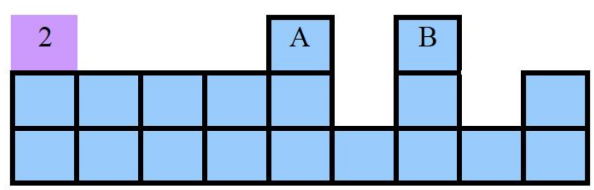
Nashville Skyline from the South	Nashville Skyline from the East
	

The 4-Views problem elicits this sense of vertigo by putting the problem-solver on a path similar to my drive around Nashville, where they are given the opportunity to realize their implicit assumptions about the provided views to understand something deeper about the structure that the views represent.

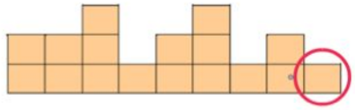
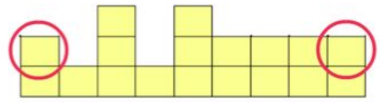
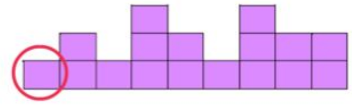
Figure 1

The designed impasses of the 4-Views problem add another layer to the natural vertigo that is elicited from attempting to combine different views simultaneously. These impasses arrive when participants perceive the diagram to represent facades of a construction (see Figure 1). In this quasi-2.5-dimensional perceptual construction (i.e., the views are not flat, but they are also not projections), participants face two impasses that support the initiation of a semiotic breakdown.

Height Impasse: North and West are perpendicular views, both of which have a height of three. Suppose these views are constructed as walls that are connected to form a perimeter, as in Figure 1. When viewing the construction from the North, block A from the West wall would be visible on the right edge of the North view. Similarly, block 2 from the North wall would be visible from the West view. This would render each view incorrect compared to the diagram.

North View	West View
	
When viewing the construction from the North, the West wall's A block is visible.	*When viewing the construction from the West, the North Wall's 2 block is visible.*

Corner Impasse: The 9th column of the South view and 1st column of the North view have a height of 1, but the East view has a height of 2 in both of those corners. When participants attempt to have each wall share its edge columns with the adjacent view, the walls have contradictory heights that must be reconciled.

South View	East View	North View
		
The 9th column of the South view only has a height of 1.	*The 1st and 9th column of the East View, which ostensibly are shared corners with the North and South View, have a height of 2.*	*The 1st column of the North view only has a height of 1.*

Having an understanding of these impasses is crucial as they are the designed instantiations for semiotic breakdown in the 4-Views problem, which by design anticipates participants to perceive the views as walls. But how do participants then move from their perceptual construction of the views as walls to understanding the necessary depth of the 3-dimensional construction? In the following literature review, I frame the unique collaborative problem solving process that results from the 4-Views problem by first providing a brief overview on the value of collaborative problem solving. Then, I detail how mathematical ambiguity, evident in the 4-Views diagram, stimulates the social construction of ideas. Finally, I build on the literature of ambiguity by discussing the role of semiotic breakdown and diagrammatic reasoning when encountering a mathematically ambiguous diagram.

Literature Review

Collaborative problem solving is both heavily researched and increasingly emphasized as a productive and potentially equitable learning opportunity for all students (Cohen, 1990; Lubienski, 2000). While much of this research has focused on the teaching pedagogy needed to support equitable group work and mathematical problem solving (Freudenthal, 1971; Smith & Stein, 2011; Schoenfeld, 1998; Boaler, 2008; Esmonde, 2009) as well as the ingredients and challenges for designing particularly group-worthy mathematical problems (Doorman et al., 2007; Lotan, 2003; Buell et al., 2017), a recent call has been made to research the role of public displays of uncertainty and unknowing in collaborative problem solving (Jordan & McDaniel, 2014; Watkins et al., 2018). More so than just a coincidental side effect of the at times arduous task of problem solving, a student's positioning as not-understanding has been found to both initiate and sustain collaborative inquiry (Watkins, 2018). I hope that this paper responds to and amplifies the call for research that validates and explores the role of uncertainty by analyzing the discursive function of breakdown in collaborative mathematical problem solving.

Research that asserts the epistemic state of uncertainty as valuable for inquiry and mathematical problem solving is a radical departure from philosophies of mathematics rooted in logicism and absolutism. Rowland (2003) summarizes the history of this departure by positioning mathematical ambiguity under fallibilist philosophy and constructivist epistemologies, both of which inform this paper. More so than just the acknowledgement of mathematical ambiguity, a theoretical grounding in the social construction of knowledge, and therefore the social situatedness of mathematics lassos the discipline down from Platonism in which mathematics is pure, objective, and beyond humanity. As Ernest (2008) writes,

> Within the unified world there are, among the myriads of things and beings, humans with minds and groups of humans with cultures. Human minds, the seat of the mental, are not a different kind of 'stuff', but are a complex set of functions of self-organizing, self-aware, feeling moral beings. Mathematical knowledge, like other semiotic and textual matters, is made up of social objects. These are simultaneously materially represented, given meaning by individuals and created and validated socially. (p. 4)

Under this framework, mathematical ambiguity has been staunchly defended as not only inherent due to the social nature of mathematics as a discipline, but also an important resource for the learning and doing of mathematics (Abrahamson, 2007, 2009a, 2009b, 2014; Barwell, 2005; Byers, 2007; Foster, 2011, Mamalo, 2010; Rowland, 2003).

Working within this philosophical and mathematical framing, I first outline research on mathematical ambiguity and provide examples of the social nature of mathematical ambiguity. Then, I describe research on the construction of mathematical signs and diagrams, as the 4-Views problem revolves around how participants enact their perceptions of the ambiguous diagram (i.e., the yellow squares that form the views). Finally, as this paper addresses the moment in which participants experience a breakdown in their perception, I detail what it is I mean by "breakdown" and situate breakdown into the literature on diagrammatic reasoning.

Ambiguity as Learning: A Semiotic Analysis of Conceptual Change

Byers (2007), who writes extensively on ambiguity, defines ambiguity as involving "a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference" (p. 28). Building on this definition, Foster (2011) draws parallels between this definition of ambiguity and the arts, where ambiguity is not only easily seen as a joyful aspect of the arts but more importantly intrinsic to the discipline. So where does ambiguity exist in the doing of mathematics? As a starting point, Foster (2011) provides four categories of mathematical ambiguity that appear throughout mathematics.

(a) symbolic ambiguity (e.g., $\frac{si\cancel{n}\,x}{\cancel{n}} = six = 6$)

(b) multiple-solution ambiguity (e.g., multiple roots of a quadratic equation)

(c) paradigmatic ambiguity (e.g., 3+2 represents the process of adding and the product)

(d) definitional ambiguity (e.g., does "radius" represent the line segment itself or the length?)

Expanding upon Foster's (2011) categorization, I propose that these different types of ambiguities are plaited together, in that an ambiguous mathematical situation can subscribe to multiple, or all, of Foster's categories. For example, the 4-Views problem contains a symbolic ambiguity (e.g., what do the little yellow boxes mean?), a multiple-solution ambiguity (e.g., there are many solutions to the 4-Views problem), a paradigmatic ambiguity (e.g., the views represent both the 2-dimensional diagram and the 3-dimensional projection), and a definitional ambiguity (e.g., what does "reconstruct" mean?). The 4-Views problem as a whole could be described as a case of ambiguity, where participants address many layers of self-consistent but mutually incompatible frames of reference.

What kind of resource is ambiguity for learning mathematics? In this section, I use a few examples from the literature that illuminate ambiguity as the fulcrum of the social construction of mathematical knowledge. For example, Foster (2011) describes a spontaneous episode with his students, in which ambiguity between volume and capacity led to deeper mathematical questions. The students were exploring the following problem:

> Find the total surface area of a solid hemisphere of radius 5 cm.

Ambiguity arose after an initial misconception was clarified and the correct solution obtained, in which students had initially divided the surface area of a sphere and halved it, forgetting to account for the base. Despite arriving at the correct answer, hence "finishing" the problem, students began to engage in a topological exploration, as they debated whether or not their first answer, which did not account for the base, was correct for a "hollow" hemisphere. Their debate centered on whether or not the "inside" of the hemisphere must be accounted for, which then led to a discussion about whether this inside surface area would be slightly smaller than the outside surface area, whether or not it was even possible to see the inside of the hemisphere, and whether or not the volume of a hollow object is the same as the volume of solid object. Foster (2011) argues that clarifying these ambiguities at the outset of the conversation would have “killed the episode” and thus take away the opportunity for students to engage in the mathematically discursive practice of defining. I would add that opportunities to explore deeper content knowledge surrounding geometry and topology would have also been taken away.

Barwell (2005) uses a similarly geometric example to support the idea that ambiguity is an important resource in the mathematics classroom. In the episode, students are learning about one, two, and three-dimensional shapes. After discussing the flatness of two-dimensional shapes, the teacher picks up a plastic circle to illustrate to students the two-dimensional version of a sphere. However, she prefaces the illustration with ambiguity, saying that she doesn’t “like these [the plastic circles] ... coz they look three-dimensional don’t they. They’re thick but they’re not meant to be, they’re meant to be two dimensional” (p. 122). The teacher’s acknowledgement of visual ambiguity - that the plastic circle can be seen as both two-dimensional and three-dimensional - leads to a discussion in which a student claims that one-dimensional shapes are impossible because “a line is kind of like a rectangle filled in.” Barwell (2005) argues that this observation, which is validated by the teacher in future turns, opens windows to explore sophisticated mathematical ideas about dimension and the “crucial aspect of mathematical discursive practice, namely that what is ‘meant’ is rarely the same as what things ‘look like” (p. 123).

Byers (2007) portrays ambiguity with the number zero: “the nothing that is” (p. 24). Using his definition of ambiguity (see p. 14 of this paper), zero offers a simple, yet exquisite example of a single idea (zero) that has two self-consistent but mutually incompatible frames of reference (nothing, existence). Defending the value of this ambiguity, Byers (2007) continues to write that mathematicians who wish to be creative

> "...must continually go back to the ambiguous, to the unclear, to the problematic, for that is where new mathematics comes from. Thus ambiguity, contradiction, and paradox and their consequences - conflict, crises, and the problematic - cannot be excised from mathematics. They are its living heart." (p. 24)

Turning towards the 4-Views problem, ambiguity is indeed at the living heart of the problem, as participants must negotiate how a 2-dimensional diagram, which by definition *lacks* depth, represents the projection of a 3-dimensional construction with depth.

While collaboratively building a construction that satisfies the 4-Views completes the task at hand, the core learning in the 4-Views problem is in re-constructing the perceptual structure of the provided diagram (i.e., the yellow squares that represent the North, East, South, and West views) to be a projection, which precipitates the assembly of the final product. Undeniably, the process of re-constructing the diagram and the final 3-dimensional construction are intertwined, as the physical construction provides necessary feedback, such as the designed impasses discussed previously, that prompts participants to reconsider their implicit assumptions about the diagram. However, it is in the collaborative reconstruction of the diagram that participants learn to question their assumptions about 2-dimensional views, face uncertainty, and ultimately see the diagram as a projection - all of which support the understanding of mathematical concepts of dimensionality and are necessary to solve the 4-Views problem. For this reason, I turn towards literature on the role of ambiguity in mathematical signs and diagrams.

To start with, there are many divergent applications and meanings of mathematical symbols. Mamolo (2010) uses the case study of the + sign to illustrate this point, where + carries various and comparatively discordant meanings when adding natural numbers, adding rational numbers, modular arithmetic, and transfinite arithmetic. For example, $1+2=3$, but $\frac{1}{4}+\frac{2}{4}\neq\frac{3}{8}$. Moreover, using the example of 5 + 4, Gray and Tall (1994) show that + operates as both a process (add 4 to 5) and a concept (9), suggesting that these polysemous mathematical symbols be termed “procepts”. Fluency in mathematical symbols therefore determines not on one’s ability to deeply understand a singular meaning, but rather “learning a meaning of a symbol, learning more than one meaning, and learning how to choose the contextually supported meaning of that symbol” (Mamolo, 2010, p. 259).

To make matters more complicated, mathematical signs do not arrive on our doorstep with objective meaning. Rather, as members of particular discourse communities, we socially construct the meaning of the sign externally (i.e., the meaning is not objectively within the artifact itself). This becomes particularly complex as signs become couched in tightly bounded communities of practice. As Sfard (2002) writes:

> “Seeing things in displays is not a matter of just looking. What cardiologists can see in electrocardiograms and what architects notice in blueprints often remain invisible to the layperson. This means that seeing what is regarded as relevant for a given problem requires learning….In mathematical discourse, this is what underlies, for instance, our instinctive decisions to attend to the degree of a variable in any algebraic expression and ignore other features, such as the shape of the letters in which the expression is written.” (p. 320, 324)

The discursive practices within specific discourse communities, such as the cardiologists, architects, or mathematicians, determine these “instinctive decisions” as to how certain artifacts and signs should be used and understood. Vérillon & Rabardel (1995) describe this process as instrumentation, in which members of a discourse community construct utilization schemes for how an artifact can be used.

Goodwin (1994) uses the Rodney King trials as a powerful display of how discursive practices can be used to legitimize, or “professionalize” a certain way of seeing and provide those outside of a discourse community with utilization schemes. Goodwin (1994) describes how the defense attorneys in the first Rodney King trial used three discursive practices - coding, highlighting, and producing graphic representations - to instruct the jury how a professional police officer would see [the video of] Rodney King. In so doing, the defense attorneys successfully developed and shared a utilization scheme for the jury. Thus, rather than seeing an African-American motorist being violently beaten, the jury was provided with an alternate perspective that while building off of racism, relied on the so-called discursive practices of police “experts” as justification. This example is powerful not only because of the socio-political context and the ethical implications, but because both the defense and prosecution used the Rodney King video as an instrument. The result of the trial, in which the jury determined that the officers were not guilty, goes to show that if beauty is in the eye of the beholder, so is apathy.

Constructing Perceptions of Mathematical Diagrams in Collaborative Problem Solving

The examples provided detail how discourse communities construct meanings from signs and diagrams. Yet, how do students, who are emerging members of the mathematical discourse community, construct the meaning of mathematical signs, particularly in collaborative group work? Sfard (2002) argues that artifacts carry semiotic potential in that both personal and mathematical meanings can be related to the artifact in use, and that it is the teacher’s job to bridge these two meanings. However, as Abrahamson et al. (2009) have shown, students carry implicit mathematical meanings to polysemous mathematical signs, which they are often unaware of. In the case of a designed compound probability experiment, Abrahamson et al. (2009) found that some students were able to shift between two contradicting, but simultaneously valid mathematical perceptions of a mathematical sign: one in which the order of the elements was attended to, and one where the order did not matter. Rather than seeing the probability experiment as necessitating one perception *or* the other, the deeper mathematical thinking and individual empowerment arrives when students are able to acknowledge the validity of both mathematical constructions of the ambiguous sign.

With collaborative group work, co-construction of a novel and ambiguous mathematical sign, such as the 4-Views diagram, demands different resources than the discursive practices of a professional community. Sfard (2002) argues that the process of student co-construction of new mathematical signs operates cyclically as students apply former discursive practices in a series of what she calls intimations, which are then assessed through implementations. That is, students enact their naive ideas regarding to a mathematical problem situation, receive feedback, and then re-examine their initial ideas. Sfard's (2002) intimations and implementations parallel Peirce's work on diagrammatic reasoning, in which he argues that students first construct, experiment with, and then observe a diagram in a new way that professionalizes their perception (Bakker, 2007). While Bakker (2007) describes Peirce's construction step as a the physical creation of a diagram to "represent the relations that students consider significant to the problem," (p. 17) I expand this interpretation to also involve students' perceptual construction of a provided diagram.

Peirce's *hypostatic abstraction* (Bakker, 2007) adds an additional layer to the collaborative co-construction of mathematical diagrams, offering an explanation as to how participants arrive at a new perception from the observation step. In short, hypostatic abstraction is the process by which certain features of a diagram become a new object of investigation. That is, when returning to observe the diagram after experimentation, participants may find that a particular part of the diagram or a common characteristic throughout the diagram become salient. As participants work collaboratively, the features of the diagram that become salient are open for debate and interpretation. This is particularly important in the 4-Views problem because participants do not know what the 3-dimensional construction they are creating is "supposed" to look like. Thus, what becomes salient within the diagram must be collaboratively negotiated.

Weaving Sfard's (2002) and Peirce's (Bakker, 2007) frameworks together with Abrahamson et al.'s (2009) findings in the probability experiment, I propose the following framework for considering how participants reason about novel diagrams. When participants collectively encounter a novel mathematical diagram, they first individually form an initial perceptual construction of the diagram by applying previous discursive practices (intimations), thus assigning implicit mathematical meaning to the diagram. These initial perceptions are often tacit, as participants are not aware that they have assigned these implicit meanings (Abrahamson et al., 2009). As participants collaboratively experiment with the diagram by enacting their perceptual constructions (implementations), they may receive feedback from acknowledging emergent conflicts in the actual or imagined material assembly of the representational system (e.g., the impasses in the 4-Views problem). In collaborative group work, this feedback might involve realizing individuals within the group have different initial constructions, or that their collective perceptual construction of the diagram has reached an impasse that cannot be reconciled without collaboratively changing it. This is the moment in which participants observe the results of the experimentation and reflect on what has happened. To address the impasses or failures of the constructed diagram, participants may feel the need "to construct a new diagram that better serves a purpose" (Bakker, 2007, p. 18).

Unlike creating a diagram from a provided representational structure, participants in the 4-Views problem are asked to create the representational structure from a provided diagram. Given that participants are often unaware that they carry implicit mathematical perceptions of the diagram, what motivates participants to reconstruct this perception? That is, what draws participants' attention to the fact that they are perceiving the diagram, making perception itself the ubiquitous quality called into question?

Semiotic Breakdown

To address these questions, I consider the discursive function of semiotic breakdown in bringing participants' awareness to their implicit perceptions. Koschmann et al. (1998) synthesize the independent works of Heidegger, Leont'ev, and Dewey to define breakdown as "a disruption in the normal functioning of things forcing the individual to adopt a more reflective or deliberative stance toward ongoing activity" (p. 26). To illustrate the concept of breakdown, Koschmann et al. (1998) cite Heidegger's example of a carpenter using a hammer. In this example, the carpenter uses the hammer as a means to some end, and does not take note of the hammer as a thing (i.e., the hammer is "ready-to-hand"). When the carpenter reaches for a hammer and it is broken ("un-ready-to-hand"), suddenly it is no longer a means to an end but an object of concern that must be accounted for ("present-at-hand"). This shift from un-ready-to-hand to present-at-hand signifies Heidegger's breakdown, in which normal functioning is halted due to a disruption that enables individuals to perceive the activity in a new way.

Using this definition, a breakdown in a mathematical sign - such as the one-dimensional line the student determines to be two-dimensional - occurs when the perception of the sign is disrupted. For example, when considering the definitions of dimensionality, the student realizes that the line can no longer be considered one-dimensional. This breakdown in perception acts as a catalyst for outwardly expressed uncertainty that motivates the group to renegotiate explicit or implicit assumptions about the sign. In so doing, students may come to realize the ambiguity of the mathematical sign; that the line drawn on the board can be both one and two-dimensional in toggling perceptions.

In the 4-Views problem, participants experience a semiotic breakdown in their co-construction of the provided diagram (ready-to-hand) when they realize there is an impasse within their distributed perceptions (un-ready-to-hand) as they build the construction, and thus attend to the diagram that requires interrogation (present-at-hand). In this case, it is as if four of Heidegger's carpenters have each used the hammer (i.e., the diagram) to collectively build a cylinder, and realize they have ended up building a box (i.e., the 3-dimensional construction). The realization of something askew precipitates the breakdown, which in the case of the 4-Views problem is made public to the group and collaboratively reconciled. While the diagram (the hammer) remains intact in its literal form, the semiotic breakdown causes participants to express their uncertainty, renegotiate their collective mathematical perspective of the meaning of the diagram, and determine a new way in which the views must be "reconstructed", or in the case of the hammer, a new way that the hammer can be used to build the desired construction. Notably, something is learned about the diagram (the hammer), as well as the intersubjective implicit assumptions, as its meaning is renegotiated. Participants are thus called to attend to the ambiguous nature of the diagram, in which it is both a 2-dimensional diagram that lacks depth and the representation of a 3-dimensional construction that has depth. Consequently, it is the semiotic breakdown that serves as the pivotal learning moment for participants as they must collaborate in order to resolve the breakdown in their construction of the diagram.

So how does the semiotic breakdown transpire? In this literature review, I first described ambiguity as central to mathematical learning. Ambiguity, however, is not visible to the problem solver until two mutually incompatible perceptions are realized. Literature on diagrammatic reasoning suggests that new perceptual constructions of a diagram form through a process of experimentation and observation with the diagram's representational structure. In the 4-Views Problem, the experimentation takes form as participants construct their views as walls and form a 3-dimensional structure. The designed impasses, when realized through the participants' experimentation, induce a semiotic breakdown that draws the participants' attention towards the diagram. As participants collaboratively reflect on and observe the diagram, they reconstruct their implicit perceptions of the diagram and learn a new and collective way of seeing.

Through this discussion of ambiguity, uncertainty, and breakdown, I have aspired to set up the conversation so as to analyze the discursive function of breakdown in the following 3 different case studies. As is the case with all forms of collaboration, the 3 different groups in this study consist of unique individuals who despite working on the same problem and ultimately arriving at the same breakdown and similar solutions, experience the breakdown, express their uncertainty, and resolve the impasse in distinct ways. By detailing how the breakdown unfolded in each case, I will draw connections between the 3 groups so as to determine the quality of the different breakdowns. As will be shown, the breakdown in the diagram does not necessarily result in collaborative problem solving. Thus, pedagogical implications for how to facilitate collaborative group work in the context of breakdowns are discussed.

Methods

Participants

Data were collected during three separate one hour sessions. Two sessions took place at participants' homes, and the third took place in a U.C. Berkeley classroom. Each session consisted of four participants who self-identified as friends or acquaintances from school. Participants in the first session included Alex, Edmund, Taylor, and Sean, all 8th grade boys. Second session participants were Sofia, Noah, Will, and Mason, a co-ed group with Sofia being the only girl, and took place at Sofia's house. Sofia was also the only 9th grader, while Noah, Will, and Mason were all in 8th grade. The third session participants included Aisha, Laura, Katie, and Iliana, all 8th grade girls, and took place at Aisha's home. All names are pseudonyms.

Materials

In each session, participants were given their own copy of the 4-Views Problem, and if possible, sat around a square table. Pencils and scratch paper were provided. Once participants reached a certain point in the problem (see Procedure), they were given blocks and the option of using grid paper.

Procedure

Before each session, participants were told they would be working on a fun math problem in an effort to motivate them to participate. In Sessions 1 and 2, participants sat around a square table, while in Session 3, participants worked at a lightly rectangular table. In all three sessions, participants were given no verbal explanation as to what the problem consisted of and instead were provided with the 4-Views Problem on a sheet of paper. This was an opportunity for participants to discuss their preliminary ideas with other group members, and for me to collect data on how participants understood the problem before they had materials (blocks and grid) to support them in making the structure. Particularly, I was interested in how participants understood the word “reconstruct”.

Once participants had a chance to think about the problem and have preliminary conversations aired to the group, I provided participants with blocks and grid paper. The moment of introducing these materials was subjective with each group. I provided blocks when either (a) group members had repeatedly motioned that there must be some other materials to the problem or (b) group member collaboration had been exhausted. Participants then used the blocks and grid paper to build the construction. Once participants told me they were finished, I asked them how they knew and to show me how they are finished. After agreement that the construction was correct, I engaged participants in a de-brief that allowed them to discuss what was difficult about the problem and reflect on their process of problem-solving.

Data Collecting

For the third session, multiple video and audio tracks were collected by myself and an assistant using handheld cameras. A stationary video recorder also ran but due to a malfunction did not retain any of the data. In Sessions 2 and 3, video data was collected using two cameras: one handheld, one stationary. I personally filmed the groups using the handheld while also facilitating, such as handing out materials or answering questions. Video data was uploaded to a computer, synced with audio data, and then transcribed. Participant work was also collected.

Data Analysis

Video data was used to track participants' gestures in conjunction with dialogue in order to understand each group's complex journey to the solution of the 4-Views problem. Video data from Session 3 was analyzed extensively with particular attention to how participants communicated their understanding of the problem with gesture, dialogue, and use of the provided blocks and grid paper. These frameworks of analysis were provided, adapted, and supported by multiple presentations of the video data in the Embodied Design Research Laboratory (EDRL, Dor Abrahamson, Director) and Gesture Group (Eve Sweeter, Organizer) at UC Berkeley. A synthesis of these collaborative presentations and individual analysis of the data led to a course paper that outlined the uniquely collaborative process encouraged by the design of the 4-Views Problem, in which the participants from this first group distributed the 4 views, one view to each participant, that they then constructed using blocks. This collaborative move off-loaded the conceptual complexity of the problem and gave each of the participants a defined perspective of the problem.

In the course paper, I asserted that this distribution of the 4 views enabled individuals within the group to notice inherent features of the 4-Views Problem (i.e., the Northeast Corner) that problematized the groups' naive understanding of the problem, in which the provided diagrammatic views are interpreted as 2-dimensional facades of a construction rather than projections of a 3-dimensional construction that has depth. In an interplay between the distributed subjective ("my view") and intersubjective ("our construction") experience, participants collaboratively negotiated how to maintain their view while resolving the impasses made salient in their effort to combine their views. While maintaining their distributed perspective, participants disrupted their naive understanding of the provided diagrammatic views by pushing the blocks towards the center of the construction, thus unwittingly occasioning for each other opportunities to realize that views are projections of a structure that has depth.

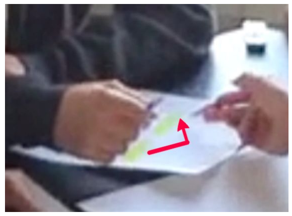
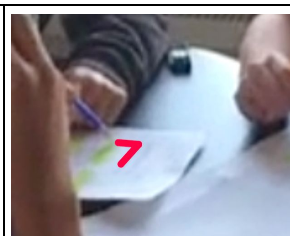
This paper builds off of the graduate course paper described by analyzing the work of 3 different groups on the 4-Views Problem. To grasp the similarities and differences between how each of the 3 groups solved the problem, Sessions 1 and 2 were independently broken up into chapters and coded for the participants' use of metaphors in their dialogue, group social dynamics, and subjective and intersubjective conceptual understandings of the problem. Chapters of the data were broken up by what I considered to be milestone moments in which the group made a considerable gain in their approach to the solution. Each chapter contained moments of high collaboration and interaction, as well as participant turns towards their subjective understandings, in a type of waxing and waning process.

Multiple avenues of analysis and ways of looking at the data presented themselves in collaborative data sharing sessions in the EDRL. Noticeably, all groups successfully solved the 4-Views Problem, while only the third session of participants distributed each view to a particular group member. Given the variations in strategy, the question of how each of the groups coordinated their collaborative actions to solve the 4-Views problem became salient. What emerged as common between the groups is that each group needed to reconstruct their perception of the diagram in order to arrive at the solution. In other words, each group needed to experience a semiotic breakdown.

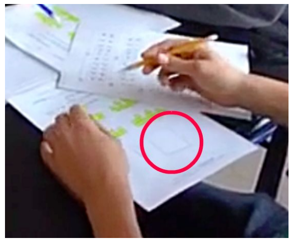
The semiotic breakdown was located by first analyzing each group member's initial perception of the diagram. Then, I located the moment when participants noticed the impasses. For some groups, this happened multiple times before a semiotic breakdown occurred and participants attended to the diagram. I viewed the video data of the semiotic breakdowns many times individually and also with others, specifically in the EDRL laboratory. Key word choices and gestures were coded as symbolic of the participants' perceptual construction of the diagram as a projection (e.g., background, silhouette, metaphorically, etc.). Particular attention was paid to group dynamics when the breakdown transpired, as the nature of collaboration in each session differed significantly. These differences are discussed further in the discussion and implications section of this paper.

Case Study 1: *This is the Background That You See*

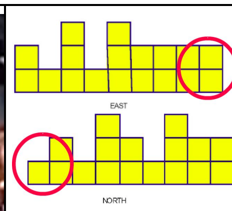
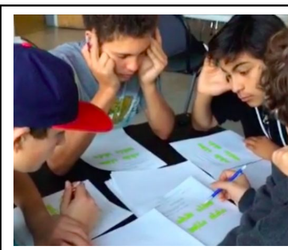
The first case study involves Alex, Taylor, Edmund, and Sean, who at this point in the data have been working on the problem for a few minutes. From the outset, the participants were actively discussing their perceptual constructions of the diagram, particularly surrounding whether or not the views "fit together." Prior to the following exchange, both Alex and Taylor relate the diagram to the game of Tetris and speculate that the problem will involve the type of "fitting" in Tetris. This conversation is elaborated on as participants take a closer look at the North and East views on the diagram.

			
Sean: OK but, I-I, these won't go together, because this slot has no spot to go into [points with pencil between North and East.] **Taylor**: Yeah it doesn't.	**Alex**: Well, but how. Does one of these fit into one of that? [points from East to North view] **Sean**: I think this one would end up going here. **Alex**: Does east fit into north? No, east doesn't- **Taylor**: No, it's not that. **Sean**: Oh! Then, so then it will end up like this, and then it will be. **Alex**: Here i'll try to draw it.	**Sean**: It will be like that [draws box around North wall] **Edmund:** No they're all the same building, they're just at like, different angles. **Taylor**: You don't understand. **Sean**: It will end up like that. **Taylor**: OK. North and South- **Alex:** OK. Do they have the same type of, same number of blocks on all of them?	**Taylor**: Shh shh. This is the talking stick.

This topic of "fitting" opens up a conversation about the connection between the views. Alex and Sean engage in a conversation across the table about how the North and East views might fit together. It's not possible to completely discern their perceptual construction of the diagram in these lines outside of the fact that the views must "fit" together in some way. However, it is clear that Edmund and Taylor have different ideas than Sean and Alex. With considerably less talking time, Edmund describes his perception of the diagram as a "building" that elicits 3-dimensionality: "No they're all the same building, they're just at different angles." This comment goes unacknowledged, despite its definite contrast with Alex and Sean's statements throughout the exchange. Meanwhile, Taylor disagrees but has not been given the space to speak. To ameliorate his position, Taylor turns his pencil into a talking stick to keep the group's rapid discussion organized, where he shares his own perceptual construction.

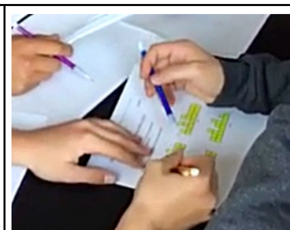
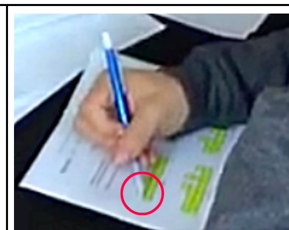
			
Alex: Alright, ok. Alright, What do you have to say, Taylor? **Taylor**: So, north and south aren't going to be touching. It, it's a square, it wouldn't work because *this*, this. You guys are saying like, north and south, you think they are going to be like touching and that's why they won't fit. **Alex**: Yeah	**Taylor**: It's gonna be like, it's a square [draws square on paper], so, uh, east and west are going to be like along the edges so there's going to be space in the middle. **Sean**: Oh, yeah you're right.	**Alex**: Oh! so the - only the edges will be touching. There's there gonna be like a- **Taylor**: Yes! **Alex**: Oh, okay! **Sean**: Well, then how are **Taylor**: I'm, I don't. Yeah I think -	**Edmund:** Wait but how is that, like? **Taylor**: That's why I think we're approaching this the wrong way. I think we're approaching, um. Uh.

Now having the attention of the group, Taylor is able to articulate that he believes the diagram represents a square so that "East and West are going to be like along the edges so there's going to be space in the middle." His development of the talking stick (which turns into a talking paper) has successfully garnered the attention of the group. As will be discovered in future turns, Taylor's perceptual construction of the square does not necessarily mean that he perceives the construction to be 3-dimensional. However, Alex and Sean's agreement with the square ("Oh, yeah you're right"; "Oh, okay!") creates a space where the impasses can be realized, as now all 4 views on the diagram must "fit" in a particular way to form a square where the edges are "touching." Note, the square grounds both a 2-dimensional and 3-dimensional perception of the views.

Sean: So I don't what exactly know what reconstruct means, **Edmund**: I don't know, yeah **Taylor:** Yeah **Sean:** So, oh may- maybe like give the south **Alex**: I think they do actually fit into each other **Sean**: Rotate, yeah, they fit sort of but they **Alex**: They, that one, that one, **Taylor**: Yeah, but, if we're **Alex**: Oh, wait, wait, wait. **Taylor**: It's looking-	**Alex**: Does that sh- **Sean:** So It makes, it makes- **Alex**: Does this sh*t [pencil points to circled area of East view] fit into there [pencil points to circled area of North view] ? **Sean**: Yeah that will fit into that **Alex**: Yeah so we just have to flip- **Sean**: And then East will have two left over.	**Alex**: These two spaces will be left over but where, what happens with those spaces. **Sean**: Well then do these fit together? Um. [pencil points to area within East view] **Alex:** Well maybe Taylor's right, we should use the talking paper.

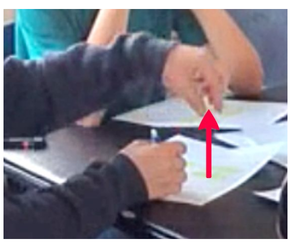
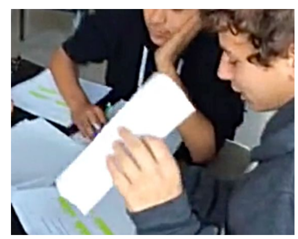
It is unclear in this clip whether the participants are imagining a 3-dimensional construction or a pattern tile where the views fit together in a 2-dimensional drawing. Regardless, the Northeast corner of the construction is highlighted by Alex, who decides that the right side of the East view "fits into" the left side of the North view. In a back and forth, Alex and Sean discover that there will be "two left over" (the "two" is what is indiscernible here from the dialogue - what "two" do they see?) However Sean and Alex perceive the diagram, whether as a 3-dimensional construction or 2-dimensional pattern, the "two spaces left over" create an impasse that leads to a breakdown in their perceptual structure of the diagram: "what happens to those spaces?". While muted due to the lack of material resources (i.e., blocks) to make it more public, this breakdown in Sean and Alex's perceptual structure of the diagram engenders uncertainty that halts the "fitting into" perception and draws the participants' attention to the diagram in a different way. As the breakdown in the diagram transpires, participants must reconfigure a way in which the views "fit" without there being "left over" squares. To address the elicited confusion, Alex attempts to bring Edmund and Taylor, who have been actively listening to Alex and Sean's back and forth, into the conversation by advocating for the use of the talking paper. As the conversation slows down momentarily, the participants turn their attention to the 4-views diagram to search for alternate perceptions that address the semiotic breakdown.

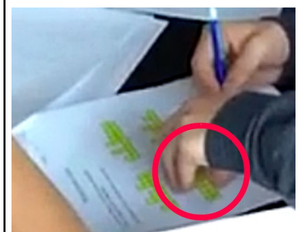
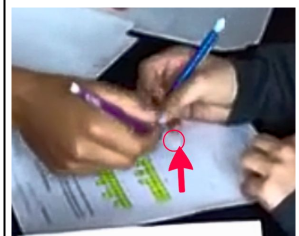
			
Sean: I feel like there's gotta be something else on here [flips paper over to back side] **Alex**: Are we missing something? [flips paper over to back side] So, what does, what does it mean by reconstruct? Do we fit them into each other? {Alex is questioning their initial perceptual structure in which the views must somehow fit with one another.}	**Taylor**: Edmund here, take- [Edmund has started drawing something on his paper and counting the number of squares on the diagram.] **Sean**: The name is 4-Views **Edmund**: I'm not saying anything! **Sean**: So that's, there are 4-views. **Alex**: There are 4 views of this so if this - **Sean**: Paradigmatic Didactical Mathematical Problematic [reading the words at the top of the 4-Views paper] **Taylor**: No no, can we imagine it to be three-dimension-? I think. **Alex**: That's some big words right there. It's a paradactal. **Taylor**: I think. I think are we trying to reconstruct it in three dimensions? **Sean**: Pterodactyl. **Taylor**: Are we trying to reconstruct it in three dimensions? **Sean**: Oh, that's an idea **Alex**: Oh, Taylor. **Sean**: So then you'd have to make a 3-dimensional shape? **Taylor**: I'm gonna supervise. **Sean**: Yeah, yeah. **Alex**: Supervise. **Sean**: Yeah, I assume that each of these would be one unit cubed. [gestures to squares on diagram] **Taylor**: Yes. Now, that, that would make sense. But, hold on nooo. I guess **Alex**: Dor Abrahamson **Sean**: I mean the height **Taylor**: Wait. so then it'd just be like, it'd, there'd be another side. [begins to draw lines coming out of North view to draw a 3-dimensional shape] It's hard to like. It would work if you did it from your angle. **Sean**: Yeah, it's hard to draw it. So like this.		**Sean**: Like that **Alex**: Oh that, yeah okay. So we're trying to reconstruct it like that with the things we have right here? **Sean**: I'm not entirely sure that what we're supposed to be doing **Taylor**: Yeah I'm not entirely..

Sfard (2002) writes that "seeing what is regarded as relevant for a given problem requires learning" (p. 320). In this collaborative exchange, the participants return to the 4-Views problem in search of something new to resolve the breakdown in the “fitting into” perception. This is noticeable as Alex questions the assumption that the views must fit: “Do we fit them into each other?” Just as many students in school are encouraged to return to the directions when they are confused, the uncertainty caused by the breakdown motivates the participants to revisit the 4-Views paper as if a new perception of the diagram is right under their noses. As the semiotic breakdown occurs, the participants find other elements of the 4-Views problem that they initially assumed were irrelevant, yet now warrant revisiting.

One of these elements is the paper itself, as Sean and Alex both flip over the 4-Views paper and declare that “there’s gotta be something else on here” and ask if the group is “missing something”. Alex then turns to the word “reconstruct,” which as previously discussed, conjures multiple perceptions with its ambiguity. Interrogating the word for its meaning would indeed provide a clue as to how the reconstruction can be achieved. Yet, as intentionally designed, to “reconstruct” the construction (or construct it, as there was never a real original construction), the participants must “reconstruct” their perception of the diagram. Thus, the meaning of the word “reconstruct” is only discovered through enacting their perception of the diagram. Sean continues his search into the elements of the paper that were initially assumed irrelevant by reading the words on the page that have not been read. He reads the words at the top of the paper, which was the title of the class where I originally encountered this problem: “Paradigmatic Didactical Mathematical Problematic-” Alex echoes this strategy later by reading “Dor Abrahamson” at the bottom of the page, who is a reader for this paper!

Taylor, who tries to involve Edmund in the conversation, perhaps to explain what he has been drawing on the paper, looks at the diagram for a few seconds and then asks a pivoting question that needs repeating to gain the attention of Alex and Sean: “Can we imagine it to be three-dimension[al]?” Alex and Sean’s validation of this idea prompts Taylor and Sean to try and draw a 3-dimensional structure out of the North view on Sean’s paper. As drawing a 3-dimensional structure on a 2-dimensional page proves difficult (particularly upside-down for Taylor drawing on Sean’s paper), Alex explains his continued uncertainty: “So we're trying to reconstruct it like that with the things we have right here?” Sean and Taylor respond that they, too, are not “entirely sure.” After a short discussion about the plausibility of the construction being 3-dimensions, Sean realizes a way in which it can be done. In this next episode, Sean shares his new perceptual construction of the diagram in which the diagram represents a “background.”

			
Sean: Oh wait. Oh, I got a. I think um, this is a three-dimensional thing so it's flat. **Alex**: Oh wait! **Sean**: So this is a flat area [draws square] and then, there's a bunch of, each of those is a raised a bit. [gestures up and down with hand] **Taylor**: That's not three dimensional. **Alex**: These things. Do all of these- **Taylor**: That's not very three-dimensional. **Alex**: -have the same number of blocks? **Sean**: Yeah, no **Alex**: North, west	**Alex**: Well look at this one. **Sean**: Well basically that goes there [gestures between East and North Views], that goes there, there's one missing. [gestures to left side of North view] {Note: the “two” missing has now turned into “one” missing}	**Taylor**: But yeah, so it won't work- **Sean**: No no no wait. No no no. **Taylor**: -because those won't match up. [gestures between North and South] **Alex**: Yeah that's true. **Taylor**: So that's what I did.	**Sean**: No, no. actually I think, I think. No. Can I have this? [grabs talking paper] Yeah, I think I know what this is.

Sean: No north has, uh. It- **Alex**: Do north, do these? **Sean**: This is 17, 18, 18, 17. **Alex**: Are you sure? Wait. **Sean:** Yeah I already counted. **Alex**: Well these. **Taylor**: No yeah we have.			
			
Sean: So you, so you **Taylor**: Hey, hey [snaps at Alex to get his attention] **Sean**: Alex. I, I think. I think this is a square area [draws square] **Alex**: Like what Taylor said? **Sean**: Uh, sort of, yeah it's three-dimensional actually. **Alex**: 3-dimensional? How? **Taylor**: That's what I said. **Sean**: So, so then. Yes, like Taylor said. Taylor was right. **Alex**: Good job, Taylor. **Sean**: Ok so then we- **Edmund**: For once. **Taylor**: Whoah! **Alex:** Roasted **Sean**: Then we've got a grid, and then on each grid space- **Taylor**: Posting this as cyber bullying.	**Sean**: -is raised up by a little bit.	So this [gestures to diagram] is the *background* that you see from each of these views. So this is North. So when you look at it. **Alex**: Explain it in plain english please. **Edmund**: Yeah **Taylor**: No that should be West. **Sean**: OK fine. No that's North. I'm just saying this is the North side. **Alex**: Oh that's the North side. Oh! **Sean**: North, East, South, West, so then like let's say.	**Alex**: I see what you're saying. That's like raising up like that **Sean**: Yes so that it's. **Taylor**: Yes! No but it is not 3-dimensional, Sean! **Sean**: No I'm not, we're not trying to put these together. **Edmund**: Why? **Sean**: So we can see it from when you. So like let's say there's some **Taylor**: OK yeah. You're right **Sean**: let's say there's 3 here, and there's one here **Taylor**: But that's not 3- **Alex**: 3 blocks are going up, 3 blocks are going up. **Sean**: 3 blocks are going up. 1 block going up. and then let's say there's two here. **Alex**: How are we supposed to write that?

Throughout this large chunk of dialogue, Sean attempts to explain not only how the construction can be 3-dimensional but his perceptual construction of the diagram in which the views are projections. Sean draws a square that represents a grid where “each grid space is raised up by a little bit,” which builds off of Taylor’s original perceptual construction of the diagram as “fitting” to make a square. Sean elaborates: “so this [gestures to diagram] is the *background* that you see from each of these views...we're not trying to put these [gestures to diagram] together.”

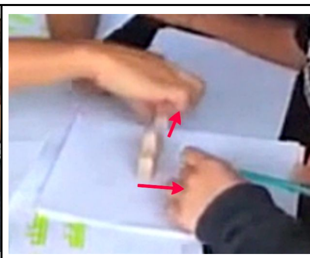
Sean's use of the word "background" to describe the diagram is key; he understands that the diagram represents a projection of the 3-dimensional construction. However, describing this new perceptual construction is difficult without the support of blocks which his peers must imagine. His peer's confusion is explicit: "Explain it in plain English please." Using the example of the North view, Sean is able to successfully coerce Alex into this perceptual construction, coupled with hesitation: "How are we supposed to write that?" Taylor's response is that Sean's perception of the diagram is "not 3-dimensional!", and Edmund, who is listening to the conversation, does not share his assessment of Sean's perception. At this point, I decided to give the participants the blocks and grid paper to work with. What results are Sean's continued attempts to explain his perceptual construction with the support of the blocks.

			
Alex: North would go up here. **Sean:** So then, **Alex**: Like that **Sean**: There has to, so basically the highest one here has to be two. **Alex**: 2 up like that? **Sean**: Because, Alex, remember we're not actuall - it might not be this pattern.	This is just the silhouette. [drags hand from left to right across blocks] Remember? **Alex**: Oh, yeah.	**Sean**: So it might be like two here, and there's one here. [rearranges blocks] What then we would still see two. **Alex**: So how are we supposed to figure it out? **Sean**: Well, I think it's possible. Otherwise we wouldn't have gotten this problem, but. So this is, one of these has to be two and none of them can be greater than two. So.	**Sean**: So what we can do is we have two, and then we can just slide it along here.

Sean emphasizes again the projection of the diagram as he says "It might not be this pattern. This is just the silhouette." With all eyes on the table, Sean's movement of his hand across the construction animates the projection of the diagram. In so doing, Sean's gesture constitutes a *phantasm* (Nemirovsky, Kelton, and Rhodehamel, 2012), as his gesture becomes a resource for other participants to examine through their collective imagination. Although Alex acknowledges the silhouette by uttering “oh yeah,” Sean continues to elaborate the gesture by rearranging the blocks and again moving his hand across the construction space from Alex’s point of view. In moving the blocks and and then reiterating the previous gesture, Sean concretizes the phantasm in the form of the durable material resource at hand. With both his gestures and the blocks on the shared grid at the center of the table, Sean has attempted to transform his peers’ perceptual structure of the diagram. Taylor, however, had a different perceptual construction of the diagram that must be voiced, which he elaborates on in the following exchange. Noticeably, this alternative way of seeing is encouraged by Alex and Edmund, who has remained on the periphery of the conversation thus far.

			
Taylor: Oh I was thinking a different. Nevermind, nevermind. **Alex**: Wait what were you thinking, Taylor? **Taylor**: So I was thinking it's like. **Alex**: Taylor, here you go. **Taylor**: I wasn't thinking 3d I was thinking like **Alex**: 2d	**Taylor**: I was gonna use this one. I was just gonna like, I think we're seeing it from like the view that we should be seeing at it, so like this one. **Alex**: So it's on the ground? **Taylor**: So it's like, like yeah. **Alex**: Where it's a bird’s eye view.	**Taylor**: Hold on, 1,2, this is right. **Edmund**: Kind of like a side angle. **Taylor**: So this is East. This is east you see, like, I don't think.	I think we're not doing it vertically. I think. What's the opposite of vertical again? **Alex**: Horrri- **Edmund:** Horizon **Taylor**: Horizontally, alright. **Alex**: You know like the horizon? **Sean**: Let's just start this way and if it doesn't look like it has a solution then we can do something else.

Taylor enacts his perceptual construction in the form of a flat view. This enactment of his perception hints that previously when Taylor imagined the views as forming a square, he did indeed see it as a 2-dimensional pattern in which the views fit in a flat jigsaw. Alex and Edmund both engage with this perception of the diagram and encourage it by elaborating: “it’s a bird’s eye view” and “kind of like a side angle.” This alternate enactment exposes a central ambiguity in which multiple perceptions are plausibly valid, but only one can be enacted on collectively. In an effort to validate Taylor’s idea (which is at least in theory supported by Alex and Edmund) while moving forward with his own perception, Sean uses his steering power in the collaborative problem solving to “start this [his] way and if it doesn’t look like it has a solution we can do something else.” No one questions Sean’s authority to determine the plan of the group or his assertion that there must be only “one way” to perceive the problem. Rather, Alex, Taylor, and Edmund reify their uncertainty in the plan (e.g., “I don’t get how we’re supposed to get a solution…” “Magic!”), before engaging with Sean as he continues to build the construction he envisions.

			
Alex: I don't get how we're supposed to get a solution for this but. [Sean continues building the North Wall] **Edmund**: I don't either. **Alex**: That's fine. **Taylor**: Magic! **Edmund**: This is just confusing on my brain. **Alex**: Just, **Taylor**: So wait, this is reconstructing it? **Alex**: Wait what are you drawing, West?	**Alex**: North and this would be gone. [Lifts up block off of left side of wall] **Sean**: This would slide [moves block stack from North wall, pauses to assess what Alex has done] So remember this is. **Alex**: Which side are we looking from? **Taylor**: No, Alex! Wrong view. **Sean**: Here. So then. **Alex**: Ok I'm looking. **Taylor**: Look at it from his perspective.	**Sean**: Here, so. **Alex**: That's what I'm trying to do. **Sean**: Like this is the north. **Alex**: Oh yeah you're right. So that's gone. Then we need to push this over there. **Taylor**: Oh I thought you were looking at it like this. [turns Sean’s paper over in his hands] **Alex**: Like that.	**Edmund**: Well, then, then we can each work from like a different angle. He, he can look from the west- **Taylor**: Or I can supervise. **Alex**: Yeah. **Edmund**: Why don't you do something Taylor? **Alex**: Like that. **Taylor**: Like supervise?

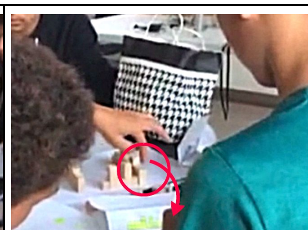
Sean: Cus I'm, I'm starting with North so then this-			

Sean juxtaposes the construction of the North wall with the paper, holding it up so Alex can see the construction from his perspective. Alex participates in the construction by lifting a block off of the North wall. While Taylor and Edmund watch, Taylor clarifies the way in which the walls are being formed in terms of which way the 4-Views paper that Sean is holding is facing. Edmund, understanding that each view must be constructed and that there are 4 views, suggests that each person can "work from like a different angle." This idea is not taken up by the group as Alex and Sean continue to work on the North wall and Taylor decides to "supervise." This confidence in supervising the construction is perhaps a camouflage for uncertainty, in that "supervising" permits the ability to question the actions of the group. This is precisely what follows.

			
Taylor: It's opposite **Sean**: What do you mean it's opposite? **Taylor**: Look.	**Sean**: So if you look at, Taylor, come over here. [Taylor slides chair over to Sean to look at diagram from Sean's angle] So like see it, so South it, it has to line up with that, see? **Taylor**: This is North. **Sean**: Yeah, but we're looking at the South side now. **Taylor**: Yeah I know **Sean**: Here let's put this in the center. [moves grid to center] **Taylor**: Oh you're looking at it from different ways! First you were looking at North but from there but now	but now you're looking at it from South from there. [Sean and Taylor gesture with their hands] **Alex**: Oh! No no no yeah, we were looking at it from South from there. **Sean**: Yeah, because south from the south side.	**Taylor**: Right, Sean. OK. Alright. **Alex**: Do you understand it now, Taylor? **Taylor**: Yeah I see it now. **Edmund**: Yeah, okay. **Alex**: So just a little bit. **Taylor**: I'm just gonna say, when I'm right. **Sean**: When you're right, I'll give you some credit, okay? **Taylor**: Ok alright.

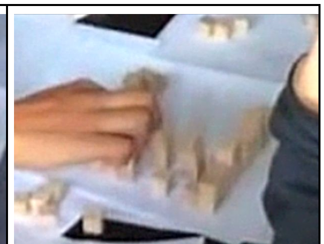
In this exchange, Sean invites Taylor to move his chair closer to “see it” - the connection between the diagram and the construction in the form of the constructed walls. Just as Alex had to “see” from his perspective on the North side, Taylor had to see from the South side in order to understand the Sean’s strategy for constructing the views. Thus, while Sean’s initial explanations of projection used verbal resources (background, silhouette), it was only through the embodied perspectives that Taylor and Alex understood how Sean’s perceptual construction of the diagram is a projection of the 3-dimensional space. However, while Sean and Taylor have successfully tapped into the strategy (i.e., build the views up as walls), it is unclear if they understand the depth of the construction.

It is also worth noting the power Sean has over the group, as he “jokes” to Taylor: “When you're right, I'll give you some credit, okay?” Whether conscious or not, this comment harkens back to Taylor’s perception of the diagram that was not enacted. Moving forward, Alex attempts to bring Edmund into the conversation by looking at the diagram from Edmund’s point of view (East), but needs further clarification on the strategy. This prompts him to shift from Edmund’s East side to Taylor’s West side of the construction, where he and Taylor collaboratively build the West side. Edmund watches.

			
Alex: What are you doing now are you doing west? What's happening? [whispers] **Sean**: Um **Edmund**: Why did everyone get so quiet?	**Alex**: 3, wait, which way are you looking? **Edmund**: Wait and how would it work for East? **Sean**: So then East is just the mirror image because.	**Sean**: Oh, and that's that thing. [gestures from West wall to West view on diagram] **Taylor**: Then you guys screwed- oh, no. Ok, alright.	**Sean**: Yeah. **Alex**: We understand, we understand. **Taylor**: Alright, Sean. Are you sure? No you actually, you screwed up.

Sean: These are all two. **Taylor**: You're talking by the way. [hands Sean talking paper] **Sean**: OK Thanks. Here, you can put that down. [puts talking paper down] OK so then, here, let's grab these things.	**Alex**: Wait West, I don't know how you're doing this. **Sean**: So then here, um, Alex. Alex come over here, and then see like that? [Alex moves over to look at folded up Diagram that Sean and Taylor are looking at on West side] So that's what we're trying- **Taylor**: Is that east? **Alex**: No that's a two **Sean**: So this is West. This is what we're trying to get it to look like. **Taylor**: Oh.		**Alex:** No, coz- **Taylor:** Oh no, alright. **Alex:** It's good, it's good.

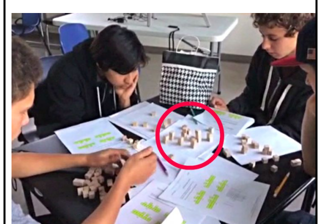
As Taylor and Alex work to reconstruct the West wall, Sean, who has finished constructing the North and South walls, begins to enact his perceptual construction by suggesting that the tallest blocks must be in the "same places." This comment creates confusion voiced by Edmund and Alex. Taylor, who initially argues that part of the structure is "reversed" incorrectly, supports Sean as Sean illustrates what he means by moving the blocks that have a height of 3 to the center of the construction.

			
Sean: The tallest ones have to be in the same places. **Edmund**: I'm still really confused. **Sean**: OK. Oh, I see. **Taylor**: Nope, no you. Sean, you reversed it. **Sean**: OK **Taylor**: No because that's not gonna work. OK so. **Sean**: This isn't gonna work? **Taylor**: These need to be opposite. **Sean**: Yeah, these are opposite.	But we're gonna have to slide these around [gestures across construction], because right now these- **Alex**: How are we gonna slide them around? **Sean**: We can, just move them. **Taylor**: Ready? **Alex**: But what is this supposed to be? **Sean**: So then when you look at it **Edmund**: The building!	**Taylor**: No, you're right, you're right. Sean, it's okay you got it [Sean moves blocks with height of three towards the center of the construction]. **Sean**: Yeah **Alex**: Because it's the silhouette, yeah. **Sean**: Yeah it's the silhouette. **Alex**: Got it, Oh I know what we're doing now! **Sean**: Yeah **Alex**: We're matching it up. **Sean**: So then	**Alex**: So like this has to stay here. Does this have to go over. No, go back. [begins moving blocks] **Sean**: Ok, then we're gonna need [moves hand to East wall] **Alex**: And, that will go there **Edmund**: Or left. **Alex**: Well, where did these go? **Sean**: So if the- **Taylor**: I'm not entirely sure that this is what we're supposed to be doing. But.

	Sean: So, okay. So when you look- **Alex**: Are we trying to make a building? **Sean**: Yeah it's like- **Edmund**: I don't even know at this point.		

In the episode, there is a wealth of uncertainty expressed: "I'm still really confused," "How are we gonna slide them around?" "But what is this supposed to be?" "Are we trying to make a building?" "I don't even know at this point" "Well, where did these go?" "I'm not entirely sure that this is what we're supposed to be doing." These expressions, heard from Alex, Edmund, and Taylor throughout the sequence, illustrate that despite the group's proximity to a valid solution, it is only Sean who has a strong grounding in the perceptual construction of the diagram as a projection. While Alex engages with the terminology introduced previously, "Because it's the silhouette, yeah," Sean dominates the physical construction by moving blocks towards the center and shifting the attention to the East wall.

			
Sean: No. So maybe this goes here, and then, [begins to construct East wall] **Alex**: They were looking at all the wrong side. **Sean**: Yeah **Taylor**: Yeah **Sean**: Here, and then um. **Alex**: What do they mean by reconstruct? **Sean**: Then look at the east side. **Edmund**: I don't know! **Alex**: Here are four views.	**Sean**: No, no, no. so, so we're supposed to create the grid. [waves hand over construction] **Alex**: of the same construction. **Sean**: So, okay. so when we're done, each of these is gonna have 1,2,or 3 blocks and when you- **Alex**: But it's gonna stay the same, you're gonna see the same thing when you're looking like that [gestures back and forth over construction] **Edmund**: You're gonna see the same thing	It's gonna look like North [gestures hand in direction of North] or it's gonna look like West- **Sean**: Yeah so when you look at- **Alex**: -or it's like east depending on which side you're looking at. **Taylor**: I see. I see Sean: Yeah so when you. Yeah, there. **Alex**: I got, I see. That was kind of funny. {referencing the "see" pun}	**Alex**: Perfect. Are we just reconstructing- **Edmund**: It doesn't have to be perfect or anything **Sean**: Yeah we're just reconstructing. **Alex**: And then we'll move it around once we're done. **Sean**: Yeah. Oh that's 3, then 1, there. [continues to construct East wall: notice that the blocks are already pushed in to account for the Northeast and Southeast corner impasses]

	Alex: Yeah **Edmund**: but they're all gonna be in a different place **Alex**: It's gonna-	**Sean**: So 1,2,3,4. S	**Alex**: Um and then we need that to be. We need this to be 3 as well.
			
Sean: OK. So then we have to slide this here. **Alex**: No. We need to slide that over there. **Edmund**: Yeah we need one more. **Sean**: Oh right right. I'm sorry. **Alex**: So you can see. And this one goes the same place. **Taylor**: Yeah. **Alex**: Yeah.	**Sean**: yeah I think we get rid of these. **Taylor**: But **Sean**: We can just. Because. **Taylor**: But, So is this like something where you can do each? Could have we? I'm not saying we should. **Alex**: No because if we look by the south then those won't be there. We need to keep those there. **Taylor**: I'm, I'm. **Sean**: No because when you look at the South you see those ones instead. **Alex**: Oh we're gonna see those. Yeah, yeah, yeah. **Edmund**: Yeah.	**Sean**: And then we can get rid of these, too. **Taylor**: Yeah. **Alex**: So it's gonna end up showing one line. Oh- **Sean**: Because this **Taylor**: No **Sean**: doesn't show depth.	**Taylor**: No but then, why are these relevant then? [gestures to two of the 3-block stacks in the middle]  Success!

Doubt and uncertainty continue to surface until the participants arrive at a solution. Alex's question, "What do they mean by reconstruct?" which has been repeatedly asked throughout the entire problem solving process is continually left unsatisfied as he seeks for a deeper answer. Sean, who is trying his best to share his perception with his peers by explaining it to them, struggles to do so in a way that sticks. While Alex's dialogue and actions show that he might share Sean's perceptual construction of the diagram as a projection at the end, Edmund and Taylor do not touch the construction at all as it reaches its final stages. Because Sean already sees that the views are projections, impasses are preemptively navigated without confrontation.

For example, as Sean constructs the entire East wall for the first time, he *begins* constructing the wall as a projection from the East as it is already shifted along the grid towards the center; this move avoids any impasses along the corners.

As Alex supports the movement of the 3-block stacks towards the center, Sean swiftly removes all other blocks at a height of 3 from the North and South walls to avoid the other impasse. When Alex protests, “No because if we look by the south then those won't be there. We need to keep those there,” Sean provides a quick justification “No because when you look at the South you see those ones instead.” The fact that this exchange happens so close to the end of the problem suggests that while Alex is engaging with projection, he is operating within a crucial limbo state in which he is in between two perceptions of the diagram: they are not walls, but we cannot change the walls.

Taylor’s final question, “No but then, why are these relevant then? [gestures to two of the 3-block stacks in the middle]” poses an interesting turning point in which the participants begin to assess which blocks are “relevant” to the problem. Indeed, the two block stacks that Taylor gestures to are “not relevant” in that they are not necessary to maintain the projected views. Rather than acknowledging that they have satisfied the views and “reconstructed” a viable construction for the problem, this question of relevancy is explored for another 15 minutes as participants eliminate blocks that are “irrelevant.”. While ultimately I believe all participants in this case study arrived at a perceptual construction of the diagram as a projection, it is unclear at which point they did. What is certain is that the initial breakdown in the diagram was one of an individual nature rather than collaborative. This led to Sean’s perceptual structure taking the lead and impasses being avoided.

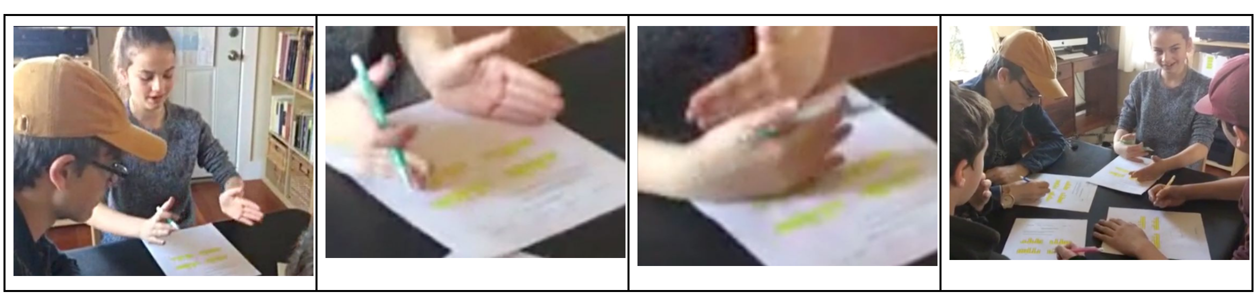
In this first case study, participants centered their perceptual constructions of the diagram before they receive the blocks around the question of how the views "fit" together. When Taylor suggests that the views will form a square with "space in the middle," Alex and Sean notice the corner impasse as they are unable to figure out how the views fit using Taylor's square perceptual structure: "These two spaces will be left over but where, what happens with those spaces?" This realization initiates the semiotic breakdown, in which a perceptual construction where the views "fit" together can no longer hold. In a collaborative moment of uncertainty, the participants coordinate their attention to the diagram as they investigate features that were initially deemed unimportant to the problem, such as extraneous text at the top and bottom of the 4-Views paper. Suddenly, Sean is able to see that the diagram represents a "background," and thus attempts to communicate his new perceptual structure in a series of phantasms and movements of the blocks. As evident through his peers' permeating uncertainty, Sean was only able to in part successfully communicate his new perceptual structure as he built the physical construction. When the construction is finished, we are unsure that all of the participants share the understanding that the views are projections.

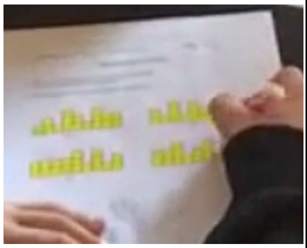
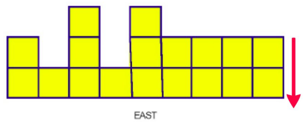
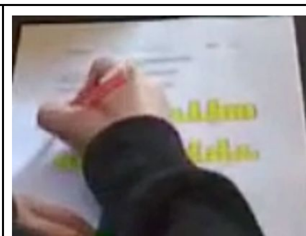
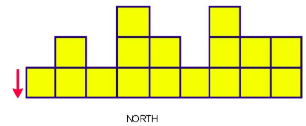
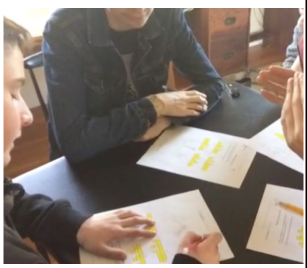
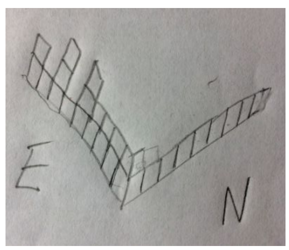
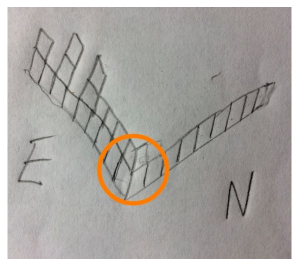
To conclude, this first case study illuminates the impact of one participant experiencing a breakdown in the diagram before material resources are available to support other participants in realizing the breakdown fully. That is, without the material resources at hand, the impasses can only be experienced in a restricted way that affects the participants' ability to experiment with their perceptual constructions. Despite attempts to explain his perceptual construction to others using gestures as well as verbal and material resources, Sean's peers were not able to see the construction as he did; the phantasm never became co-constructed.

This may have occurred due to the participants' lost opportunity to experience the breakdown in the diagram through manipulation of the blocks. For example, had Taylor's perceptual construction of the diagram from "birds eye view" been accepted by Sean, other group members who supported Taylor's perception may have arrived at the impasses through the material resources provided. The valuable moment in which multiple participants share a perceptual structure but arrive at a contradiction was lost, as by the time the construction is built, Sean had already committed to avoiding these contradictions.

Case Study 2: *Does the Inside Have to Be Filled In?*

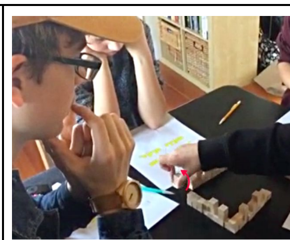
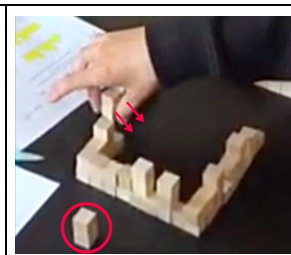
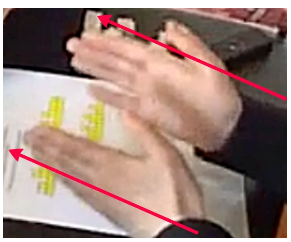
In the second case study, I turn to Sofia, Mason, Noah, and Will, who have been working on the problem for just a few minutes. After asking me almost immediately if the construction is supposed to be a 3-dimensional shape, to which I responded that they should sort it out themselves and I might jump in later, the group operated under the auspice of 3-dimensionality, localizing their attention on the word "reconstruct." Will, for example declared that if they need to "reconstruct" it, this could be done with "building blocks." This statement was made well before blocks were introduced to the group, and led to the following discussion as participants began to try and draw the construction on their papers.



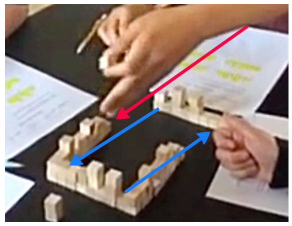
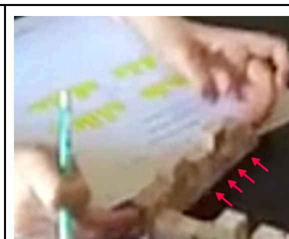
<table>
<tr><td colspan="3">Sofia: If you put all 4 views together, so that like, you’re looking at it from one direction so that these-
Noah: One Direction isn’t a good band.
Sofia: I’m not talking about the band!
Noah: I don’t care.</td><td>Sofia: But then if you like, took all 4 views and kind of arranged it so it would be a square
Noah: No
Sofia: Then like..
Mason: This wouldn’t work because if this is here, like
Noah: They are reversed.</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="2">Mason: this line</td><td colspan="2">and this line should be the same, so this square should actually go in there {The gesture for“there” is not discernible from the video data}</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="3">Noah: This is north and east!
Mason: Yeah. North, like pretend this is like up
Noah: Yeah?
Mason: and you make it to the right.
Noah: Yeah?
Mason: North, East.
Noah: And?
Mason: So this is East and North doesn’t have a square over here, so it would have to, it just starts there
Noah: So?
Mason: So if this square goes in here, it doesn’t work.</td><td>Will: No, no, no. The East, the east one, is like, it’s, it’s that way.
Sofia: I don’t think that they’re cub-
Mason: What, what makes it that way?
Sofia: I don’t think they’re cubes because of that [points to Northeast corner of Mason’s drawing with pencil]</td></tr>
</table>

Sofia begins the conversation by describing a relationship between the views, which is without a beat denied by Noah, who doesn’t seem interested in listening to Sofia. Yet, she continues. When she says “arranged it like a square” and gestures with her hands, she proposes a perceptual construction of the diagram in which the views are connected to form a cohesive shape. Mason disagrees, and moves to the diagram to cite evidence as to how this perceptual construction “wouldn’t work.” In so doing, Mason gestures to the impasse in the Northeast corner, referring to the different height of the squares as justification: “this line and this line should be the same, so this square should actually go in there.” Mason’s statement is rejected by Noah, who is unable to discern what Mason’s contribution is important (“So?”). Will also rejects Mason’s statement, “No, no, no,” offering his own perception. However, Sofia validates Mason’s critique of her perceptual construction of a the diagram as representative of a square, gesturing to Mason’s drawing and saying “I don’t think they’re cubes because of that.” Thus, Sofia is able to temporarily accommodate Mason’s critique by suggesting that the materials they will use to create the construction are not cubes. While Mason’s critique does not gain the approval of the collective group, his statement, which highlights the North and East sides of the diagram, represents the preliminary, pre-block breakdown in which Sofia’s perceptual construction of the views as forming a square does not work because of the impasse.

When blocks are introduced, the impasse reappears as participants engage in a conversation about the relationship between opposing views (North/South; East/West). At this point, the group has distributed the job of constructing the views, one per person. Mason has constructed the South wall and Will has constructed the West wall. Note, despite Mason’s previous critique of Sofia’s square idea, all of the participants begin to combine their views to form the square shape, including Mason, describes the construction as “like a castle.” In this first exchange, we see the beginning of the conversation about the relationship between opposing sides. Mason and Will work together to combine the West and South walls before moving Sofia’s North wall in to make the square. Meanwhile, Noah continues to work on building his East wall. Note, the impasse appears only in corners shared with the East wall.

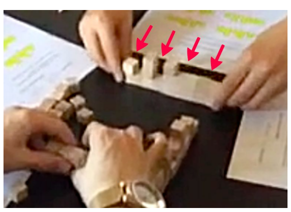
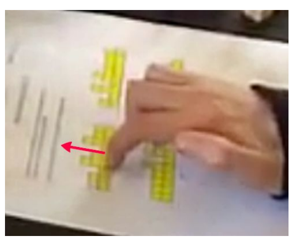
			
Mason: Ok so now I'm going to shave this off [removes end of his South wall that will be shared with West wall, so it doesn't become a 9 x 10.] because this and that **Will**: Yeah, yeah. There, okay.	**Mason**: It's like a castle. **Noah**: Wait guys, why are you taking them off? **Mason**: Because **Noah**: You're right each side should have 9	**Will**: Wait, wait, wait… **Sofia**: That wouldn't work. **Will**: Do we, are you - no. Do we, are you doing that right? [Mason begins to move Sofia's North wall in to combine with the West wall, by first removing her corner piece.]	**Noah**: They're flipped! Each side is flipped. **Sofia**: Yeah. North and south, and east and west. **Noah**: Shhh {Note, the circled corner piece that was removed from Sofia's North wall.}
			
Mason: Coz we're looking at it from the side that it says. I think.	So like if you look at this [uses Noah's East wall as an example.]	from this side	it looks opposite from this side. **Noah**: Yeah.

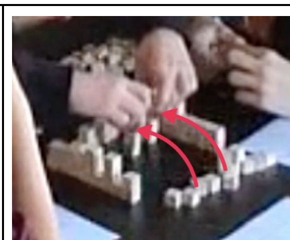
Mason and Will easily combine their views at the corner because both the West and South views share a height of 2 blocks in the corner. However, as soon as Mason begins to move Sofia's wall in, both Will and Sofia express hesitation, as if completing each other's sentences: "Wait, wait, wait," "That doesn't work," "...are you doing that right?" Noah chimes in to the uncertainty with his perceptual construction of the diagram, in which opposing views are "flipped." The word "flipped" in this context carries dual meaning. For Sofia, the flipping occurs as opposing sides are mirror images of one another. For Noah, the flipping means that the views are constructed as reverse images. This explains why Noah has taken longer to build his East wall, because he has been reversing the image on the page to construct the 3-dimensional wall (i.e., the yellow squares on the right represent the blocks on the left). Mason agrees and elaborates on Noah's statement. This prompts a collaborative discussion, as these different perceptions are incompatible.

			
Mason: So that [gestures to East wall] should be flipped. And then this one should be. [gestures to his own South wall] **Will**: No it's the other way around, it's the other way around.	**Noah**: Um, no. This is right. Because this [gestures to East wall] is opposite of that. [gestures to West wall] {Although small, it's easy to see in this diagram Noah's perception of the "flipped" nature of the diagram}	**Will**: No, no, no, no, no, no, no. No, see, look at, **Mason**: Does East? **Will**: look at how they did it. They did that right. [hovers hand back and forth between North and South]	**Mason**: No we did it wrong. **Will**: See **Noah**: No you guys did it wrong. **Will**: Oh, okay [Sofia moves her East wall away from the construction.] **Noah**: Yours is supposed to be flipped. Re-reverse opposites of the other side **Will**: Okay.
			
Mason: But, so if you flipped yours then you're gonna have one over here [gestures to the end height of 1 on the North wall]	and two over there [gestures to Northwest corner] which doesn't work,	because his originally had this [moves the corner piece back to the West wall], and you can't just take this one off. Because, that's, if we flip that, that's going to be one	**Sofia**: So and also if we were looking at it from this side and this was like this, kind of, then, you would see this block. [gestures with pencil sideways, and points to third block.] **Will**: Waaait. **Sofia**: I don't think it's gonna **Will**: I had an idea, but then like, nice. [Sofia destroys her wall.] Ok. Wait. Ahh. Ok.

Noah's version of "flipped" wins the favor of both Mason and then Will, who perhaps succumbs to the peer pressure of agreement. This perception of the diagram successfully relocates the impasse from the Northeast and Southeast corners to the Northwest and Southwest corners. Imagining the flipped South and North walls, Mason is able to see the impasse again, this time in the blocks: "If you flipped yours then you're gonna have one over here and two over there which doesn't work." Sofia acknowledges Mason by adding on another layer of uncertainty in the perceptual construction of the diagram as forming a 3-dimensional square, in which each view would see a height of three in the corners: "if we were looking at it from this side and this was like this, kind of, then, you would see this block." Both Mason and Sofia's statements, which refer to impasses within the current construction, induce a breakdown in the diagram, in which the views can no longer be constructed to form a square or share corners. Will, who previously did not accept the impasse when brought up by Mason, expresses his uncertainty, acknowledging the breakdown in the diagram: "Waaaaait. I had an idea, but then like...Ok. Wait. Ahhh. Ok."

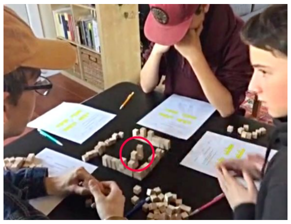
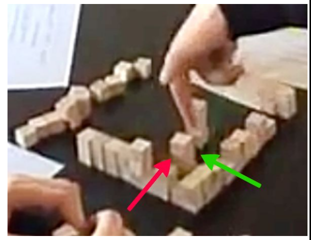
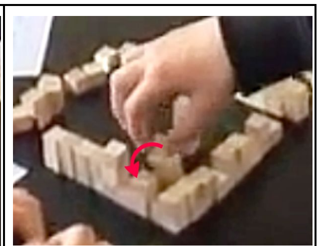
Noah, who has missed this crucial discussion, finishes his flipped East wall and pushes it forward to the group. Accepting the breakdown in the diagram, Mason, Sofia, and Will begin reconstructing their perceptual structure by throwing out new perceptions of the diagram that would avoid the impasses. While all three tackle the same problem, they do so in different ways.

			
Noah: So [pushes his East wall towards the center of the table.]	**Mason**: What if we only put like one side goes up there [gestures up with his finger over the height of 3], like, one side, each side- **Sofia**: Wait **Mason**: -has one on the bottom.	**Will**: It doesn't necessarily, is it a cube? It's not, like, it doesn't have to- **Sofia**: No, I think **Will**: -be a cube **Sofia**: Okay. Let me see, I think I have an idea [grabs Will's paper.] **Will**: Like a square, not a cube, but a square. Maybe it's not a square. **Sofia**: We're, here. Maybe it's- **Noah**: Maybe it's a tesseract.	**Will**: No, my point is like, like this one could be here [points with right hand to West wall, and slides left hand in the center, insinuating that the West wall could move towards the center of the construction.]

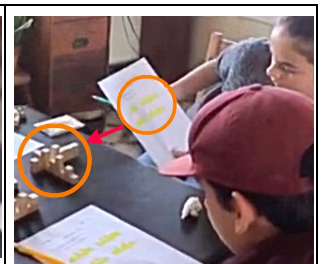
			
Mason: It can be, as long as this side, [points to North view on diagram]	like if you look at it from there [moves head sideways to see the construction from a ground level]	so we could literally have things in the middle... [moves blocks from Sofia's destroyed North wall.] **Will**: We could **Mason**: ...that go up 3,	as long as it's in line with this one. [removes blocks with height 3 from his South wall, gestures projection]

Will verbalizes the breakdown in the diagram explicitly: "it doesn't have to be a cube. Like a square, not a cube, but a square. Maybe it's not a square." He elaborates further by gesturing the movement of the west wall towards the center of the construction to make a T-shape. "This one could be here." While Noah engages with Will by offering, perhaps sarcastically, that the structure might be a tesseract, Sofia and Mason have other ideas. Mason goes back to the diagram and reconstructs it as a projection, "as long as this side, like if you look at it from there," moving his head sideways to physically "look at it." This realization of the views as projections leads Mason to state that blocks can be located in the middle "that go up by 3," resolving the impasse Sofia previously made public to the group. When Mason wraps up his idea with the constraint "as long as it's in line with this one," he articulates the projection of the diagram as a rule that could be applied to all sides.

Despite the fact that both Will and Mason provide valid solutions to the 4-Views problem that signal they have both reconstructed their perception of the diagram, Will's idea has been ignored, and Mason's idea is not accepted by Will or Noah. Sofia, who has left the table temporarily to get a tissue, misses both of their ideas. What ensues is a demand from Mason for justification as to why his idea does not work. Unsatisfied with the response, Will turns towards me for validation.

			
Noah: What? **Will**: Noooo **Mason**: Yeah! **Noah**: Wrong **Mason**: Why not? Why not? **Will**: That doesn't sound right.	**Mason**: Yeah, because if we look at this from this side [gestures to West view] then this one has three	and this one [gestures to South view] has three	instead of having one each [moves blocks to fill in where the walls previously had heights of 3] **Will**: Well, yeah. But like, my thought is like, here wait
			
Mason: Does the inside have to be filled in? [turns towards me] **Lizzy**: That's. I'm going to leave that question for you to ask maybe the people in your, in your team	**Will**: Yeah, because I mean, I feel like **Sofia**: Ok. I have another idea that maybe like, oh wait nevermind **Noah:** That could be for something **Will:** No I have an idea, I have an idea, wait one second. **Noah**: (?) that could be potentially irrelevant whether or not the f-, middle should be filled in.	**Noah:** Whatchya doin? **Will:** I have an idea. **Noah:** That's a first. **Will**: Aw, thanks.[laughs] Mason, can you make yours really quickly? [Mason makes the South wall] (...) What if, this doesn't have to necessarily be here? [moves South wall to touch corners with West wall] Couldn't it be like… **Sofia:** What if it's like an X-shape? **Will**: Couldn't it be here? And then I need to take this off. Wait no, actually not this one. But like, does that makes sense? Maybe it doesn't have to be a huge… **Noah**: Yeah, that actually makes a lot of sense. Wait...well Wills's X shape theory works out perfectly.	**Sofia**: That was..okay. **Noah:** What is it? [Sofia begins building her X-shape] {Noah and Mason engage in a short conversation about the plausibility of the idea, in which Mason doesn't think it will work.} **Will**: Well, cus it's just from the view. **Mason:** What if they don't even have to be touching? **Noah:** This is the same construction. **Mason**: Yeah, but. What if you just had blocks (?) somewhere.

Mason never receives a justification for why his perception is incorrect. When Mason turns to me for validation of his perception, I tell him to ask the people in his team, a somewhat hopeless suggestion considering Mason has already tried to do that. Will moves on to elaborate on his perceptual construction, which takes the form of a T-Shape. Enlisting Mason to support his idea by building the East wall, Will is able to successfully put the shape together: "Couldn't it be here?" He has the attention of Mason and Noah, but Sofia is visualizing something different: "What if it's like an X-shape?" As Sofia leaves the conversation, Will becomes uncertain in his perception, despite the validation that Noah provides him: "Yeah, that actually makes a lot of sense." Noah incorrectly gives credit to Will, instead of Sofia, for the "X shape theory," and Sofia begins to build out her theory, or perceptual construction, of the views. Mason tries one more time to get his peers' support, "What if they don't even have to be touching?" to no avail. As Sofia constructs her "X shape theory," she receives almost immediate validation from Noah.

			
Noah: I think Sofia is going somewhere. **Sofia:** Kind of, maybe not though. **Noah:** You're going somewhere. It's, yup...definitely. You've got it right. Guys, watch. You see what she's doing? **Mason**: Yeah, she's just crossing them. **Noah:** and that works perfectly because it can be several different views in the same problem. Because it's not like multiple, it just has to be the same.	**Mason**: The only thing that won't work about that is that if you put another one here it has to the same. **Will**: We don't have to put another one. **Mason:** Yeah there is another one that's exactly the same. **Will:** No there is no other one. It, it's- **Sofia**: Look, look, so if you- **Will:** There is no, **Sofia**: If you- **Noah:** Shhh. **Will:** there is no other one.	**Mason:** Oh yeah! Because you're looking at it from that, and that way. [moves hand back and forth over construction] **Sofia**: So if I'm looking at it from North right here, it looks like this. If you [Mason] look at it, you should see this.	**Sofia**: And then if you can see West, and you can see East. **Noah:** Yeah. **Mason**: Is that it?  Success!

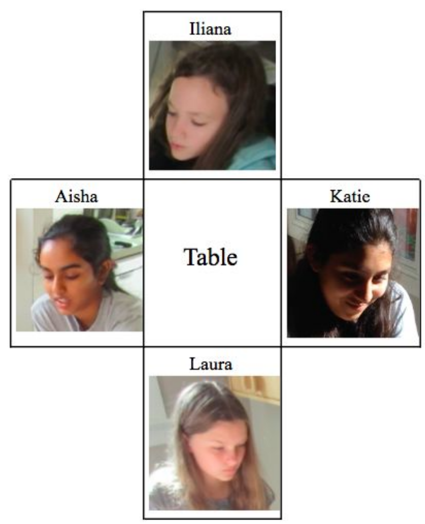
Noah's strong affirmation of Sofia propels the attention of the group towards Sofia's construction. Mason initially disagrees with the construction because two of the walls are missing. If placed, where would they go? Will has accepted Sofia's construction, and engages with Mason's questions about the "other ones [walls]." Mason eventually agrees that the

X-Shape works as a perceptual construction: “you’re looking at it from that, and that way.” Sofia jumps in, and justifies her construction by matching each student’s view that they had previously been assigned with how they would see the construction by leaning to their side. Turning to me, “is that it?” the participants collectively agree that the construction is finished.

Unlike the first case study, participants in Case Study 2 were able to collaboratively experience the designed impasse as they pushed their distributed walls together. When the impasses are vocalized first by Mason, and then by Sofia, a brief moment of uncertainty holds space, as Will announces that “maybe it’s not a square.” The semiotic breakdown is thus experienced simultaneously by Mason, Sofia, and Will (Noah is busy finishing his East wall), with each participant searching for alternative perceptions. Yet, despite the rich conversation in which the impasses are articulated so precisely, the collaboration of the group devolves to where the group’s coordination revolves around competing enactments of new perceptual structures, all of which address the necessary depth of the 3-dimensional construction. Mason proposes that blocks be moved to the inside which can be “filled in,” demonstrating with the shared construction at the center of the table. However, his idea is immediately shut down by Will and Noah, who holds abrasive authority over the group (“Wrong!”). Will’s following suggestion goes unacknowledged by any member of the group, and Sofia, who struggles to be heard as she is repeatedly gets cut off, resorts to silently building the construction she imagines to gain the respect and attention of her peers.

The group’s shift from a collaborative to more individual use of material resources leads them to a populist solution framework, in which only one construction can be correct and those that are incorrect do not require justification. This phenomenon is perhaps the result of a disequilibrium in the face of ambiguity; if *you* have a different enactment of the diagram than *me*, then we must have different perceptions of the diagram, in which case we are seeing things differently. In other words, there is no way that both of our constructions can be correct. The participants' rapid shift to enacting alternative perceptual structures with the material resources provides a distinct contrast to Case Study 1. Whereas in the first case study, one participant struggles to articulate his perceptual construction using any communicative resource possible (gestures, words, diagram, blocks), participants in Case Study 2 ground their enactments within the material resources. The blocks thus enable the participants to discuss concrete structures that can be deemed as correct or incorrect as opposed to searching for a collective and new way of seeing the diagram. Notably, the grid paper was not used in Case Study 2, which had it been the centerpiece of the table may have supported participants in coordinating their actions in a collaborative way.

Case Study 3: It's Like "Metaphorically"



For the third case study, I turn towards the collaborative group work of Aisha, Iliana, Laura, and Katie. We enter 23 minutes into the task. After receiving the blocks and distributing the views (one view per participant), the group built a 9 x 11 construction without the grid by squeezing the East and West walls in between the North and South walls (see Figure 2). At this point, participants engage in a short discussion about the flaws within the construction, one of which is that the East and West walls

Figure 2

have been constructed incorrectly and must be rebuilt. As Katie and Aisha reconstruct the East and West walls to match the diagram, Iliana, who sits on the North side of the table, notices something odd.

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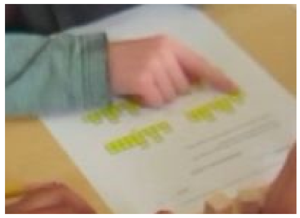
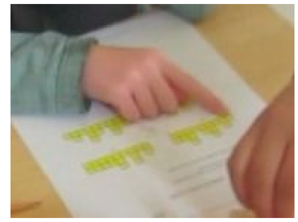
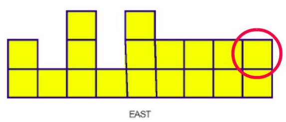
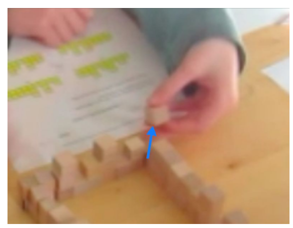
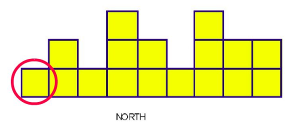
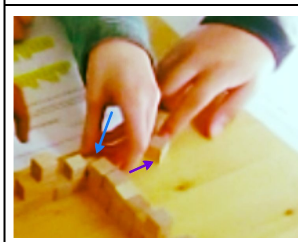
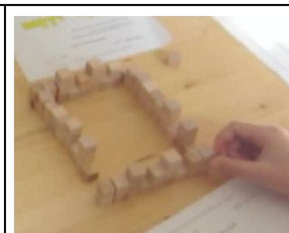
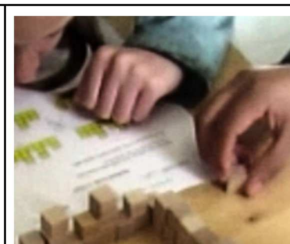
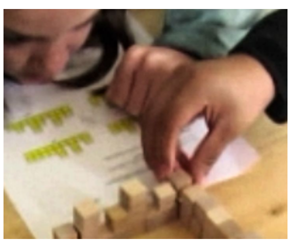
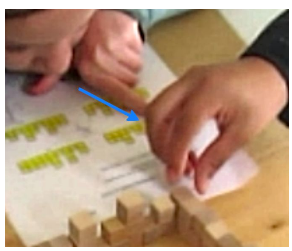
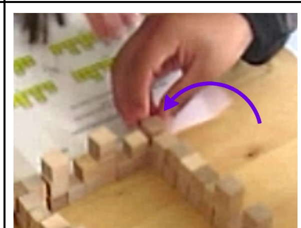
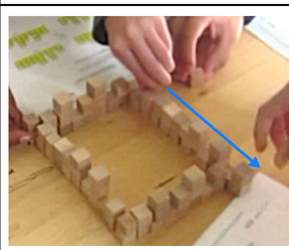
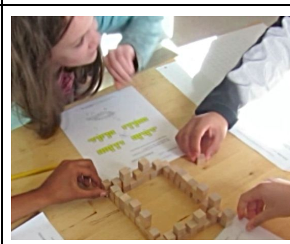
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>Iliana: That’s odd.
Lizzy: What’s odd? [off camera]</td><td>Iliana: So if you’re looking North,</td><td>you’d also see the corner of East</td><td>but there’s only one square in North. {Iliana has recognized the impasse, in which the North and East views seemingly contradict one another at the corner.}</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="2">[Katie, who is rebuilding the East Wall, places 2 blocks down on the right side of the East Wall, where Iliana has just referred to the oddity]</td><td colspan="2">[Iliana lifts the corner block that Katie has just put down, which matches Katie’s view with a height of two but contradicts Iliana’s North height of one.]</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>[Katie removes the block that remains]
Iliana: You need one square there [Iliana replaces her block, which she had dangling in her hand, back into the empty space.]</td><td>but it’s like you don’t see it. {Meanwhile, Aisha has been discussing the issue of the construction being a 9x11 instead of a 9x9, suggesting blocks be removed to “smush” the construction together. Her hand lingers on the North wall to gesture this smushing effect}</td><td>Aisha: Oh, huh. [in response to Iliana’s previous statement.]</td><td>Laura: What? {Laura touches the Southeast corner that currently has two blocks, which conflicts with her South view. This suggests that she realizes the discussion just held calls into question the corner her South view shares with the East view}</td></tr>
</table>

In this exchange, Iliana has recognized the impasse, in which the North and East views seemingly contradict one another. Her use of the word “odd” and following gestures towards the corner locations on the diagram signal the breakdown, in which the group's current enactment of the diagram results in the impasse. Note that the diagram itself cannot be objectively odd. The "that" Iliana refers to when she says "that's odd" is the participants' collective co-constructed perception of the diagram. When I prompt Iliana by asking "What's odd?", she describes the breakdown in the diagrammatic sign. In doing so, she connects the diagrammatic sign (i.e., the views) to her embodied distributed perspective from the North side, by simultaneously gesturing towards the diagram and describing the contradiction as something to be seen: "So if you're looking North, you'd also see the corner of East, but there's only one square in North."

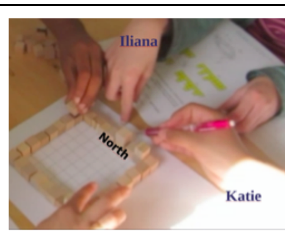
Coincidentally, Katie, who is rebuilding the East wall due to a previously realized mistake, places a second block in the exact corner where Iliana has noticed the impasse so as to satisfy her East view, which sees a height of 2 in that corner. This leads to a block tango, where Iliana removes the block that has just been placed by Katie, Katie then slides the leftover block towards herself (perhaps to support Aisha's unrelated suggestion to remove blocks and "smush" the East and West walls in between the North and South walls to resolve the issue of the construction currently being a 9 x 11) , and Iliana places the block she has just removed back down. Iliana justifies her placement of the block in the corner and elaborates on her previous statement: "You need one square there, but it's like you don't see it." Iliana hovers her fingers over the adjacent blocks in the East wall as she says "it," suggesting that what "you don't see" are the rest of the blocks in the East Wall.

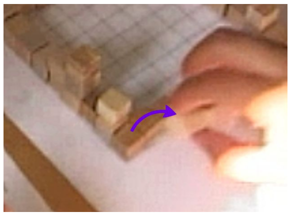
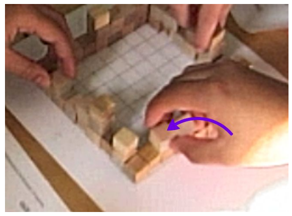
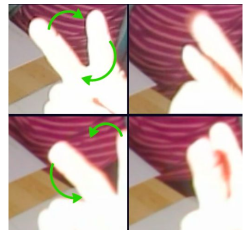
At this point, Iliana's statement is acknowledged by Aisha ("Oh, huh.") and Laura ("What?"), who gestures towards the Southeast corner, which as a mirror image of the Northeast corner shares the same impasse. Before the participants can get any further, however, I interject to ask the participants if they think the construction is finished. Each participant tacitly responds "no" by pointing to an issue with the current construction that is noticed from her perspective. The impasse between Katie and Iliana's views becomes transparent as they engage in a dispute about the Northeast corner in response to my question.

			
Aisha: Well kind of, for me it looks like this shouldn't be here. [grabs Northwest corner] {Aisha is addressing the issue that the construction is currently a 9x11}	**Iliana**: Except for that. [points towards Northeast corner] {The Northeast corner at this moment has a height of 1. Iliana's gesture of a persistent problem suggests that she is not just "viewing" the problem from her own side.}	**Laura:** Wait, I think this should, this was… [points towards Southeast corner which has two, where Laura's South view only "sees" one]	**Katie**: I think this [Katie picks up block]
			
should be here. [places block in Northeast corner. Iliana's head rests on table, checking diagram against construction.]	**Iliana**: No but then for me that doesn't look like North. [points towards diagram of North view]	[Katie lifts block off of Northeast corner] **Aisha:** Oh I have an idea why don't we like correct each, like, correct our view	[Katie places block back onto Northeast corner.] and like take out some things
			
[Katie removes the block again from the Northeast corner. Iliana gets her hand set up to push East wall] that don't look right for us.	**Iliana:** So basically just like, clear this out [pushes second row of the East wall away, laughs, looks towards Katie as she pulls her hand away.]		{Per Katie's suggestion, the group starts over, this time using the provided grid.}

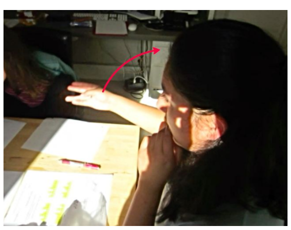
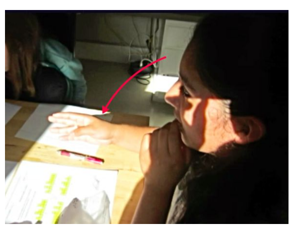
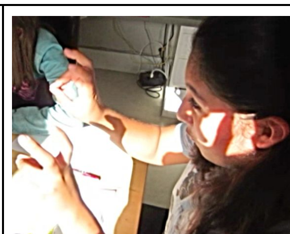
Katie's placement, removal, replacement, and re-removal of the block in the Northeast Corner signifies the incongruous dance of the Northeast corner: the block *"should"* be there but also cannot be there. Iliana, who verbalizes the dispute but does not receive a verbal response, sarcastically responds to Aisha's suggestion of "correcting" the views (i.e., walls) by pushing off the entire second row of the East wall. Her use of sarcasm is not slated purely for irony, as Iliana looks towards Katie for either a consequence or validation of her action. Of course the entire East wall cannot be pushed off, but is there any other option?

Iliana's brazen action is the result of realizing a impasse and articulating it publicly without the hoped-for reciprocal action of the group, in which the significance of the breakdown in the diagrammatic sign would be addressed directly. Katie, who is aware of the impasse, but perhaps unsure how to move forward, suggests that the group start over, this time with the provided 9 x 9 grid. As each girl reconstructs her view as a wall, the impasse relentlessly shows up again, this time in the Southeast corner. Now experiencing an impasse in both the South and North corner of the East wall (a impasse now shared by 3 members of the group), Katie develops a level of uncertainty that must gain the attention of the others.

			
Katie: We should just first find out before constructing it what would like go next to each other. {This idea is not taken up by the group; each girl begins to construct her wall.}	[Katie labels each side of the grid with its respective direction. The other 3 girls begin constructing the walls.]	[Katie places a second block in the Southeast corner to satisfy her East view.]	**Katie:** Do you have one right here? **Laura:** Mm I only have one, yeah.

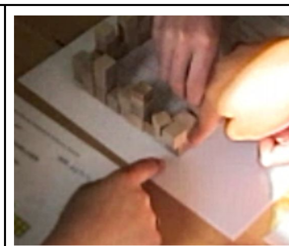
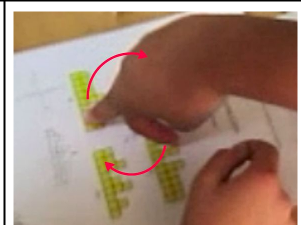
			
Katie: Oh one. [Katie removes the block she has just placed and places it at a different section of her East wall.] I feel like we're matching them up in the wrong way.		**Aisha:** Well just- **Katie:** Like they're supposed to share different sides. [gestures with fingers] **Aisha:** Wait. Well, wait…wait what do you mean by that?	**Iliana:** Well north and south look pretty good. [Iliana rests her chin on the table, assessing the North and South views.]

Unlike Iliana's "odd" comment, Katie's declaration that "we're matching them up in the wrong way" halts construction and draws the attention of the other participants with its directness. Aisha asks Katie what she means, and Iliana engages with Katie's uncertainty by providing information from her perspective about the validity of the construction. Katie continues to describe the breakdown in the diagrammatic sign, grasping for words that accurately describe just *what* is making her feel uncertain.

			
Katie: That I know it's *supposed to be* north, west, east, south, and I feel like it's *not direct*, and slightly…I don't know how to explain it	**Katie**: but does that kind of make sense? **Aisha:** Oh, like the structure may not be exactly north, south, east, west?	**Katie:** Like it could be close, but…Like um…The best way I can describe it except it's completely wrong, it's like "metaphorically" [uses quotation gesture] kind of.	**Katie**: Does it make sense? Sorry if it doesn't, but… **Lizzy:** Keep going. So you're saying metaphorically… **Katie:** Kind of. I don't know how else to explain it because it's kind of the wrong word. But…

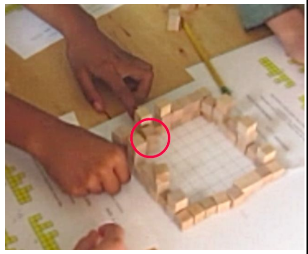
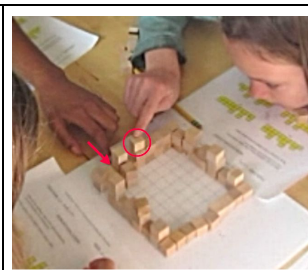
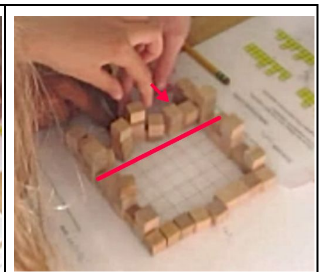
In this sequence, Katie addresses the ambiguity of the diagrammatic sign head on. She knows the diagram is "*supposed to be* north, west, east south" (the diagram says so), *and* she feels "like it's *not direct*." The assertion that the diagram is not direct challenges the group's co-constructed meaning of the diagrammatic sign: that the views represent 3-dimensional facades of a construction. Discussing the ambiguity of the diagram further proves difficult, and Katie searches for the affirmation of her peers: "I don't know how to explain it, but does that kind of make senses?" Aisha attempts to both affirm and clarify Katie's uncertainty, which prompts Katie to use a different resource to explain the breakdown. Prefacing that the word is both "the best way" and that "it's completely wrong", Katie uses a simile partnered with a quotation gesture to describe the breakdown: "it's like 'metaphorically' kind of". Her use of "metaphorically" is an astounding word choice that is simultaneously self-referent (both its use and the word itself are literary devices) and descriptive of the relationship between the diagram and the construction. Indeed, the diagrammatic sign acts as a metaphor for the 3-dimensional construction in that the 3-dimensional construction is not literally depicted in the diagrammatic sign. The depth of the construction is unseen, hidden, abstract; hence, the diagram operates only "metaphorically".

One might argue that Katie's description of the breakdown using a simile provokes only further confusion, as these realizations remain undiscovered by the group. Quite the contrary, her persistent attempts to communicate the sheer uncertainty in the diagrammatic sign unsheathes the pivotal learning moment in the 4-Views Problem, in which a collaborative negotiation process begins. Noticing the gravity of Katie's hesitancy ("Does it make sense? Sorry if it doesn't, but…") and the potential of the fleeting moment, I insert myself into the situation and ask Katie to continue, affirming her "best wrong" word choice: "So you're saying metaphorically…" With further prompting, Katie is able to articulate what makes her feel like the diagram is like a metaphor, drawing all four participants to the diagram's ambiguity, and then to the impasse within the construction.

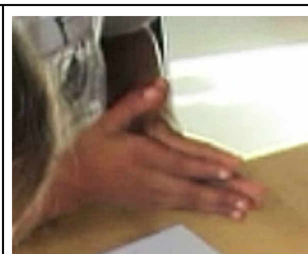
			
Lizzy: Like you're saying this is a metaphor for how the building should look? Is that what you're saying? Or… **Aisha:** Oh like they're not exactly north, east, south, west. It's just a positioning I guess?	**Katie:** Kind of? Like I don't know how, but yeah. Hopefully, yeah, you can read my mind and it will all make sense.	**Lizzy:** Do you – do you see an issue with this? [points to construction] **Iliana:** Yes. **Lizzy**: Like what, are you noticing an issue and that's what's making you feel like- **Katie:** Yes. **Lizzy:** -that's a metaphor? What's the issue that you're seeing?	**Katie:** That you have to, for example, she [Laura] only has one here **Laura:** I only have one here. **Katie:** But I would have two. **Iliana:** Yeah. **Aisha:** Yeah, and then-
			
Iliana: That's what the weird issue is. [Lifts SouthEast corner blocks up and down. Notably, this is the opposite corner from where Iliana sits.] **Aisha:** oh wait. **Iliana**: And west ends with two here but I only have one here.	**Aisha:** Can I just see that? I want to see what you guys are seeing.	**Aisha:** Wait so okay if you're only seeing that, that only makes one **Iliana:** And even if you, woops [knocks over block]. **Katie**: The only way that it would make sense is if we sear these together, but then that wouldn't (?)	**Iliana:** Even if that one's gone you still see this one. Even if that one's gone you still see that one. **Aisha**: Oh, yeah **Iliana**: So unless it's just all made of ones **Aisha**: Now, that doesn't make sense **Iliana**: - it doesn't work **Aisha**: Huh. **Iliana**: I really don't know **Aisha**: Yeah yeah. That doesn't make sense at all.

Katie points directly towards the impasse in describing her uncertainty, connecting two incompatible views through the distributed perspectives: "*She* only has one here, but *I* would have two." All 4 participants echo the concern immediately, and Iliana illustrates with the Southeast corner by lifting the 2nd block up and down twice. This gesture concurrently draws attention to the second impasse and embodies the ambiguous nature of the diagram: both the Northeast and Southeast corners must have two blocks and one block. Pressed to understand the issue more deeply, Aisha gets up out of her seat to "see what you guys are seeing" and stands behind Laura to see things from her perspective. As Aisha narrates her understanding of the impasse and Katie provides a potential fix by "searing" the North/South and East/West views together, Iliana reveals a new impasse, in which even if the corner impasses were somehow addressed, the blocks along the entirety of the East wall would still be seen from the North perspective (see height impasse on p. 11). Thus, "unless it's [East wall] just all made of ones, it doesn't work." Aisha, who has verbally engaged with Katie's uncertainty from the outset, reaffirms that something is awry: "Now, that doesn't make sense. Huh...That doesn't make sense at all."

The uncertainty that begins with Iliana at the beginning of this episode comes full circle as the one student (Aisha) who by her perspective (West) is not in a position to see the impasse validates the feeling that something is not right. Thus, the participants are in a position to collaboratively problem solve, as all members of the group are on an even playing field. The discussion circulated from concretely stating the impasse ("Iliana: So if you're looking North, you'd also see the corner of East, but there's only one square in North") to the necessary reimagining of the diagram (Katie: "I feel like we're matching them up in the wrong way...it's like 'metaphorically'"). Katie's comment in particular allows the group to break down their hitherto implicit perceptual construction and begin exploring toward reorienting their way of seeing. She pushes the group beyond the uncertainty grounded in the construction, asserting that something is "metaphorically" about the problem itself. Within seconds, a potential solution to the impasses is stated, and participants work together to enact a new perceptual construction of the diagram in which the views are not walls, but projections.

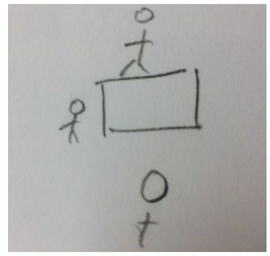
			
Aisha: Oooh! What if this entire row starting here was shifted back so that it was hidden [gestures towards inside of grid] **Iliana:** Oh! **Aisha:** by this **Laura:** What do you mean by shift it? **Iliana:** Yeah but even still-	**Aisha:** Like if you shift [pushes blocks from South wall towards center] **Laura:** Oh. **Aisha:** Like this **Laura:** Oh!	**Iliana:** -you would see those top threee. **Aisha:** But then that wouldn't work because from then, you'd still see that. Oh. Huh.	**Iliana:** But if you look at it here, then it would work. Maybe? **Aisha:** It kind of looks. **Iliana:** Oh no, but then you would see these two. Or maybe you wouldn't so. Hold on let's try moving it to here. [Aisha, Iliana, and Laura push West wall in towards the center of the construction to 'hide' behind the 3 block height on the North and South walls.] **Aisha:** Wait, where? **Iliana:** Here. **Aisha:** Wait, all the way there? **Iliana:** Yes, all the way here.

In an effort to make sure that what you see is what you get, Aisha latches onto a perceptual construction in which blocks are "hidden" behind others. This language of concealment illustrates that in combining all four of the walls, certain blocks must remain unseen. Only through acts of concealment will they be able to form a construction that avoids the impasses. While pushing the West wall towards the center to hide eliminates the impasse of the North and South views seeing the height of the West wall on their edges, the West view still sees the heights of 3 on its edges from the North and South walls. A few ideas emerge to resolve this problem.

Aisha: Yeah but the thing is from my side I still see this block. **Iliana**: Yeah, and this one.	**Katie:** The only way it would really make sense is if we were to pair North and South together, then everything would fit because the rest are twos.	**Aisha**: Yeah, no what it's kind of seeming to me is that, this is kind of impossible, but it feels like if you smashed - yeah, what Katie's saying is if you paired them together. If you smashed West and East together, and North and South together, and somehow made it weird and multidimensional and then made it so the West and East were measured with two and North and South - No, that doesn't make sense. It's confusing now. Um...	**Iliana:** If we removed that. **Katie:** Can I say an idea? That- **Aisha:** Yeah **Katie:** It's not, but I'm just gonna say it anyway **Aisha:** Okay. **Katie:** Um, [looks at diagram], just go like that, then bend it [everyone laughs], then do the same thing to the other side and make a square! [Katie laughs] **Aisha:** Yeah, except I don't really know about that.
			
Aisha: Oh, wait. Iliana, you had a good point, and I'm going to build off your good point. Wait, is this in the third row though? **Iliana:** Because that's the only way those would stay hidden. **Aisha:** Okay. **Iliana:** But then from your side you have the one extra.	**Aisha:** So we get rid of this! [lifts third block from South wall off of construction] Simple. **Iliana:** Yeah but then again my side doesn't look right.	**Aisha**: Yes it does because it's like **Iliana:** Oh yeah! **Aisha:** You get it? **Iliana:** Yeah! **Aisha:** It's kind of weird, but I don't get it either, but it kinda makes sense, but it doesn't make sense.	**Iliana:** Let's move these now, to, over here. **Aisha:** And then get rid of these blocks [lifts other 3rd block off of North wall] **Laura:** All the way up here? **Aisha:** Wait, to like over here? **Iliana:** Yeah, to right here.

Katie and Aisha toy with the idea of "smashing" the opposing walls together, but are not able to arrive at an alternative perceptual construction than the one they have already begun to construct on the grid. Iliana, however, recognizes that with the West wall "hidden," the blocks on the North and South walls that create a height of three can be removed. Aisha affirms this idea, and both impasses can now be resolved by pushing the East and West walls towards the center of the construction. Iliana, Katie, and Laura move to do so with the East wall. However, somewhere along the way, Katie has constructed the East wall as a reverse image of the diagram. When Aisha begins to check the diagram against what she sees on 'her side', an issue arises that leads Katie to a new perceptual construction.

			
Aisha: I want to see, okay my side looks right. It's only two high, one, and then three. Oh, but then, but then, but then, [Aisha notices that the East wall was constructed as a reverse image of the diagram.] **Iliana:** But then. **Laura:** But then what? **Aisha:** Oh wait, Katie? **Katie**: What **Aisha:** Is your side? Oh wait, your side and my side are switched. How does that work? **Iliana**: Well we just need to put them..[begins to fix the East wall to mirror the West wall] **Aisha:** But what if they are switched?	**Katie:** Aren't they? Because they're just reflections. It's like, it would be. What if, we're thinking of this, so we're thinking of this as like facing our way. Are we thinking of this as facing it like um, each side is, we're looking at it through each individual person like that? [Draws box with two stick figures on either side.] Or are we just doing it with one person looking at it like that? [Draws larger stick person on outside of box] **Aisha**: I think its from each different view. **Iliana**: It's like here [gets up and moves behind Katie] **Katie:** What if we tried it with one person and they'd be like that, instead of...Does that make sense? **Aisha:** I mean if we were doing it with one person, but my guess is… **Iliana**: Well let's all check out sides.		**Aisha:** Well my side looks right. **Katie:** Wait my side doesn't look right. **Laura:** I have south. **Aisha:** Wait. Does it look like... **Laura:** My side...my side looks right. **Katie:** Wait, no! It does look right. [Aisha puts her head on the table to check her side, Iliana gets up to move and check Katie's side.] It does, I'm wrong. **Iliana:** Yeah, it does look right. Okay! So I think we did it. **Aisha**: We did it.  Success!

Right before the group finishes the construction, Katie articulates a realization akin to what improv comedians call "groupthink." She says, "Are we thinking of this as facing it like um, each side is, we're looking at it through each individual person like that? Or are we just doing it with one person looking at it like that?" Her question is a philosophical one that lies at the inner core of productive group work. While her question goes mostly unnoticed, its timing as the participants arrive at their final construction is perhaps a call to get at something deeper in the problem: are we doing this as four collective individuals, or one cohesive group?

In summary, the participants in Case Study 3 experience the impasse collaboratively through their construction of the walls, just as in Case Study 2. The participants' use of the material resources and series of back and forths ("Do you have 1 here?") support the participants in realizing that the group's perceptual construction of the diagram is not plausible. Unlike the other two case studies, however, the participants in Case Study 3 are collaboratively able to observe these impasses as Katie bravely describes the semiotic breakdown, despite her struggle for the "correct" words. With the support of Aisha, who encourages her to continue describing that which she is experiencing, Katie draws attention not to a new concrete perceptual construction but to the fact that a new perceptual construction is needed (which, arguably, *is* a new perceptual construction). Katie's public deliberation that something is "metaphorically" draws her peers' attention to the diagram. This collective observation of the diagram, where implicit assumptions are open to be questioned, supports the participants in coordinating their actions so as to collaboratively accept or deny decisions that would resolve the previously latent impasses. The collaboration is evident as all four participants mend the construction by pushing the walls towards the center.

Discussion

Through these three case studies, I have sought to explore how participants coordinate the collaborative reconstruction of a mathematical diagram when faced with a semiotic breakdown. Through my sequencing of the case studies, I have shown that the process by which semiotic breakdown is handled in collaborative problem solving varies significantly. In the first case study, we see the attempts of one participant to explain to others his perceptual construction. In the second case study, we see how the impasses motivated participants to reconstruct their perceptions of the diagram but enact upon those perceptions individually. In the third case study, participants encountered the impasse collaboratively, listened actively to a group member who described the semiotic breakdown and her resulting uncertainty, and tinkered with the construction together as they re-envisioned the meaning of the diagram. While I believe there are many reasons why the semiotic breakdown occurred differently in each group, I will discuss three different dimensions that had a noticeable impact on the participants' coordination of their actions.

Material Resources

As stated in the Methodology section, my decision as to when the groups would receive blocks and grid paper was subjective. In fact, I entered the filming of Case Studies 1 and 2 with the idea that withholding the resources for as long as possible would support participants in coordinating their actions. My rationale was that I wanted participants to have a chance to discuss the diagram and create a plan. Reflecting on this decision has led me to a different understanding of how the material resources play a role in the collaborative problem solving process.

In Case Study 1, Sean perceives the diagram as a projection before the material resources are handed out to the group. The moment when Sean realizes the diagram is a "background" significantly changes the group dynamics and the coordination of the group, as Sean uses his authority to get others to see the diagram as he does. Compared with Case Study 3, in which the impasse is experienced by three members of the group through the use of the material resources, the delay of providing the blocks and grid in Case Study 1 withheld the opportunity for the other three participants to experience the impasses. In future iterations of the 4-Views problem, participant collaboration can be supported by providing participants with the grid and blocks before they have realized the impasse in the diagram itself.

In Case Study 2, I did not require that the participants use the provided grid when problem solving. Although they combined their walls together and collaboratively realized the impasse, their observation of the impasses did not initiate a collaborative response. The blocks, which were scattered around the table, had no central binding or defined location where the construction had to be built. Had the grid paper been required, I anticipate that the participants would have needed to coordinate their actions more collaboratively and perhaps listened to each other's ideas as to how the construction can be modified to accommodate a new perception of the diagram. The grid provides an important resource as it localizes the attention of the group to the center of the table.

The Role of Uncertainty

Research on the role of uncertainty suggests that an individual's ability to recognize their uncertainty and bring it to collective attention is vital in collaborative problem solving and inquiry (Jordan & McDaniel, 2014; Watkins et al., 2018). Mathematical ambiguity necessarily invokes uncertainty as to how an idea can be experienced in diverse ways simultaneously. For example, the student from Barwell's (2005) paper who ascertains that "there's no such thing as a one dimensional shape coz a line is *kind of like* a rectangle filled in [italics added]" (p. 123) displays uncertainty that a line is 1-dimensional as he engages with dimensional ambiguity. In putting forth the logical binary of the line being either one-dimensional or "there's no such thing as a one dimensional shape", the student is pressing the class to define dimensionality concretely so as to resolve the definitional ambiguity. Indeed, the student's statement exemplifies the logical deduction that mathematics teachers often seek to instill in their participants.

Yet, understanding the logic of the student' statement as well as the importance of being able to see the line as both one and two dimensional, the teacher responds by giggling and validating the ambiguity, rather than determining that the line must be either one or two-dimensional: "Very clever. Like a dot. It's interesting isn't it." The teacher models a reaction that invokes the student's uncertainty as a resource for learning by engaging directly with the tension of seeing a line in two mutually incompatible ways. In the 4-Views problem, a similar response is vital for student collaboration. Acknowledging uncertainty provides space for participants to hold onto their implicit mathematical perceptions while simultaneously seeing in a different way. When uncertainty is not perceived as threatening, it can support deeper collaboration and therefore deeper mathematical work.

In Case Studies 1 and 2, there are moments of uncertainty that go unaddressed. For example, in Case Study 1, Edmund announces repeatedly that he is confused. Unfortunately, as an observer we are never able to scrape underneath the surface to see why Edmund was confused. Did he have a different perceptual construction of the diagram that he did not feel comfortable sharing? Taylor similarly expresses uncertainty about how the group is approaching the problem at multiple points, but does not receive any attention. Without the acknowledgement of their peers, this uncertainty is never fully articulated, and an opportunity is lost.

In Case Study 3, Katie's uncertainty, prompted by the impasse, is voiced similarly to Taylor's: "I feel like we're matching them up in the wrong way...Like they're supposed to share different sides." Unlike in Case Study 1, Aisha immediately follows up and validates: "Wait.

Well, wait…wait what do you mean by that?" This is the crucial moment where Katie is expressing the semiotic breakdown - that the diagram is "like 'metaphorically'." The fact that her uncertainty is supported by the group and myself prompts the participants to see the construction from other perspectives. Uncertainty is seen as a strength, rather than a weakness. What begins as Katie speaking individually about something only she sees, leads to all four participants articulating what they believe the "issue" to be in the construction.

Gender Dynamics

In a case study on co-ed collaborative group work, Langer-Osuna (2011) found that female authority was positioned as inappropriate while male authority was positioned as desirable. While gender was not the focus of this paper, the data shows interesting parallels in the collaborative structures of the groups and the gender formations. Particularly interesting is Case Study 2, where Sofia is continually silenced until she constructs the final construction. For example, when Sofia initiates a conversation in the beginning of the problem about her perceptual construction, she is silenced by Noah who uses a pun to delegitimize her statement: "One Direction is not a good band!" We also see that throughout the problem solving process she is shushed by Noah multiple times. While she is able to engage in a collaborative conversation with Will and Mason, this series of negative exchanges with Noah undoubtedly affect her ability to participate fully.

This is apparent as she is continually shut out of the conversation that results from the impasse. While Will and Noah are able to air their ideas, even if no one is listening, Sofia is continually interrupted. When she is finally able to share her idea, "Maybe it's an X-Shape," her idea is attributed to Will almost immediately afterwards. This leads Sofia to silently build the construction on her own, as the only way she will receive attention for her idea is if she physically shows her perceptual construction. This forced isolation eliminates any chance for collaboration. Despite her continued silence, it is her construction that is ultimately accepted by the group as a solution.

This is not to say that gender is the only factor in collaborative problem solving power dynamics. As noted previously, Edmund and Taylor are repeatedly ignored in Case Study 1, even chastised: “When you're right, I'll give you some credit, okay?” Additionally, despite my arguments for Case Study 3 as exemplary of collaboration in the face of semiotic breakdown, we do not hear much from Laura. Thus, my point here is to remind that all collaborative learning is situated within power contexts, and that this connection cannot be ignored in an analysis of participants coordinating their collaborative actions. What motivates someone to speak or listen is contextually grounded and baked into the former discursive practices that Sfard (2002) argues students bring with them when they encounter new mathematical signs. In the face of novelty, it is a question as to whether we choose to reenact discursive practices that silence or bring forward new ways of engaging in collaboration.

Conclusion

In collaborative mathematical problem solving, semiotic breakdown provides an opportunity for participants to collaboratively problematize a diagram and acknowledge their implicit mathematical assumptions. While the 4-Views problem is quite novel, the lessons learned from this thesis can be applied to any mathematical problem solving situation. As we demand participants to engage with mathematically ambiguous artifacts, it is vital that we simultaneously value and explicitly teach them how to navigate uncertainty. In the context of collaboration, what appears as one participant's uncertainty or unknowing can in fact be the precious driving force that supports all participants in coordinating their collective actions to reconstruct their perceptions of a diagram. When uncertainty is voiced, participants are invited to dwell in an epistemic limbo where their implicit mathematical assumptions are brought to light. This is particularly powerful in collaborative mathematical problem solving, where participants must somehow coordinate their actions. How do they do it? Tolerance for mathematical ambiguity, and uncertainty as well as a willingness to see from multiple perspectives are critical for the coordination necessary to collaboratively solve the problem. Only when participants embrace these features of collaborative problem solving are they able to cohesively reconstruct their perception of a diagram. Thus, the function of semiotic breakdown is initiating the participants' reflective stance towards the ambiguous diagram. In the case of the 4-Views problem, as participants attend to the diagram and question their implicit assumptions, they are able to reconstruct its meaning and therefore reconstruct the physical construction.

Limitations

This work on semiotic breakdown is limited in scope. First, only 3 case studies have been analyzed, and therefore most of the findings in this paper cannot be extrapolated from their original context. As I look towards the future of my teaching career, a major limitation of this thesis is the location outside the classroom context. On the one hand, conducting this study outside of school permitted a different learning environment where participants were more relaxed. On the other hand, giving 3 groups of 4 students the 4-Views Problem in a classroom setting would have been a more 'realistic' application of what I might experience as a future teacher who plans to give this problem to students. Finally, all three groups of participants were self-proclaimed friends, which unquestionably affected how they solved the problem. As this is not always the case in classroom settings, what might the project have looked like with students who are less familiar with one another?

Implications & Future Work

My findings from these three case studies have illustrated the pedagogical importance of supporting students as they navigate uncertainty. In this work, I attempted to keep my role as a facilitator/teacher limited, as I sought to give the participants agency as they collaboratively problem solved. However, handing over to students complete agency in the problem-solving process and saying “go for it” does not necessarily lead to equitable outcomes where each student is empowered to share their view. One reason why this happens is because some students, particularly those who hold authority within the group, may not feel open to multiple perceptions of an ambiguous diagram. The variations in the case studies presented illuminate the power a teacher holds in validating multiple perceptions, particularly those from students who are voicing uncertainty in an implicit perceptual construction. How can teachers disrupt the mathematical narrative of competing perceptions (e.g., only one of us is correct) and equip students to see from multiple views simultaneously?

Additionally, the striking differences between the three case studies in their response to the semiotic breakdown demands closer attention to the diagrammatic reasoning discussed in the literature review (Sfard, 2002; Bakker, 2007; Abrahamson et al., 2009; Bartolini Bussi & Mariotti, 2008). As participants collaboratively reason about a diagram, how do they participate in the experimentation and observation phase of Peirce’s framework (Bakker, 2007)? If the role of semiotic breakdown lives at the intersection of these two steps in diagrammatic reasoning, what can we say about participants navigating these breakdowns in a collaborative way? This thesis has shown the divergent responses to semiotic breakdown in collaborative problem solving and speculated the reasons for these striking differences. However, while the case studies in this paper provide examples of semiotic breakdown, the need for a more robust theoretical framework for semiotic breakdown in collaborative mathematical problem solving is salient.

Finally, as complex instruction and collaborative group work in mathematics classes make headway, we must consider how the design of the task affects the type of collaboration that will ensue. As stated in the beginning of this paper, the 4-Views problem anticipates participants to at first reconstruct the views as walls. In anticipating this, the 4-Views problem has a designed impasse at the corners. This impasse was noticed by participants from all three case studies and formed the basis upon which participants experienced the semiotic breakdown. Coupled with supporting participants to embrace uncertainty during collaborative problem solving, designing these impasses is crucial to making a problem worthwhile to do together. While teachers don't often have the power and time to design tasks that elicit the rich collaboration we desire for our students, teachers often make incredibly subtle instructional tweaks that change the outcomes of tasks like the 4-Views problem. Making what are described in this paper as the impasses, breakdown, and reconstructed perception into a cohesive framework could support teachers in considering how collaborative group work will unfold in their classrooms. Finally, it is not enough to just tell students how valuable their peers' perspectives are. If educators truly believe this, we should give students the opportunity to experience that value.

Reflection

I began this paper by asking the following question: If we construct our own meaning from our prior knowledge and experiences, how can we construct something new, as opposed to reconstruct what has already been? In the 4-Views problem, constructing a new perceptual structure requires participants to embrace the semiotic breakdown and question their implicit perceptions. In the case studies that formed this paper, this embrace was evident as participants expressed doubt and uncertainty, even if they struggled to communicate their state of quandary to their peers. At the core of their uncertainty was an object previously unclouded as a means to an end that necessitated inspection as its ambiguity emerged.

I do not presume to have an all-encompassing philosophical response to my original question. Rather, I hope to offer a small byte of reflection. Constructing new perceptions requires that we not only revisit our implicit assumptions but also realize the ambiguity that is. As with the 4 different views, it is not that your view, or my view, or even their views, are correct. Should we reach an impasse in our collective views, an opportunity arises for us to re-coordinate our actions and face the latent ambiguity, constructing a new perception in which we see from many sides.

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