# I *See* What You Mean: A Dimensionalization of Multimodal Revoicing Interactions

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## ABSTRACT

Learners use both gestures and words to develop and communicate their mathematical ideas, and in so doing they provide opportunities for a teacher to respond to and build on their ideas. Flood and Abrahamson (2015) have identified and documented several forms of multimodal revoicing (MMRV) interactions, examining the ways in which a teacher can rephrase a student's verbal or gestural utterances. In this paper, I both corroborate and expand on Flood and Abrahamson's work. My data corpus consists of six videotaped task-based semi-structured clinical/tutorial interviews that were conducted in the context of a design-based project investigating probabilistic cognition and instruction (Seeing Chance, Abrahamson, 2009, 2012). Across all interviews, I focus on instances in which the student and interviewer negotiate the meaning of a set of artifacts, 16 cards that represent the sample space of a binomial randomness experiment. Students frequently experienced confusion or contradicted themselves as they tried to make sense of this experiment. I examine how the interviewer responds to these confusions in both gesture and speech, and I analyze the occurrence, structure, and dynamics of cases of MMRV. I demonstrate that, in distributing their response over two modalities, tutor–interviewers often repeat the student's utterance in one modality while qualifying it in the other modality. In so doing, I propose, interviewers are able to maintain the student's voice "verbatim" even as they modify it. The thesis thus both elaborates on the forms of MMRV delineated by Flood and Abrahamson and introduces new dimensions for future research on these types of interactions.

# INTRODUCTION

This paper describes my Master's project in the Masters and Credential in Science and Math Education (MACSME) program at the University of California, Berkeley. I am a second-year graduate student and pre-service teacher. Upon graduation, I plan on teaching mathematics at the high school or college level.

I have always found the topic of gesture to be fascinating. I tend to be very gestured when I speak and communicate with others, and oftentimes I am not intentional about the way I use gestures. When I started my education at UC Berkeley, the idea of studying how teachers can use gestures strategically was a novel concept to me. I decided that I wanted my Master's project to encompass gesture in some way, because I see this as an excellent way of developing my teaching practice.

I am pursuing a career in mathematics education because it allows me to help shape future generations of mathematical thinkers. Teachers play a crucial role in shaping the ways students experience a subject; however, our society's understanding of this role has evolved over the past several centuries. An old approach to teaching views students as 'empty vessels,' where the teacher's job is to 'fill' their students with knowledge. The constructivist perspective emerged in response to this teacher-centric approach to education. Constructivists argued that students should learn primarily through guided self-discovery, activating their prior knowledge and experiences.

I see these differences present in the ways that students see the subject of mathematics. For students who have been taught as 'empty vessels,' mathematics becomes a collection of isolated facts to memorize. Over the years, the sheer number of facts to memorize can get overwhelming, leading these students to believe that they are not 'math people,' eventually giving up on the subject. However, students taught through a constructivist perspective develop a sense of core concepts that apply in many different situations, leading to a deeper understanding of mathematics. Constructivist teachers allow their students to learn from their mistakes and make their own discoveries. Freudenthal (1971) argues that any other approach to teaching would be 'criminal.' He claims, "Telling a kid a secret he could find out himself is not only bad teaching, it is a crime" (p. 424). Students need many opportunities to explore mathematics to truly make sense of the subject.

However, the constructivist perspective does not explicitly discuss the role of the teacher in helping guide students to these discoveries. If students are not provided any support during their learning, there are many mathematical conventions and techniques that they might not ever discover. Simply stepping back and letting students flail can lead to an unproductive struggle rather than the desired learning outcomes. I believe that the role of the teacher should be to have students engage in mathematical discourse with their peers and the teacher. Talking about a subject can help students establish what they know and what they don't know. It requires them not just to apply formulas and procedural knowledge but also to develop mathematical reasoning skills. Discourse can also serve as a form of individual assessment; if a student cannot explain their reasoning to a peer, it is likely that their understanding is currently superficial and can be deepened through further study or discourse. The Common Core State Standards include "Construct viable arguments and critique the reasoning of others" as one of the Standards for Mathematical Practice, showing a national push toward having students engage in discourse in mathematics classrooms (National Governors Association Center, 2010).

Teachers can use discourse to help guide students in the process of self-discovery. A teacher can revoice a student's argument through repetition, reformulation, or elaboration. Such an act values the student's way of making sense of the material, placing their reasoning at the center of the discussion. However, the teacher can be selective in their revoicing of a student's utterances, both verbal and gestural, in order to steer the student's thinking.

In this project, I examine instances of revoicing in an educational context. I chose to work with videos from the Seeing Chance project. This project included an empirical effort, in which interviews were conducted with two populations of students: $4^{th}$ through $6^{th}$ grade students and undergraduate/graduate students at UC Berkeley. I chose to focus my analysis on the latter population. This is the age group I envision myself teaching over the coming years, so my selection of a focal data corpus made the project particularly relevant to my career goals.

Within this data corpus, I examine interviews where students drive a mathematical conversation without corrective feedback from the interviewer. Such an interview acts as an excellent formative assessment of the students' current understanding of probability, and several misconceptions emerge. The protocol of the interview leads the students through different tasks, and students are asked to predict outcomes, make observations, and revise their thinking. This is a low-stakes environment that is conducive to helping students develop a deeper understanding of probability through discourse.

## LITERATURE REVIEW

### Gesture Frameworks

Investigating teachers' multimodal revoicing of students' multimodal utterance is closely affiliated to research on gesture more broadly, so it is important to start with an overview of existing research in this field. Students gestures provide insight into their thought process. A commonly-cited framework for examining gestures is McNeill's (1992) typology. This typology breaks down gestures into four main categories:

a) *Pointing gestures* (also called *deictic gestures*) are gestures which are used to indicate a location or an object. These gestures often use an extended index finger for the purposes of pointing, but could also incorporate an entire hand. For example, a student pointing at a card to refer to that card is using a deictic gesture.
b) *Iconic gestures* are gestures which represent an idea directly through the shape or the trajectory of one's hand(s). For example, if a student is talking about a normal-shaped distribution, they might trace a bell-curve shape in the air. This is an iconic gesture referring to the semantic content of the normal distribution.
c) *Metaphoric gestures* are gestures which represent a metaphoric idea. For example, if a student were cupping their hand as if to 'hold' an idea, they would be making a metaphoric gesture which represented the metaphor *Ideas Are Objects*.
d) *Beat gestures* are gestures which align with a person's speech to add emphasis. These gestures are usually simple and rhythmic.

Oftentimes, iconic and metaphoric gestures are considered together as forming a single category of *representational* gestures (Alibali & Nathan, 2012).

In later research, McNeill (2005) suggested that this typology describes dimensions rather than categories. He stated that some gestures can belong to more than one of these dimensions. For example, a deictic gesture referring to something in the room might be co-timed with verbal content that the speaker finds significant. If this gesture were made sharp and rhythmic, and the speaker inflected their voice at this time, this could be seen as both a deictic gesture and a beat gesture. Regardless, these dimensions provide a framework for making sense of the way individuals use their gestures in combination with their verbal utterances to communicate their ideas.

**Multimodal Revoicing**

When examining an interaction between a teacher and a student, one dimension that can be studied is the way in which the teacher possibly *revoices* what the student says. This revoicing can happen through *repeating* (re-stating information verbatim), *reformulating* (modifying content/form of original contribution), or *elaborating* (adding new information) the student's contribution. O'Connor and Michaels (1993) were the first to examine this phenomenon, examining the way in which it can be used to credit students and provide them with ownership of their position within a classroom. Further studies expanded upon the study of revoicing (e.g., Forman et al., 1998; Oh et al., 2008). However, these studies tended to focus exclusively on the verbal content of both the teacher and student, ignoring the role that gestures play.

Shein (2012) expanded this concept of revoicing to encompass both verbal and gestural contributions that students and teachers use during an interaction. Shein described the study of revoicing as "seeing with two eyes," gesture and speech, which provides a multimodal perspective into a student's mathematical thinking and understanding. Shein's study examined how multimodal revoicing was used by a fifth-grade teacher working with English language learners. English language learners do not always have access to the language which they need to verbalize their thought process; an examination of both their gestural and verbal utterances can provide a researcher with a better understanding of what the students understand.

Arzarello and Paola (2007) identified a particular form of multimodal revoicing, in which a teacher used a student's unconventional gesture while providing an explanation with more conventional mathematical terminology. They describe this pedagogical move as "the semiotic game." Flood and Abrahamson (2015) offered a taxonomy for this form of multimodal revoicing:

*Form 1*: Gestural repetition co-timed with elaborated verbal content.

Flood and Abrahamson go on to dimensionalize this concept further by introducing two additional forms of multimodal revoicing:

*Form 2*: Selective gestural repetition with co-timed elaborated verbal content

*Form 3*: Elaborated gestural content with co-timed repeated verbal content

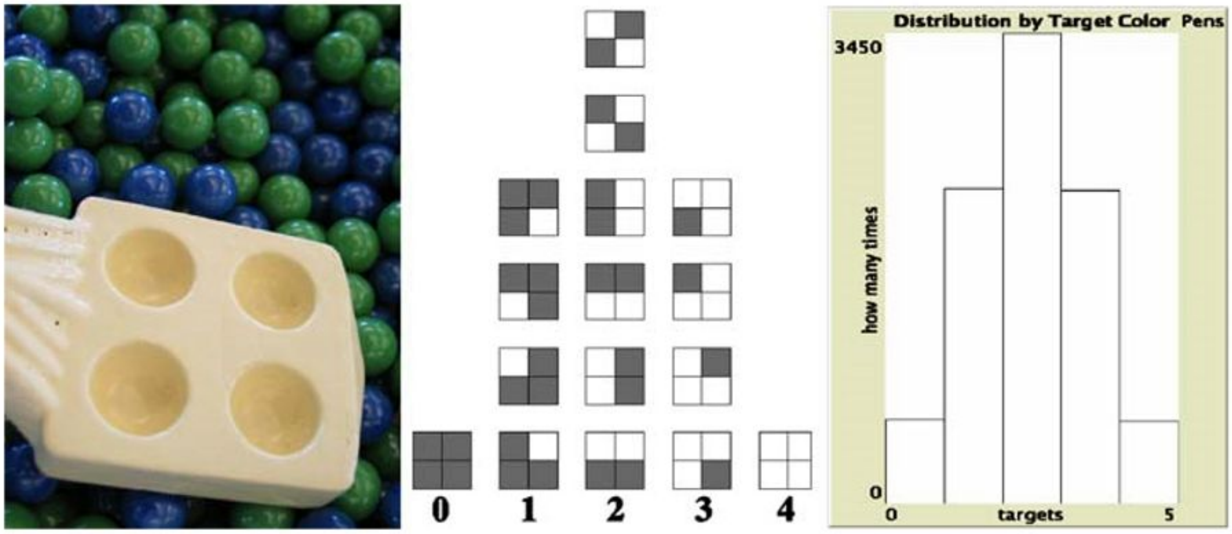
These forms identify different ways in which a teacher can simultaneously revoice a student's verbal content and gestural content. Flood and Abrahamson identified instances of these forms of multimodal revoicing interactions within a corpus of video interviews from the *Kinemathics* project that centered on developing and evaluation the Mathematical Imagery Trainer for Proportion (MIT-P) (see Abrahamson & Trninic, 2014, for a comprehensive overview of this project). Flood and Abrahamson called for further research on this form of interaction, stating that doing so would provide a "better understanding of how responsive teaching supports embodied learning" (p. 2).

The forms of multimodal revoicing identified by Flood and Abrahamson provide some insight into the variety of ways in which these interactions can take place. However, as those authors themselves emphasize, a need exists for a more elaborated taxonomy of multimodal revoicing interactions. I will suggest dimensions for such a taxonomy of revoicing. Two dimensions are the ways in which both gestural and verbal content is revoiced (whether it is repeated verbatim, reformulated, or elaborated). Another dimension to examine is the agent of the original statement and the agent of the revoicing. Namely, the interaction need not be a teacher revoicing a student's idea. Lastly, additional factors which might play a role in a multimodal revoicing interaction, including the use of physical resources and the passage of time between key discursive events. In this paper, I will examine instances of multimodal revoicing and identify how these instances match up with the new dimensions here proposed.

## METHODS

My data corpus for this paper is the Seeing Chance project, conducted by Dor Abrahamson and colleagues at the University of California, Berkeley. Seeing Chance was a design-based research project that investigated students' intuitive probabilistic reasoning. The two phases of this project involved different populations of learners. In the first phase, the researchers worked with $4^{th}$ through $6^{th}$ grade students; in the second phase, the subjects were undergraduate and graduate students. I chose to examine only data from this second phase, because of my interest in working with this age group.

Subjects participated in semi-structured clinical interviews that lasted around 70 minutes each. In these interviews, subjects were shown a plastic tub containing equal numbers of blue and green marbles and a scooper designed to extract exactly four marbles from the tub (see Figure 1a). Subjects were asked "What will we get when we scoop?" Each interview followed the same protocol, which consisted of three stages. In the first stage, the subject engaged with the tub of marbles and scooper, using only these materials to engage with the question. In the second stage, the interviewer brought out a set of empty 4-block cards which allowed the subject to draw




a. b. c.

*Figure 1.* Materials used in the Seeing Chance study: (a) the marbles scooper; (b) the combinations tower; and (c) an experimental outcome distribution produced by a computer-based simulation of the marbles-scooper probability experiment

out the possible outcomes of one scoop (see Abrahamson, 2008, for a detailed structural analysis of the 4-block). The subjects used these cards to create the sample space for the experiment, and the interviewer guided them to use the cards to form the ‘combinations tower,’ the experiment’s sample space consisting of 16 possible compound outcomes arranged in five stacks (see Figure 1b). In the third stage, the subject participated in several computer-based activities that engaged them in thinking about what they would get after many trials of the experiment (see Figure 1c for an example of what one of the computer-based activities looked like).

**Participants**

Within my data corpus, a total of 24 undergraduate or graduate students participated in this phase of the Seeing Chance project. Twenty-two of them were undergraduate students majoring in statistics, economics, or mathematics; the other two were graduate students in similar fields. These participants were self-selected, and received a compensation of $20.00 for their participation in the study.

The interviews were conducted by Dor Abrahamson as well as other members of the Embodied Design Research Laboratory (EDRL), all of whom had a background in mathematics and computer science or cognitive sciences. Five of the six case studies that I selected involve Dor in the role of the interviewer. In the other case study, the interview was conducted by Rose Cendak, a member of the Undergraduate Research Apprentice Program (URAP) within EDRL.

**Materials**

All materials and activities used in my data corpus were part of *ProbLab* (Abrahamson & Wilensky, 2002), a middle-school probability-and-statistics unit initially created at the Center for Connected Learning and Computer-Based Modeling, Northwestern University, under the umbrella of Wilensky's *Connected Probability* project (Wilensky, 1997). Physical materials used include the plastic tub of blue and green marbles, the marble scooper, the blank 4-block cards, and crayons. Additional materials include the computer-based simulations of theoretical and empirical probability authored in *NetLogo* (Wilensky, 1999) and participatory simulation activities for networked classrooms using *HubNet* (Wilensky and Stroup, 1999).

**Procedure**

I examined a set of video data gathered during the Seeing Chance project. These videos captured the interviews with the 24 undergraduate and graduate student participants. Within these interviews, I hoped to find instances of multimodal revoicing similar to the instances documented by Flood and Abrahamson and described above in the Literature Review.

First, I identified which interviews had a suitable camera angle. The original intention of the Seeing Chance project was not to capture interactions between the student and the interviewer, so some of the interviews were filmed with only the student in the line of sight. These interviews were discarded as not relevant to my project. After this, I scrolled through the remaining interviews in fast-forward, observing the way in which the student would use gestures during the interview. This scanning allowed me to identify students who were very gestured in their explanations; interviews with these students would be more likely to capture instances of multimodal revoicing. I also mainly focused on interviews conducted by Dor Abrahamson, because he tended to be more gestured than the other interviewers.

From there, I watched the interviews that seemed promising. Some of these interviews were watched from beginning to end; for others, I jumped around to sections of the interview protocol that I found interesting due to their conceptual content or the forms of productive confusions they engendered. I identified many moments where gestures were being shared between student and interviewer. In consultation with my advisor, I underwent an iterative process wherein the target class of episodes was articulated and further refined. A set of six case studies were identified that capture different types of multimodal revoicing.

My purpose in sifting through the data corpus was not to achieve an exhaustive enumeration of target events. Rather, due to the exploratory nature of the study, I was focused on identifying phenomena as proof of existence for these education forms of interaction. Thus, the particular cases I analyze are ‘as-is’ -- they should not be taken to represent typical distributions in these populations and settings. By way of analogy, it is as though I were a zoologist trekking through a little-known land in search of new species of animals rather than surveying the abundances of well-known species.

**Data Analysis**

I used microgenetic analysis in examining my selected case studies. Microgenetic analysis allows a researcher to examine a student's real-time experience with learning in ways that other research methods (e.g., pre- and post-tests, or surveys) do not allow. Siegler (2006) argues for the utility of this research method by stating:

> If learning followed a straight line, [microgenetic analysis] would be unnecessary. Yet, cognitive change involves regression as well as progression, odd transitional states that are present only briefly but that are crucial for the changes to occur… and many other surprising features. Simply put, the only way to find out how children learn is to study them closely while they are learning. (p. 468)

For my project, microgenetic analysis allowed me to look at the different ways in which I saw the interviewer revoicing a student's claims. I focused on short interactions (typically less than a minute) between a student and an interviewer, closely detailing each person's verbal and gestural contributions. I categorized the case studies based on how the student's gestural as well as verbal content were revoiced. I then analyzed each clip closely, identifying significant features of these interactions.

## RESULTS

I now present six case studies as findings of my project. Each case study provides an example of multimodal revoicing. These instances provide evidence that corroborates the findings of Flood and Abrahamson (2015). Yet the instances go farther to expand on the forms of multimodal revoicing they had enumerated. Each of my case studies provides insight into a new form of revoicing, demonstrating variations along dimensions of agent and modality. I will argue that these findings, while not comprehensive, further dimensionalize the study of multimodal revoicing.

***Case Study 1 – Elaborated gestural content co-timed with elaborated verbal content*** [1]

For my first case study, I examine a clip involving Genji. In this clip, Dor and Genji discussed the relative probabilities of each outcome of the marble experiment. Genji initially stated that all 16 cards represent equally-likely outcomes, which is mathematically correct[2]. However, she changed her mind after the cards were arranged into the combinations tower. She claimed that the cards in taller columns (like the 2-green column, see Figure 1) represent more probable outcomes than the cards in shorter columns (like the 4-green or 0-green columns). (Below I will be using abbreviated form: '2G' means 2-green, etc.) Genji explained to Dor:

> G: These [traced out '1G' column, see Figure 2a]… these get [traced out '2G' column]… getting all [traced out '3G' column]… these patterns [traced out '1G' column], they have the same chances [repeated '1G' gesture], but it's not like this one [touched/moved 'BBBB' card, see Figure 2b] equals to this one [touched/moved

[1] Video clips from each of these case studies are available here: https://drive.google.com/a/berkeley.edu/folderview?id=0B7sweoeKUAbhYl9mT2NFZGlyMW8&usp=sharing

[2] Technically speaking, the cards represent outcomes that are *almost* equally likely. When the scooper is used to draw four marbles, we can think about it as equivalent to drawing four marbles out the tub one at a time. When the first marble is drawn, there is a ½ probability of drawing a green and ½ probability of drawing a blue. However, after this first marble is drawn, the numbers of green and blue marbles in the bin are no longer equal, meaning that the probabilities are no longer ½ and ½ for the second marble to be green and blue (respectively). However, given the large number of marbles in the tub compared to the numbers of marbles being drawn (four), we can say that the probabilities will be roughly ½ for each marble. This means that the probability of each outcome can be considered ½ * ½ * ½ * ½ = 1/16. Thus the Seeing Chance marbles-scooping activity embodies a hypergeometric experiment that for all purposes and intents simulates a binomial experiment.

a. b. c.

*Figure 2.* Genji stated that all of the cards in the '1G' column have the same probability: (a) she traced the length of the column with her pen, as shown by the purple arrow. She then stated that the 'BBBB' card did not have the same probability as the 'BBBG' card: (b) she touched to the 'BBBB' card; (c) and immediately afterward touched the 'BBBG' card.

‘BBBG’ card, see Figure 2c]. This one [repeated ‘BBBB’ gesture] does not equal this one [repeated ‘BBBG’ gesture].

D: To this specific one? [pointed to ‘BBBG’ card]

G: Yeah. Uh… This one [pointed to ‘BBBB’ card] does not equal to this one [pointed to ‘BBGB’ card]. But this one [repeated ‘BBBB’ gesture] equals to this one [pointed to ‘GGGG’ card].

D: I see.

G: So, uh, what, what I… just now what I am saying that all they are equal [pointed to each card ‘0G’ and ‘1G’ columns]… no, that’s not right. The, uh… This thing should be right, is… these two columns [traced ‘3G’ column followed by ‘1G’ column] should get the same, uh, the possibility should be the same, these two columns, and this one [touched/moved both ‘BBBB’ and ‘GGGG’ cards] should be the same, and all these [traced out the ‘2 green’ column] should be the same. But not this with this [pointed to ‘BGGG’ followed by ‘BBGG’][3] or others.

[3] Interestingly, Genji initially pointed to the ‘BBBG’ card and the ‘BGGG’ card simultaneously. However, she quickly realized that these two cards were from columns that she believed to have equal probabilities (‘1G’ and

Through this exchange, Genji demonstrated that she thought each card within a column had the same probability as one another, and that columns symmetric about the tower's center have the same probabilities, but that not every card is equally likely. However, her verbal content alone was not sufficient in communicating these ideas, because she frequently used ambiguous pronouns (as evinced in "This one does not equal this one"). Her gestural content, mainly consisting of deictic gestures, was necessary to disambiguate and ground her verbal content and communicate her ideas to Dor. Additionally, Genji explained her understanding through considering many cases, providing examples of cards (or columns) which are equal and those which are not equal. She did not explain her understanding in a clear or concise manner, or through making statements about generalized phenomena.

Dor responded to Genji in the following manner:

D: So you are saying that *within* [open hand circled twice] the column…

G: Yeah.

D: This card is the same as this is the same as this is the same as that [pointed to each card in '1G' column].

G: Yeah.

D: Again, *within* [moved hand in a high arc toward '2G' column, see Figure 3a] this column, this is the same as this is the same as this [pointed to each card in '2G' column]…

G: Yeah, yeah.

'3G'), so she revised her gesture to indicate a card from the '2G' column in combination with a card from the '3G' column.

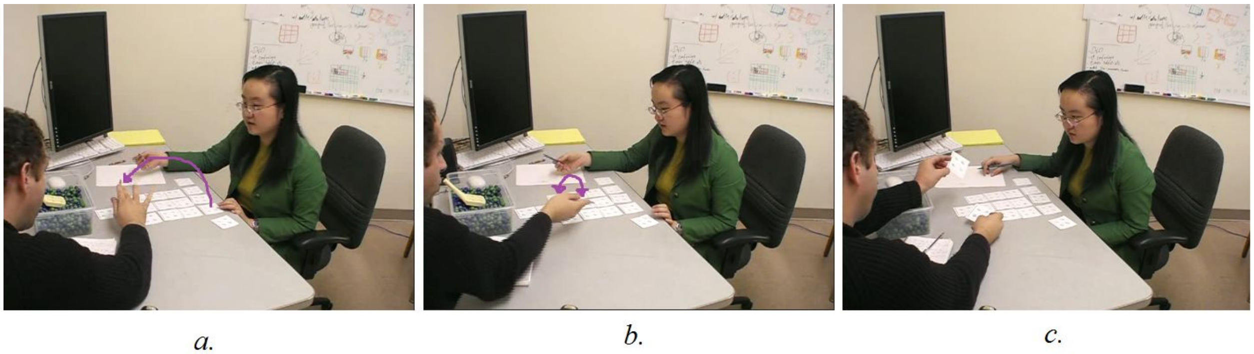


*Figure 3.* (a) Dor used a high arc gesture, shown in purple arrow, when he talked about the cards "within" a column. (b) When he asked Genji about what was happening "between" columns, he flipped his hand (shown in purple arrow) so that it was facing downward above one column and facing upward above another column. (c) When Dor asked Genji about two "particular" outcomes, he picked up the corresponding cards.

D: Within [moved hand in a high arc toward '3G' column] this column [pointed to each card in '3G' column]… But *between* [waved hand side-to-side from '3G' column to '1G column, see Figure 3b] columns, you say…

G: Between columns, these two columns [laid open hands over '1G' and '3G' columns[4]] are the same, these [laid open hands over '0G' and '4G' columns] are the same, but this is… [pointed at 'BBGG' card, waved other hand toward entire '2G' column]

D: So for instance, the chance… I'm just taking these two closest to me [picked up 'GBBB' and 'GGBB,' see Figure 3c].

G: Yeah.

D: So you're saying the chance of getting this *particular* [waved left hand, holding 'GGBB'], particular card that has two green, two blue, versus the chance of getting this *particular* [waved right hand, holding 'GBBB'] card which has one green, you are saying the chances are different.

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[4] An interesting affordance of gesture is that a person can refer to two objects (or collections of objects) simultaneously. In contrast, when a person wants to refer to two objects verbally, it necessarily comes out linearly (e.g., "[both] this one and that one") rather than being a simultaneous reference.

G: Yeah.

D: Which one is more…?

G: This one [pointed to 'GGBB'].

In this dialogue, Dor built off of the gestural and verbal content in Genji's original explanation. When referring to the cards in the '1G' column, he used consecutive deictic gestures and pronouns to refer to each card, stating "this is the same as this is the same as this…" However, Dor elaborated on Genji's explanation through the introduction of more precise vocabulary. He brought in the idea of 'within' to frame with greater precision the dyad's verbal utterances about comparative likelihoods of cards in one column of the combinations tower as compared to verbal utterances that compare likelihoods of cards across the columns. Each time Dor said this word, "within," he co-timed it with a gesture. The first time, Dor held his hand outward, with his palm facing himself, and slightly swirled his hand twice. He then pointed at each card down the '1G' column. After this, to leave one column and enter into another, Dor lifted his hand in a high arc while he says 'within.' These gestures can be seen as representational gestures, embodying the process of going 'within' each column of the combinations tower. They could also be considered a beat gesture, placing an emphasis on Dor's introduction of the new word[5]. The rest of Dor's verbal and gestural content was consistent with Genji's original explanation. He used consecutive deictic gestures and pronouns to refer to each card in the column, stating "this is the same as this is the same as this…"

Dor's second elaboration came with the idea of 'between' columns. When referring to the notion of looking between different columns, Dor flipped his hand from face down to face up,

[5] Dor did inflect his voice to add emphasis whenever he says 'within,' 'between,' and 'particular' during the clip.

roughly landing on top of two different columns of the tower. Again, this gesture can be seen as both a representational gesture for the notion of ‘between’ and a beat gesture to emphasize the word itself. After this elaboration, Genji jumped in with an explanation rather than waiting for Dor to revoice, providing her with an opportunity to build off of what she had originally stated.

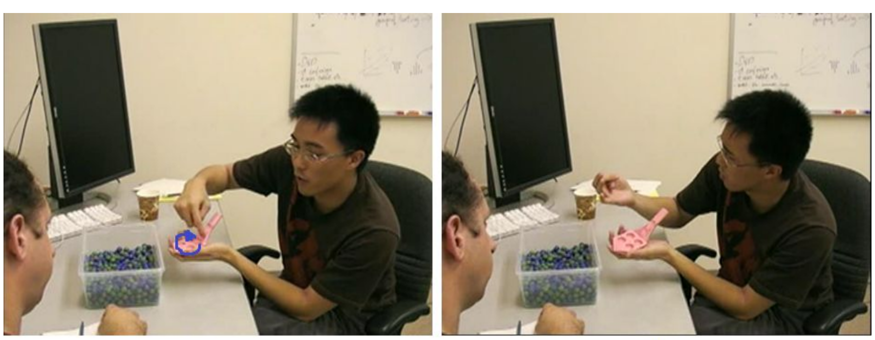
Dor’s final elaboration is the concept of a ‘particular’ card. He asked Genji to discuss the probabilities of two different cards. Rather than using a pointing gesture to refer to the two cards, Dor picked up each card, one in each hand. This could still be considered to be a deictic gesture to refer to the cards. However, through physically picking them up, Dor differentiated the cards from their respective columns, creating a spatial juxtaposition. Dor co-timed his utterances of the word ‘particular’ with a beat gesture, gently shaking the card he was referring to at the time. Genji stated that the two cards are not the same probability, and Dor pushed her to state which one represented a more likely outcome.

This clip ended with Genji still demonstrating a misconception about the probabilities of the possible outcomes for the marble experiment. In this instance, however, Dor’s multimodal revoicing allowed Genji to make her reasoning more precise and explicit. Gibbons (2002) discusses the difference between “context-embedded” language (or “here-and-now” language) and “context-reduced” language. Genji’s first explanation was very context-embedded, where she was examining particular cards in the combinations tower and using pronouns and pointing gestures in her explanation. It would have been difficult for somebody who just walked into the room to have access to her explanation. In Dor’s revoicing, he modelled a more context-reduced way of explaining the situation, introducing more abstract terms into the reasoning. Genji picked up a bit of this – after Dor introduces the notion of ‘between,’ she referred to columns rather than individual cards in her explanation. This is a small but significant step in developing more precise language.

***Case Study 2 - Selective recounting of student's verbal and gestural content***

Second, I examine a case study involving Liang. At the time of the study, Liang was recently graduated from UC Berkeley with a degree in Engineering, Mathematics, and Statistics. Liang explored the tub of marbles through the context of a gambling game. When asked to predict what he would get when he used the scooper, Liang interpreted the question in terms of betting money on outcomes.

During the initial stage of the interview, Liang had difficulty figuring out what aspect of the scooper to attend to. He sometimes focused on the arrangement of marbles within the scooper (for which there are 16 possible outcomes), and at other times he focused on just the number of green marbles in the scooper (for which there are 5 possible values). When Dor asked about the situation where he cared only about the number of green marbles in the scooper, Liang stated that there was a 1/5 (or 20%) chance for each outcome of this gambling game (see Figure 4a). In figuring out his answer, Liang listed out the possibilities, stating "… zero, one, two, three, four…" as the number of green marbles that could be in the scooper. Simultaneously, he



*a.* *b.*

*Figure 4*. Liang described the possible values for the number of green marbles appearing in the scooper. (a) He rotated his finger around the scooper, as shown in blue arrow. (b) He counted on his fingers while saying "zero, one, two, three, four" to establish that there are 5 possible values.

gestured with his hand, bending a finger in towards his palm for each word (see Figure 4b). This gesture helped him count out the number of possibilities, because he had initially said that there were four possibilities rather than five.

Later in the interview, Liang worked with the 4-block cards and set up the combinations tower. His conclusion in this stage of the interview was: “The chance of getting the ratio of green to blue equal to one is higher than the other ratios.” In other words, Liang was arguing that the probability of getting 2 green marbles (and, thus, 2 blue marbles) in the scooper is higher than the probability of getting other numbers of green marbles. This statement contradicted what he had said previously, but Liang did not demonstrate any awareness of this contradiction. Dor responded in the following way:

D: Now, I… I may have misunderstood you earlier, but, uh… earlier when we were talking, before we started drawing these things [waved at the 4-blocks on the table], you said, ‘Well, we could get, uh, when we scoop, we could get zero green, [started counting on fingers, see Figure 5a] one green, two green, three green, or four green,’ so there are 5 groups, and so you said, ‘Well, it’s 20 percent each.’

L: Let me think… [8 second pause] I need to rethink what I just said, because, yes, this may be a misjudgment.

D: Yeah… [3 second pause] But that’s okay.

L: Yes. Yes, in fact, yes.

D: Do you remember that?

a. b.

*Figure 5*. (a) Dor recounted Liang's counting gesture to remind him of his previous claim. (b) Liang rejected his own earlier statement and reasserted that "getting two greens is higher than getting the rest of the ratios."

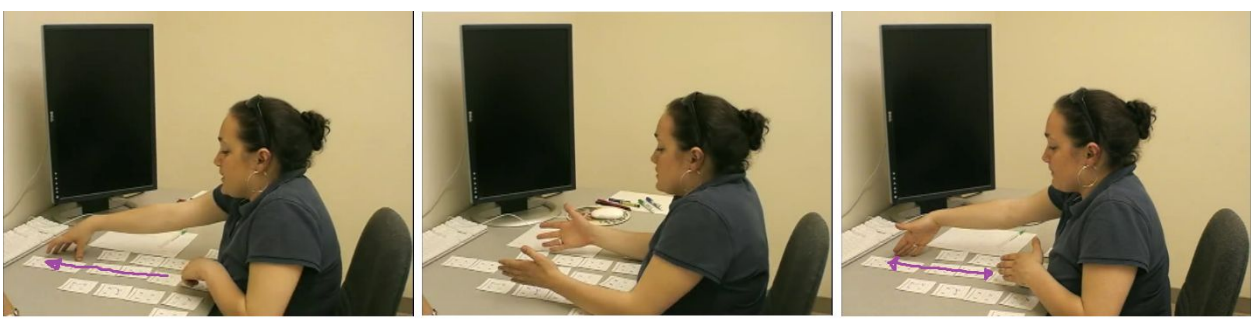
L: Yes, I remember that. I need to readjust my statement. Yes. In fact, getting two greens is higher than getting the rest of the ratios [placed hand on middle column, see Figure 5b].

In this moment, Dor recounted Liang's prior gestural and verbal content without significant modification. I have chosen to label this form of multimodal revoicing as 'selective recounting of student gestural and verbal content.' Abrahamson, Gutiérrez, Charoenying, Negrete, & Bumbacher (2012) describe recounting as when a tutor "reminds [the student] what s/he had said and done earlier by re-evoking/reenacting a previous episode, including actions, discussion, [and] inferences." The authors of that paper include recounting as one of many tutorial interaction tactics that a tutor could use to support a student's mathematical learning. This is a form a multimodal revoicing which could be thought of as 'repeated gestural content co-timed with repeated verbal content.' However, categorizing it in this way loses the essential information of time. A tutor can be very selective about *when* they decide to recount and *what* gets recounted. In this instance, Dor chose this particular moment to recount Liang's initial thinking to get him to reflect on his original thinking and resolve the contradiction himself. This case study demonstrates that time, or perhaps timing, must considered as a factor when it comes to enumerating possible forms of multimodal revoicing.

***Case Study 3 – Reformulated verbal content co-timed with reformulated gestural content***

For my third case study, I examine a clip involving Riley. At the time of the interview, Riley was a senior at UC Berkeley, graduating with a degree in statistics. In her interview, Riley had a difficult time deciding if all of the 4-block cards represent equally-likely outcomes. During the course of the interview, she wavered back and forth on this issue. Even after setting up the 4-block cards into the combinations tower, she remained uncertain of how to calculate the probability of each individual outcome. After using some equations to help her make sense of the situation, Riley concluded that each card has a probability of 1/16. She explained her reasoning to Rose[6]:

Ri: So, if overall this group [waved right hand over the length of the '2G' column, see Figure 6a] has a probability of 6/16 or 3/8, and overall this [arm hovered over the '1G' and '3G' columns, see Figure 6b] has a probability of 1/4, and then 1/16 [touched '0G'



a. b. c.

*Figure 6.* (a) Riley measured out the length of the '2G' column, as shown by the purple arrow, stating that the group has a probability of 6/16. (b) She however her arms over the '1G' and '3G' columns when referring to the probability of these groups. (c) She used her right hand to trace out the '2G' column several times, shown in purple arrow, when she talked about dividing by the number of cards in the column to figure out the probability of each card.

[6] This was the interview which was conducted by Rose Cendak, a URAP member, rather than Dor.

and '4G' columns with hands], then, basically, you just divide by the number of [moved right hand toward top of '2G' column, leaving left hand at the base, see Figure 6c]… combinations within each of these [traced out '2G' column with right hand]. So, if this is 3/8 divided by 6, then that is 1/16, and that is basically 1/16 for each of these [touched the cards at the base of each column] combinations…

Rose summarized Riley's explanation shortly thereafter:

Ro: So, to get something from [laid hands on the cards in the '1G' column, see Figure 7a] this group is not as… like, if we are scooping to get something with one green [tapped her pen on the 'BBGB' card, see Figure 7b], is not as likely as getting something with two greens [tapped her pen on the 'BGGB' card]?

Ri: Mhmm (affirmative).

Ro: But, if… it's equally likely to scoop this pattern [put her hand on 'BBGB' card, moving it slightly, see Figure 7c] as this pattern [put her hand on 'BGGB' card]?

Ri: Yes.

In her initial explanation, Riley referred both to the columns of the combinations tower and the individual 4-block cards. When she referred to the columns, she used gestures which either measured out the length of the columns (for instance, during 'you just divide by the

a. b. c.

*Figure 7.* (a) Rose initially laid her hands on the '1G' column to talk about the idea of scooping something with one green. (b) She changed her gesture to be tapping a card in the '1G' column with her pen. (c) When referring to the probabilities of particular cards, Rose touched the cards with her hands.

number of…') or actually formed the columns themselves with her arms (for instance, during 'this has a probability of 1/4'). These gestures could be seen as representational, because they are representing the columns[7] in the tower, but they are mainly deictic since they are referring to the physically-present cards on the table. When she referred to the individual cards, she did so by actually touching the cards. Riley's explanation, and all of her gestures, are grounded in the physical environment provided by the combinations tower. She never referred to the notion of 'scooping something with one green,' but rather referred to the groups/columns present in the tower.

In her revoicing, Rose reformulated the verbal content of Riley's explanation. Rose never mentioned the columns of the tower, focusing her verbal content to be about the marbles-scooping experiment. She asked about the likelihood of getting 'something with one green' versus getting 'something with two greens.' Rose also maintained one of Riley's gestures: touching individual cards to refer to individual outcomes of the experiment. However, Rose used another deictic gesture (tapping a card with a pen) to represent the notion of the columns rather than a representational gesture (tracing out the columns themselves). In so doing, Rose had the card play two different roles. When the card was tapped by a pen, Riley was being asked to attend to the number of green marbles on that card. When touched by a hand, however, Riley was asked to consider the actual arrangement of marbles of that card. Through these gestures, Rose demonstrated that there is more than one way to make sense of and attend to the properties of an object. Her selective gestural revoicing provided Riley with access to this notion.

[7] A person's forearm, due to its shape, is particularly suitable for referring to a column.

***Case Study 4 – Elaborated verbal content with reformulated gestural content***

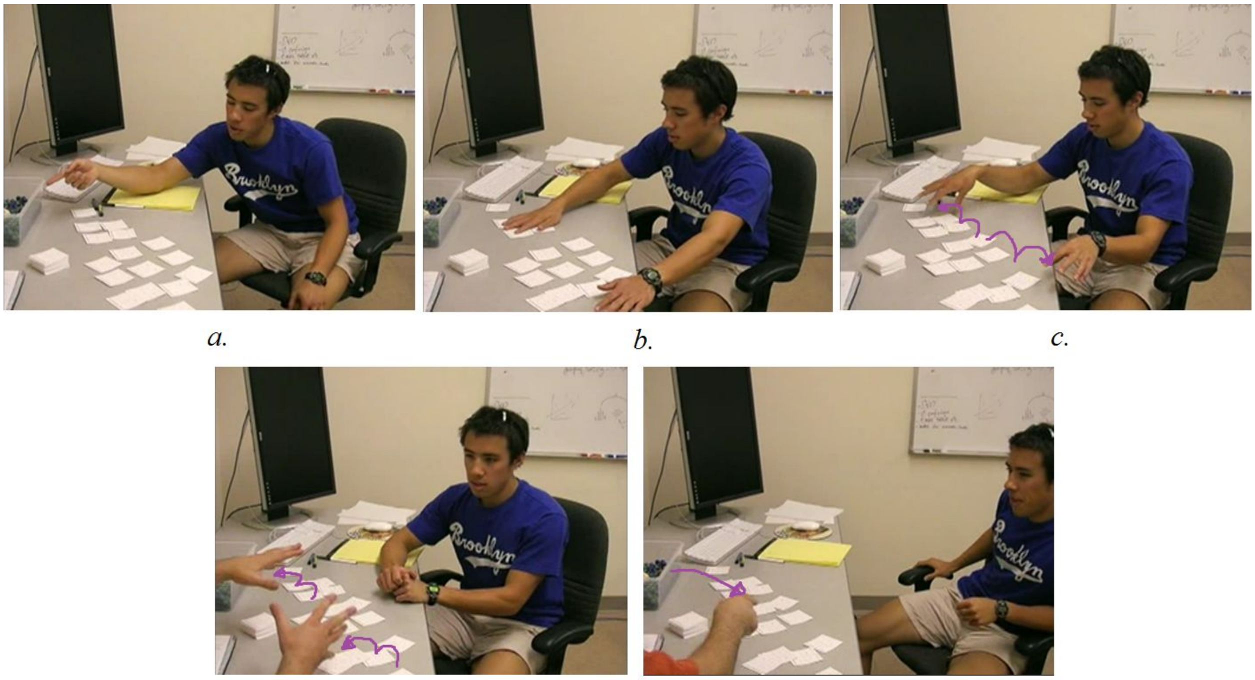
My fourth case study examines a clip of Ryan. In his interview, Ryan was easily able to identify that there was a greater probability of drawing 2 blue and 2 green marbles over drawing 4 green marbles. When he worked with the 4-block cards, Ryan chose to organize them into five piles according to the number of green marbles on each card. These piles were organized in a linear manner, with '0G' on the left side of the table and '4G' on the right side of the table. However, the cards within each pile were clumped together rather than being organized into columns (as they are in the combinations tower). When Ryan described to Dor what this card distribution meant, they had the following discussion:

R: Presumably, as you perform a lot of trials, you would get this distribution [pointed toward the piles]

D: Okay, now what do you mean by *this* distribution? [gestured toward the piles] This is how it would look, in any sense, or what do you mean by *this* distribution?

R: Uh, well, so if you took a picture each time [gestured toward tub of marbles, see Figure 8a] and then you put the picture in one of these, uh, five piles [laid hands flat on each pile consecutively, see Figure 8b], then eventually you should end up with, you know, the ratios shown here. 6 to 16, 4 to 16… [used open hands to gesture toward piles, see Figure 8c]

D: Okay, so if I just had these five slots [open hand gesture toward each pile, see Figure 8d], and each time I take a picture and [dart-throwing gesture with popping noise, see Figure 8e]… and just draw things randomly, then eventually it would somehow correspond to this?



*a.* *b.* *c.*

*d.* *e.*

*Figure 8*. (a) Ryan referred to the marble tub as he described his picture metaphor. (b) He placed his hands on piles on cards when he described sorting the pictures, and (c) he used an open-handed gesture, shown with purple arrow, to refer to the piles when talking about the ratios of the distribution. (d) Dor repeated Ryan's open-handed gesture. (e) When Dor described sorting the pictures, he used a dart-throwing gesture, shown with purple arrow, to put each picture into one of the five piles.

R: Yeah, sure. I mean, if you properly normalized the trials, or whatever.

For the purpose of making sense of the card distribution on the table, Ryan offered up the metaphor of taking pictures of trials. This metaphor built off of Ryan’s embodied experience of creating the card distribution. He created a 4-block card to represent each possible outcome, which is a form of getting a ‘picture’ of a trial. He then organized these 4-block cards into piles; his metaphor involved an analogous process of organizing the pictures into five piles[8]. While explaining this metaphor, Ryan used gestures that were grounded in the materials around him. He first referred to the five piles by laying his hands on top of each one, with his fingers spread

[8] I have proposed one way of making sense of Ryan’s metaphor, but there are many different ways to interpret metaphors offered up by students as they explain their reasoning. See Abrahamson, Gutiérrez, and Baddorf (2012) for a more thorough examination of the discursive functions that these metaphors can play.

out. Later, he used an open hand to gesture toward each pile, without physically touching the cards.

When Dor followed up with Ryan, he continued to use this same metaphor. Dor's verbal content was grounded within this metaphor, yet elaborated on the verbal content offered up by Ryan. Dor used the idea of sorting pictures, but introduced the idea of using 'slots' instead of forming 'piles.' However, Dor chose to use different gestures during his discussion of the sorting process. He acted out grabbing a picture and flinging it away from him (as one might throw a dart). This contrasted with Ryan's open hand gestures used during his description of the sorting process.

Dor's use of multi-modal revoicing served as a foreshadowing of what was to come in the interview. Ryan's choice of card distribution allowed him access to consider the likelihood of outcomes and what would happen after repeated trials; however, Dor needed to guide him toward arranging the cards in the combinations tower. Ryan's gestural content was consistent with the notion of piles: when talking about sorting pictures, his gestures involved open hands with finger spread out in all directions, forming two-dimensional clumps. However, Dor chose a gesture for the sorting process that had a one-dimensional aspect to it. The dart-throwing gesture formed a linear trajectory for each card. This gesture is consistent with the configuration of the combinations tower, where each category has cards organized in a one-dimensional column. Dor's revoicing primed Ryan to think about the problem at hand in a different way. A minute later, Dor asked Ryan to think about another way of arranging the cards so that a person who just walked in the room could make sense of what they were doing. Upon some reflection, Ryan was able to arrange the cards into 5 columns rather than 5 piles.

Dor's elaboration on Ryan's verbal content had implications later on in the interview. Dor brought in the idea of using 5 'slots' to organize pictures taken of the outcomes. This acted as priming for the first computer simulation within the protocol. In this simulation, many trials of the experiment are taken. A picture representing each trial is shown being dropped into one of 5 columns. These columns could be thought of as being different 'slots' into which the cards are being dropped. Dor's elaboration prompted Ryan to consider more normative[9] ways of thinking about the experiment.

***Case Study 5 - Student revoicing of teacher's verbal and gestural content***

My fifth case study involves Lidia. I focus on a moment during the last stage of the interview, when Lidia worked with the first computer simulation. She watched the simulation perform many trials of the experiment, sorting pictures of each trial into 5 different columns. Initially, she expressed some confusion about how this simulation related to the combinations tower that she had made with the 4-block cards. Dor provided her with a way of thinking about the connection:

D: Here's a kind of, uh, metaphor that… tell me if this makes sense to you, I don't know, alright? So we have 16 different cards here [open hand gesture to combinations tower, see Figure 9a]. Alright?

L: Mhmm.

[9] When I use the word 'normative,' I am examining this scenario from the perspective of a teacher. For any lesson, a teacher has particular learning objectives, or ideas that they want students to take away from the lesson. Part of the work of teaching is anticipating how students will initially make sense of the content with which they are presented. 'Normative' ways of thinking about the content would be those ways that best lead students to the teacher's desired learning outcomes in accord with the sociocultural norms of the mathematical discipline.

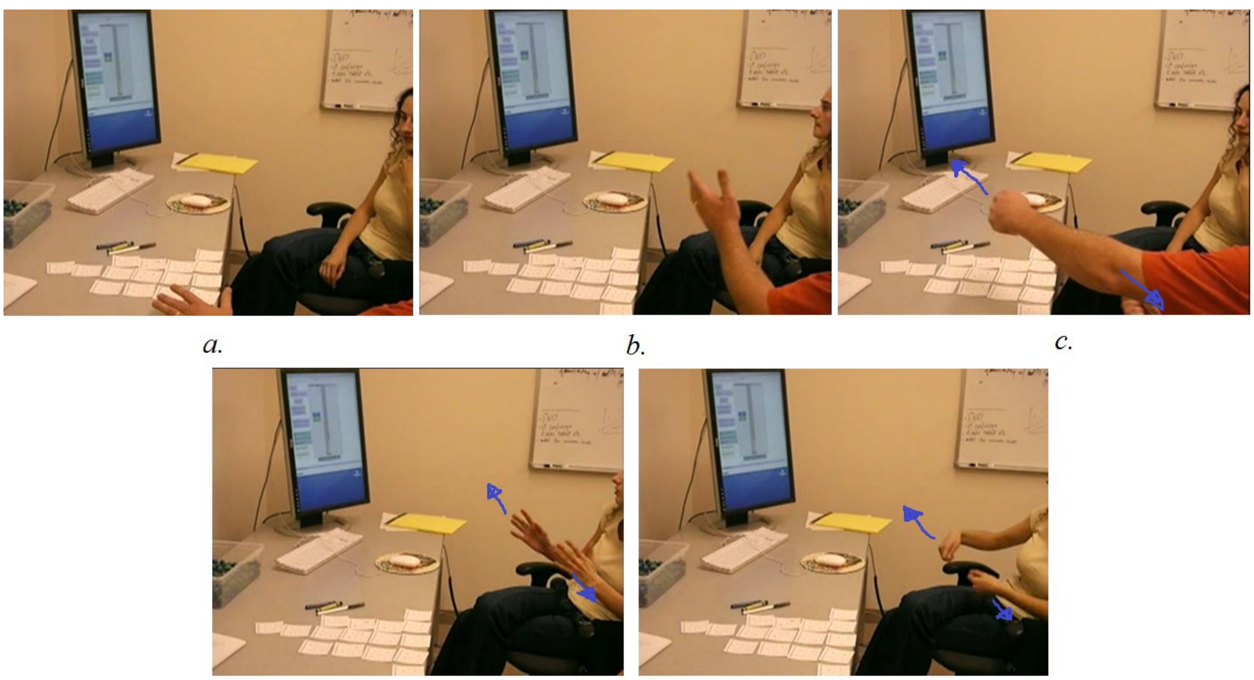
D: Can you think of the outcome distribution [pointed to computer screen, see Figure 9b] as kind of stretching this [2-handed vertical stretching gesture, see Figure 9c] up? You know?

L: Umm…

D: I guess, technically…

L: Yeah, um, I mean, that's what you do, you definitely do not stretch it this way [opened hands horizontally, see Figure 9d], right, so you definitely just [2-handed vertical stretching gesture, see Figure 9e] stretch it up. Yeah, I would say that's fair.

In this clip, Dor introduced a metaphor to help Lidia make sense of the connection between the computer simulation and the combinations tower. This metaphor, involving



a. b. c. d. e.

*Figure 9*. Dor gestured to: (a) the combinations tower, followed by (b) the simulation on the computer screen. (c) He suggested the idea of vertically stretching the combinations tower, shown in blue arrow, to make it look like what was on the screen. Lidia: (d) stated that there was not a horizontal stretch happening, and (e) agreed that she could think about it as a vertical stretch.

stretching the combinations tower to match it with the simulation, involved a two-handed stretching gesture. Dor moved his left hand upward and his right hand downward, closing both fists as if he were grabbing something and physically stretching it.

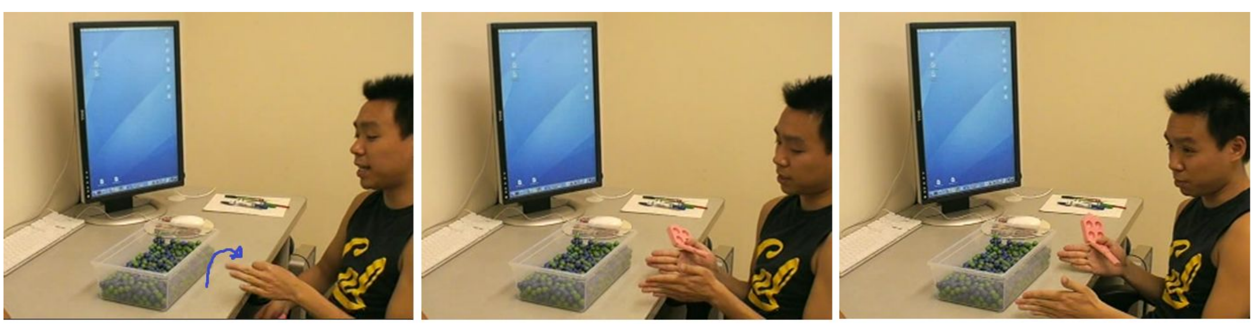
Lidia adopted this gestural content to help her make sense of Dor's idea. She initially took a few seconds to make sense of what he had said, looking back and forth between the combinations tower and the computer screen. Then she used two different gestures while she spoke. First, she stated that a horizontal stretch is not occurring. At this moment, she held both hands out in front of her, holding them vertically to measure out the bottom of the distribution. She moved both hands outward, as if to suggest a horizontal stretch to the distribution. Second, she stated that a vertical stretch makes sense. She simultaneously made a stretching gesture almost identical to Dor's initial gesture. After both of these statements, Lidia confidently agreed with Dor, stating it is was a fair metaphor for him to make.

This clip demonstrates that it is possible for a student to take advantage of gestures used by a teacher. Lidia's first gesture is a revoicing of Dor's original gesture. Lidia was using this gesture to represent horizontally stretching the combinations tower. However, based on her verbal content, she knew that this stretch would not produce what she sees on the computer screen. Her second gesture, a repetition of Dor's gesture, occurred while she affirmed Dor's metaphor. It is interesting to note that Lidia's first gesture represented the stretch itself, or perhaps the product of the stretch, while her second gesture represented her as the agent of the stretch. Her hands were both open palms for the horizontal stretch; in comparison, her hands closed into fists for the vertical stretch, as if she were grabbing the combinations tower to pull it into shape.

***Case Study 6 – Student revoicing of their prior gestural and verbal content***

My final case study examines a clip of Mark. During the first phase of his interview, Mark used the scooper in the marble tub once and drew out 3 blue balls and 1 green ball. From there, he began to discuss what he would expect to get after performing the experiment many times. He constructed the following argument for why he would expect to get 2 blue balls and 2 green balls:

> M: So when the times that you have 3 blue balls [left hand gestures to the left] will be weighted against when the times you have 3 green balls [left hand gestures to the right, lightly touching desk, see Figure 10a], and that will make it a combinations that... weighted to be 2 green balls and 2 blue balls [hands parallel on desk, see Figure 10b].... And... these two combinations in expected value[10], it's the same....as the scenario that you got 2 green balls and 2 blue balls.... /8 sec/ And, for the same reasons... the 4 blue balls will also be weighted against the 4 green balls [gestured to



*a.* *b.* *c.*

*Figure 10.* (a) Mark gestured to the left side of the tub when he said "you have 3 blue balls," and moved his hand to the right side of the tub, shown in blue arrow, when he said "you have 3 green balls." (b) He said these two possibilities would be "weighted to be 2 green and 2 blue balls." (c) He gestured to the far ends of the tub when referring to getting "4 blue balls" and "4 green balls."

[10] It is important to notice that Mark was using the term 'expected value' in a non-normative way. He used the term to make a qualitative argument about what he thought the average would be rather than making a quantitative argument involving a discrete random variable.

far ends of the marble tub, see Figure 10c], and in expected value they should be in the same ratios. So that would.... eventually give you the same expected values, when you calculate it that way.

In this instance, Mark's gestures referred to an imagined sample space where outcomes with more blue marbles appear on the left and outcomes with more green marbles appear on the right. When Mark said "the times when you get 3 blue balls," he made a deictic gesture toward the left side of the desk; when he said "the times when you get 3 green balls," he made a similar gesture toward the right side. His gestures were symmetric about the center of his imagined space (which corresponded with the center of the marble tub), and he used this symmetry to argue that those combinations would be "weighted to be 2 blue balls and 2 green balls."

In the second phase of Mark's interview, after he had set up the 4-block cards in the combinations tower, he used the tower to repeat his previous argument:

M: And also I... As I say before [left hand on the 4-blue card, right hand on the 4-green card], the 2 columns, in having.... 3 greens and 3 blues [left and right forearms on these columns, respectively, see Figure 11], will be weighted against each other in expected value terms.

*Figure 11.* Mark laid his forearms on the '1G' and '3G' columns while he revoiced his previous argument.

This instance can be seen as Mark revoicing his own argument. The verbal content of this argument was quite similar to what he said previously, referring to the notion of ‘weighting’ in helping him figure out the ‘expected value.’ However, his gestures referred to the real space on the table (the combinations tower) rather than an imagined one. When he talked about the ‘3 greens’ and the ‘3 blues’ outcomes, he laid an arm over each column.

This is an example of how revoicing can occur through reconfiguration: by providing a student with an additional resource (the combinations tower), they are able to manifest their thinking in different ways. As Abrahamson (2009) argues, the combinations tower can be seen as a *semiotic means of objectification* (Radford, 2003). Yet as a student appropriates a semiotic means, new properties emerge that in turn the student can incorporate into their reasoning. Abrahamson (2009) provides a more thorough examination of the combinations tower as a semiotic means of objectification for the case study of Mark.

**Summary**

In their study of multimodal revoicing, Flood and Abrahamson (2015) discussed three forms of multimodal revoicing interactions:

Form 1: Gestural repetition co-timed with elaborated verbal content

Form 2: Selective gestural repetition co-timed with elaborated verbal content

Form 3: Elaborated gestural content co-timed with repeated verbal content

Through my six case studies, I have documented six new forms of multimodal revoicing:

Form 4: Elaborated gestural content co-timed with elaborated verbal content

Form 5: Selective recounting of student’s verbal and gestural content

Form 6: Reformulated verbal content co-timed with reformulated gestural content

Form 7: Elaborated verbal content with reformulated gestural content

Form 8: Student revoicing of teacher's verbal and gestural content

Form 9: Student revoicing of their prior gestural and verbal content

Forms 4, 6, and 7 demonstrate various permutations of how the gestural content and verbal content can be independently revoiced through repetition, reformulation, or elaboration. These forms build off of the three forms previously enumerated by Flood and Abrahamson (2015). Form 5 introduces the notion of recounting as a method of revoicing, taking into account the amount of time that passes between the original statement and its revoicing. Forms 8 and 9 introduce the student as a possible agent for the revoicing.

These different forms are organized into Table 1, which demonstrates the variety of forms that currently exist and suggests forms that might exist but have yet to be documented.[11]

| Table 1 | | GESTURAL | | |
|---|---|---|---|---|
| | | Repetition | Elaboration | Reformulation |
| VERBAL | Repetition | Form 5 (with recounting) | Form 3<br>Form 9 (student revoicing of their own statement) | |
| | Elaboration | Form 1<br>Form 2 (with selective repetition) | Form 4<br>Form 8 (student revoicing of teacher's statement) | Form 7 |
| | Reformulation | | | Form 6 |

[11] All of the forms of multimodal revoicing listed in Table 1 involve revoicing of both verbal and gestural content. It is possible to examine the revoicing of verbal utterances which are not accompanied with any gestures, or vice versa. Therefore, this table should not be seen as comprehensive.

# CONCLUSION

Revoicing is a powerful instructional tool. As learners develop their understanding of a topic, their arguments often change to become more precise or more elaborate, demonstrating deeper conceptual knowledge. Thus conceptual development can be discerned by comparing statements uttered during instructional interaction. This thesis has looked in particular at multimodal statements uttered along instructional interactions that are similar enough to implicate what exactly has changed.

When viewed from a multimodal perspective, the phenomenon of revoicing unfolds combinatorically, with verbal, gestural, or both verbal and gestural utterance components revoiced. Along both the verbal and gestural modalities, revoicing can happen through repetition (repeated verbatim), elaboration (adding new information), reformulation (modifying what has been presented), or some other form. Another dimension which plays a role in revoicing is the relationship between the agent of the original statement and the agent of the revoiced statement. Flood and Abrahamson examined only instances where the student made the original statement and the teacher provided the revoiced statement. However, I have demonstrated an instance where the student revoices a teacher's original statement (Case Study 5) as well as an instance of a student revoicing their own previous statement (Case Study 6). An additional dimension would be the amount of time that passes between the original statement and the revoiced statement. A recounting of a student's statement after some time has passed (as in Case Study 2) plays a different discursive role than an instantaneous repetition would. After new information has come to light, recounting a student's argument allows them the space to reconsider their stance without telling them they are right or wrong. The last dimension I identify is the role that physical resources play in the revoicing process. In Case Study 6, the 4-block cards and the combinations tower influenced the way in which Mark revoiced his original argument.

**Limitations**

The forms of multimodal revoicing interactions that I have demonstrated in my case studies, as well as the dimensions I have outlined, do not by any means form a comprehensive enumeration of how these interactions may transpire. In fact such a comprehensive enumeration may not even be possible. Rather, the purpose of this project is to continue the process of identifying dimensions for multimodal revoicing interactions that may be utilized in future studies on the subject.

Additionally, I do not make any claims about the frequency of multimodal revoicing. My research methods were not aimed at documenting how often such instances emerged in the Seeing Chance data. Rather, I analyzed particular instances which stood out as demonstrating various forms that I noticed.

**Future Work**

Future research in multimodal revoicing interactions should aim to identify further instances of these interactions. My proposed dimensions could be used for the means of categorizing and distinguishing these instances. Further research is also needed to expand on these identified dimensions, to develop a more comprehensive understanding of multimodal interactions and how they influence the learning process. The blank cells in Table 1 suggest that there may exist forms of multimodal revoicing which have yet to be documented; further studies could aim at filling in these gaps.

**Reflections**

Through this project, I have gained a better understanding of the variety of ways that gesture can be used in learning and instruction. Each of my case studies highlighted gestures being used for different purposes. They can be used by teachers to provide redirect a student's thinking, or they could be used by students to make sense of new information. I have also had opportunities to reflect on the ways in which revoicing can be used in the process of guiding students to discovering mathematical concepts and identifying their own misconceptions. These are pedagogical aims for my future teaching practice.

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## APPENDIX A – *Seeing Chance* Semi-Clinical Interview Protocol

| What we say/do | Why we say/do it | Possible Responses | How to respond to those responses |
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| Hi, we're from _____ (name of university).<br><br>First of all, thanks for agreeing to take part in this study. I hope you'll find it interesting. We'd like to get your help to understand how students think about our stuff. We're going to be asking you some questions, but remember there are no right or wrong answers. We just want to hear what you have to say about these things. We'll be glad to answer any questions you have now or later. | Introduce ourselves | My mother works at ___ (name of university) etc.<br>Do you know so-and-so? | Cool, Go ____ (name of university team)! |
| You'll notice we're using a camera to video-tape this session. We use it because we don't want to miss anything you say. If at anytime you wish not to be recorded, please let us know and we'll turn off the camera. Also, please feel free to use any of the items on the table at any time to help you to explain or to think about things. | Make students comfortable; Let them understand their rights as volunteer participants.<br><br>Let students know it's fine to touch and use materials: paper, crayons, pennies, dice, etc. | Will I be on TV? | Probably not, but lots of professors will see you, maybe in some far-away countries! |
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| [Show the marble scooper] What do you think this is? Have you ever seen anything like this? What does it remind you of? Guess what we're going to do with it. | Gauge initial reactions to the instruments. Specifically, evaluate the obviousness of the design. What "goes without saying?" What aspect is abstruse? | Golf club<br>Cool; It's like a lottery; It's fair, because there is an even number of holes; where did you buy this/how did you make this?<br>Are we going to put things in | How do you know that? What do you mean? Can you show me that? Would you like to try it? If that were true, would it matter?<br>How should we call this? |

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| | | the holes? | |
| [Show bin]<br>What do you see in here?<br><br><br><br><br><br><br><br>What do you think we're going to do with this? | | A lot of marbles.<br>How many marbles are there in the box?' Are there more greens than blues? How do you know? *Dip scooper into bin to pull out marbles;* How much did this cost?/Where did you buy this?;<br>Count them<br>Use the marbles with the 'scooper'<br>Fill holes with marbles | Why is that important to you? How many marbles do you think there are? Why do you think that?<br><br><br><br><br><br>Can you show me?<br>Do you want to show me? |
| I'm about to dip [the scooper] in here—what do you think will happen? I'll make sure that each of the slots gets its own marble and I'll let any extras spill out.<br><br><br>((optional questions: How likely is it/What is the prob. that a certain slot will get a blue marble? How about a green marble? What about a different slot? [Don't imply importance of location if student has not yet attended to this.]))<br>What do you think about this pattern we just got? Do you like this pattern? What can you say about it? | Delineate details, elicit clarifications, establish common vocabulary ('pattern' 'arrangement' 'combination' 'order' 'hole/slot' ;repeat; etc.).<br><br>See if students understand that these are practically "independent" events—like flipping 4 coins. See if students think some patterns are more likely than others and elicit their reasoning for this. | Marbles will go into the slots; you'll get such and such a pattern. Patterns will be different each time.<br><br>You have to mix them first. You can't look while you're doing it.<br><br>How many green marbles are there? | That's interesting--why do you think that?<br>Is it possible to get something else? Ever? Sometimes? Are some patterns more rare than other patterns? Why so?<br><br>Why?<br><br>What would/could happen if we looked while we did it? [Maybe get a cloth to cover bin while scooping]<br>Is it important to know the number of marbles to figure this out? What if didn't know – what would we do? |
| *Let student try it out about 5 times*<br>Can you describe what we have?<br><br>*If they get an arrangement that is a rotation of a previous one*—Is this the same as before?<br>*If they get all or no green*—what do you think | Are students attending to the specific arrangement of the colors. What seems to be the default way of seeing and describing the sample–in terms of the pattern?; the number of green and blue squares irrespective of the pattern? If they attend to the number of green, do they realize that some classes | It is equally likely to get 1 green, 2 green, 3 green, or 4 green.<br><br>There are different chances | What if we specify a top to the scooper?<br>If no, why is this one different? |

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| about getting all of one color?<br><br>How about getting just one blue in the top-left corner and all the rest green–is that as rare as getting all green? | (e.g., “1-green”) are smaller than others?; ‘rarer’ than others? Do they have any sense why this is so? ((We have to be careful not to give away that we care so much for different arrangements, but we do want to exhaust the possibility that they actually are attending to the arrangements.)) | | |
| Do you think you’ll get that again? Ever? Do any [patterns] repeat scoop to scoop?<br><br>((*optional – might be too confusing/time-consuming* If this slot has a green, do you have any sense of if the slot next to it to has a green or a blue?)) | See what they feel is predictable. | Yes/No<br><br>We’ll get 2G, 2B more often than other arrangements.<br><br>We’ll get something different each time. | When? Now? Later? How are you so sure. Maybe we’ll never get it?<br>That’s interesting. Now, how could we be sure this is true?<br><br>That’s interesting. Now, how could we be sure this is true?<br><br>Do you mean that if we keep scooping it will be different each time?–always? [if not:] So how many times can we keep scooping without getting repeats? |
| *Before a particular scoop*–How many greens do you think you’ll get this time?<br><br>When enough context has been built: Can we know what will happen each time we scoop?; Is there anything we can say about what happens if we scoop many times, like, 100 times? | See where students are between ‘no idea’ and ‘best guess’. | No way to know.<br><br>Can’t know exactly what we will get.<br>It’s chance. | How should it be (marble mix up in bin) so that we always get the same pattern? (Is that possible?)<br><br>Why do you think that?<br>If no, what *can* we say? |
| How many different [combinations] do you think there are? Is it possible to find all?<br><br>(How could we keep track of | Prompting any form of combinatorial analysis.<br><br>Trying to introduce drawing 4-blocks | 4; 8; 16; 40… infinity<br>Do we have to count? | Can you show me? How did you get that number? How do you know? What makes you think of this number? How sure are you? Is it possible to |

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| the combinations we have gotten?)<br><br>*After students are "on board" with drawing*<br>Let's start drawing the combinations. How can we know if we have found all? What should we do about repeats? | Combinatorial analysis has two important qualitative principles that express its systematicity and rigor: (1) We must find all the possible combinations; (2) We must not repeat any combination. What warrants might students have to support success in (1) and/or (2)? We are probing for spontaneous methods. | [after drawing] there are 16 | prove that, or will we never know? How would you show me? Oh, I have some paper and crayons here, would this help you? Here, why don't you use this grid to draw the sample we just got?<br><br>Is that all? How do you know? How did you draw these? [probe for procedure/ method] |
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| *Once we have the all the patterns or the student cannot find any more:*<br>How could we arrange these so we could see what kinds of combinations we have?<br>Is there a way to group these? | Trying to get students to create tower. | Oh, it doesn't really matter. | Well, when you were building these, you kept on talking about the number of green squares. So maybe that's a helpful way to organize the 4-blocks – in groups, according to the number of green squares.<br>Also, let's figure out a way that a person who sees this for the very first time might immediately know if there are more combinations in one group than in another. |
| What can we say about this tower?<br>What does it represent/show?<br>Does this help us in any way with the scooper stuff? (What could we predict when scooping?)<br>How many ways are there to get 'this' [pattern]?<br>How many ways are there to get only one green? | Probing for a connection back to the marble scooper activity<br><br>Do students see that their tower answers these questions? Do they use it to answer? | Shows the different combinations we can get when scooping.<br>Shows number of occurrences for each pattern. | [We should not press too much in asking for the implications of the combinations tower for the marble scooper. Instead, we move on to the ProbLab model and see if "it clicks"] |
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| *Show students the 4-block stalagmite [hide command center]*<br>*Let students try it out*<br>What is happening here?<br>Would you like to try it?<br><br>Do these buttons have anything to do with what we have done before?<br><br>*Demonstrate the 'keep-repeats?' mode.* | Making sure the student understands how the model works. You can even slow it down for this, initially. | It's like what we just did with the other model.<br><br>What do these [buttons, sliders, etc.] do? | In what way? Are there any differences?<br><br>Would you like to try playing with them and changing them around to see? [if any time is left]<br>Would you like to see how this simulation works? [show procedures, explain overall logic, and choose an easy procedure to explain] |
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| *Show students 4-Blocks model [hide command center]*<br><br>What do you think this is?<br><br>*Show students one "tick"*<br>[make sure model doesn't go too fast]<br>What do you think just happened? | Gauge initial reactions to ProbLab model<br><br>Again, probing for a connection to earlier activities | It's a game<br><br>The computer filled in the "slots" These turned green, these turned blue.<br>How does the computer choose how to fill a slot?<br>Each square flips a coin to turn B or G | How do we play it? Any guesses?<br><br>What do you mean 'filled in'? Can you explain what happened?<br><br>How do you think it does it? |
| What does this tell us [#-target-color counter]?<br><br>What is happening here [plot] each time we press the 'go once' button? | The counter monitor may help students to bridge between picture of 4-block to growing column in histogram | Tells us how many greens we got.<br><br>Don't know.<br><br>Or<br><br>This column gets bigger because we got that many | Okay, let's slow this [the model] down to see if that helps us to see<br>[slow WAY down]<br>Scenario:<br>How many green are in this 4-block?<br>Now what does the counter say?<br><br>Let's look over here [the plot]. What do you think will happen? |

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| | | greens in the block | Interesting…why do you think that?<br>What just happened?<br>Why do you think that happened? |
| *Prompt student to take some more samples*<br>Now what do you think is happening?<br>What kinds of combinations are possible?<br><br>*Run the simulation in the fast mode*<br>What is happening?<br><br>What's happening to the columns as the model keeps running?<br>*Summarize and answer any questions* | Checking whether the student is seeing any connections between the activities, and if so, which.<br><br>Discuss with the student the nature of a computer-based simulation of random selection. Is the simulation "really really" the same as scooping by hand from a box of marbles? Can we trust the computer? | Everything/the same ones we got in the tower/don't know<br>More with one green.<br><br>The pattern is being recorded in the histogram.<br>It got all the patterns plus some repeats.<br><br>Starting to stabilize, to look like our tower | How do you know?<br><br>Is this like the thing you made with your drawings? Why or why not? |