Negotiating Mathematical Visualizations in Classroom Group Work:

The Case of a Digital Design for Proportion.

By
Rosa G. Lee

Submitted May 17, 2013
In Partial Fulfillment of a Masters of Art Degree in Education

MACSME Program
Graduate School of Education
University of California, Berkeley

Readers:
Professor Dor Abrahamson
Professor Alan Schoenfeld

Abstract:
Since Arcavi (2003) proposed visualization as a central focus for mathematics-education research, the field has made progress in articulating the role of visualization in individual reasoning and learning. Complementary research has modeled instructors' discursive actions as fostering learners' professional perception of visual displays germane to their academic and vocational practices (Abrahamson, 2009; Alaç & Hutchins, 2004; Goodwin, 1994; Sfard, 2007; Stevens & Hall, 1998). The role of visualization in authentic classroom activities, however, is yet under-articulated. This paper reports on results from a study that investigated classroom activities through the theoretical lens of visualization. A research team analyzed a corpus of videotapes collected during a classroom implementation of an experimental unit on proportion that incorporated an innovative tablet application for proportion. In particular, the team analyzed cognitive mechanisms and discursive processes implicit in students' collaborative problem solving by attempting to determine how individual interlocutors visualized a shared referent in their perceptual field and how utterances about these visualizations mobilized the discussion in directions that the teacher evaluated as aligned with the activity's pedagogical objectives. Collaborative micro-ethnographic analyses of selected episodes led to the emergence of two types of relation between students' respective visualizations: conceptually complementary and conceptually contradictory. These categories were then used to code and evaluate the effect of the instructors' mediatory utterances, as they attempted to facilitate productive conversation. When the instructors validated conceptually contradictory visualizations, students were more inclined to persevere in argumentation. Teachers should develop visualization-based formative-assessment skills as a means of moderating group work in ways that foster both grounded content learning and effective argumentation practices. In particular, teachers should be trained to elicit, interpret, and leverage student visualizations of mathematical objects as a means of deepening student understanding of subject matter content. For teachers, student behavior is the perceptual display in which they "see" student reasoning. In a sense, teachers visualize visualization.

Introduction

What does it mean to *visualize* something? Visualization is not merely the passive observation of the world, but an active engagement in seeing, "both the product and process of creation, interpretation, and reflection upon pictures and images" (Arcavi, 2003, p. 215). Ever since Arcavi proposed visualization as a central focus for mathematics-education research, the field has made progress in articulating the role of visualization in individual reasoning and learning. But I argue that there is still room yet to add to the theories surrounding visualization.

The phenomenon of visualization in wider content areas has been examined from multiple angles already. From an anthropological view, visualization is dependent on the norms formed by a group of participants. Goodwin (1994) speaks about building the practices of *professional vision*, the organized ways of seeing in particular social groups. Professional vision is built on three discursive practices, by which experts foster the professional vision of novices: coding phenomena into objects of knowledge, highlighting salient features of specific phenomena, and producing and articulating material representations. Stevens and Hall (1998) define *disciplined perception* as "forms of perception [which] differ across disciplines…. [W]ithin disciplinary communities we should expect to find interactional and organizational means through which disciplined perception is learned." Thus, the phenomenon of specialized ways of seeing has been well documented in social and professional spheres. Taking into account these different views of visualizations (or visualizations of visualizations) in varying communities of practice, let us examine specifically the role of visualization in discourse between individuals with varying prior experience.

In discourse studies, expert and novice pairs have been observed to accommodate for their differences in expertise and make themselves understood despite varying degrees in prior knowledge (Isaacs & Clark, 1987). When asked about landmarks in New York City, the expert-novice pair in conversation was able to communicate about a common reference. Each in the pair assessed the other's level of expertise, and then was able to adjust, supply, and acquire names, descriptions, and perspectives that, in turn, allowed their references to be understood by their interlocutor. From a socio-cultural perspective, Newman, Griffin, and Cole (1989) propose that systematic vagueness, or "looseness," in interpretation of an object between student and teacher is actually necessary for learning and a key component in the process of appropriation. Similarly, Sfard (2007), who treats thinking as a form of internalized communication, posits that substantial discursive change in mathematics learning is spurred by *commognitive* conflict, a situation that occurs when different interlocutors act according to differing discursive rules. These existing perspectives lay the groundwork and inform this thesis in which I discuss visualizations specifically regarding mathematical content in the classroom setting.

In previous publications, EDRL (the Embodied Design Research Laboratory research team; Abrahamson, *Director*) has drawn on empirical data from design-based research studies of mathematical cognition and instruction so as to examine interviewer–interviewee interactions in cases where the two interlocutors were referring to the same mathematical object in the world yet nevertheless assigned differed meanings to those objects, in ways that were significant to the target content (Abrahamson et al., 2012; Abrahamson, 2009; Abrahamson et al., 2009). When two people "see" the same physical object in the world, the actual referent is the same, but the mental constructions, interpretations, and consequently, inferences, can be vastly different and even contradictory. Consider the classical illusion Duck–Rabbit, popularized by psychologist Joseph Jastrow, which epitomizes the role of perceptual agency in constructing mental images.

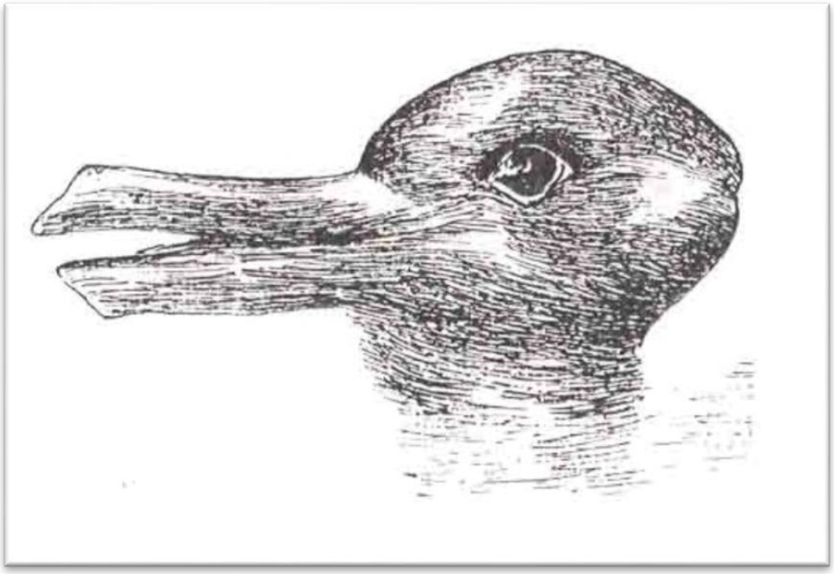


Figure 1. The classical illusion, Duck–Rabbit, in which one's perception can vacillate between a duck and a rabbit. The two visualizations are considered conceptually contradictory since they cannot both be simultaneously true.

Upon first glance, one may feel strongly confident in an interpretation of an object at hand. It swims in a lake, don't you agree? If you see my way of seeing, we might call it a day and move on. But, if you see a rabbit, you might find my comment extremely confounding, (What does she mean? It obviously hops on grass!). Our interpretations, or *visualizations*, of the shared referent might have been contradictory. But the individuals involved would not be aware of any discrepancy in seeing, or worse, think the other individual to be deranged, unless one had voiced some disambiguating feature, such as "beak" or "ears" (Abrahamson et al., 2009). Consequently, the visualization of a referent depends on critical features and aspects of the object to which the viewer attends, which Stevens and Halls (1998) call the *orientation of view*.

While the Duck–Rabbit illusion serves as a classical example of a referent which allows for two *contradictory* visualizations, ambiguous objects also allow for *complementary* visualizations. Specifically, I speak about the generative nature of mathematical objects that bear potential for mathematically rich visualizations. Take for instance, an image of six objects as arranged in Figure 2.

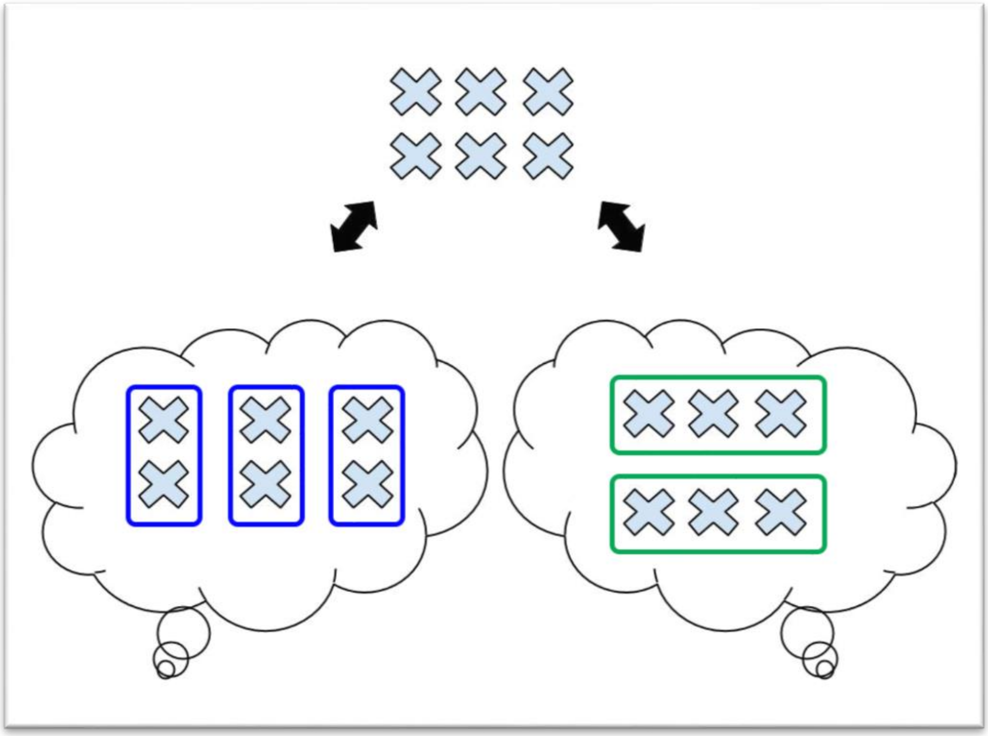


Figure 2. The referent of six objects could be visualized as three groups of two, or two groups of three. Unlike Figure 1, these two visualizations are conceptually complementary in the sense that they are partial meanings of the same mathematical object (Abrahamson & Wilensky, 2007).

One may interpret the six objects and insist that “there are three groups of two.” Another may view this image and be convinced that “there are two groups of three.” These two visualizations are different, but they are not conceptually contradictory. Rather, they are conceptually *complementary* in nature because they are both correct; the viewer is simply attending to different aspects of the same whole. For mathematical objects, the complementary conceptualizations of each individual may capture a partial meaning of the bigger object (Godino et al., 2011), leading to complementary inferences about the object (Abrahamson, et al., 2009).

It is worth noting that in these examples, I have spoken about discourse as occurring between at least two interlocutors; however, it is more than common that a single individual can hold alternating visualizations as well (Abrahamson et al., 2009), especially when student thinking can be conceptualized as discourse itself (Sfard, 2002).

Next, we examine the importance of visualizations in regards to mathematical inferences in the classroom setting. Students’ ambiguous and shifting meanings assigned to mathematical inscriptions should not be viewed as obstacles but as rich resources for teachers to build and connect multiple interpretations to canonical mathematical concepts and practices (Moschkovich, 2008). A teacher’s challenge includes allowing students to wrestle with their own generated questions as well as treating concepts explicitly such that students can actively use mathematical language to communicate and negotiate meaning. The principle of building a discursive whole that is larger than the sum of it parts is discussed also by Asterhan and Schwarz (2011), who demonstrated the conceptual affordances of structured, collaborative dyadic argumentation..

As rich as the current literature is on instructors' discursive actions in fostering learners' perception of visual displays germane to their academic and vocational practices (Abrahamson, 2009; Alaç & Hutchins, 2004; Goodwin, 1994; Sfard, 2007; Stevens & Hall, 1998), I propose that there is still room for examining the cognitive mechanisms and discursive processes in students' collaborative problem solving process in an authentic classroom activity. I argue that strong conceptual change occurs when teachers create environments in which groups of students are able to meaningfully cultivate and weave initially contradictory visualizations into eventually coordinated meanings. Specifically, I posit that teacher utterances play pivotal roles in evoking and steering student visualizations, thus mobilizing classroom conversations toward productive negotiation of visualizations, whether conceptually complementary or contradictory.

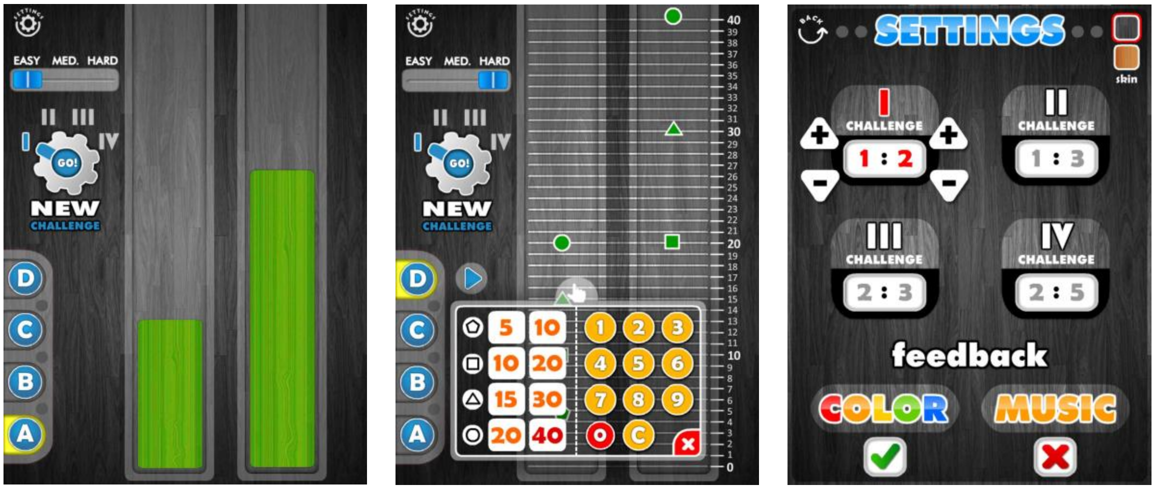
Data Source: a Digital Design for Proportion

This thesis builds on empirical data generated in the context of a design based research project that investigated the emergence of mathematical concepts from embodied learning activities. The Embodied Design Research Laboratory (EDRL) conjectures that certain mathematics concepts are difficult for students to learn because every day experiences do not provide situations on which to ground their knowledge. Thus, the research team both designs and develops theories of learning; in this design, the research team has developed an embodied learning activity for proportion which we call the Mathematics Imagery Trainer for Proportion.

The research team designed an app (built by Terasoft[1]) on the Apple iPad for the Mathematics Imagery Trainer for Proportion (MIT-P). The MIT-P iPad application, the third iteration of the MIT-P series, is intended to support mathematics teachers or tutors in facilitating discovery-based instruction of proportion. It is not designed to be a stand-alone activity; rather the research team expects that a teacher and students interact around this technology.

When students activate the application, they encounter a screen with two red vertical bars, whereupon the instructor challenges them to "make the bars green" (see Figure 3a). Unknown to the student, the bars will stay green only when their relative heights correspond to a pre-set ratio, such as 1:2. There are interface functionalities for controlling the performance tolerance (see the EASY-MEDIUM-HARD slider in the top-left of the screen in Figures 3a & 3b). Other widgets control the selection of new ratio challenges (see the I-II-III-IV dial in Figures 3a & 3b). The MIT-P iPad has integrated interaction "modes" consisting of gridlines, numerals, and a ratio-table (see the A-B-C-D buttons on the bottom-left of Figures 3a & 3b). The ratio table in Mode D, which we call the "driver," enables users to input a set of four number pairs that each, in turn, control a pair of shapes (pentagon, square, triangle, and circle; see Figure 3b). Finally, a Settings button enables the user to go "under the hood" and change the ratio settings of the four 1-II-III-IV challenges (see top-left corner of Figures 3a & 3b). Not shown in Figure 3 are further information pages, one with an overview of the design and another with detailed suggestions for teachers to use the design with their classroom students.

[1] Free download at Apple App Sore:
https://itunes.apple.com/au/app/mathematical-imagery-trainer/id563185943?mt=8&ign-mpt=uo%3D2



a. *b.* *c.*

Figure 3. Three phases of interaction with the Mathematics Imagery Trainer for Proportion on an iPad tablet (MIT-P iPad) are shown: (a) Mode A with two bars which turn green when their relative heights correspond to a hidden ratio; (b) Mode D with gridlines, numerals, and a ratio-table; and (c) The settings page, where ratios and other interaction features can be modified.

Methods

A research team analyzed a corpus of videotapes collected during classroom implementations of the MIT-P iPad. The classroom implementations took place in a small public school in the San Francisco Bay Area. The school serves a small student population of ~270 students, 84% ethnic minority students, 67.2% free or reduced lunch enrolled students, and 16.6% English Learners. Two periods of an Algebra support class participated in the study, the first and second periods consisting of 18 and 16 students, respectively.

The 70-minute lesson plan included several components: the class Warm-Up with content unrelated to the intervention, an introduction to the task, the group activity, small group presentations to the whole class, and an individual writing assessment. First, the co-researchers introduced the embodied-learning group activity. One researcher requested a student volunteer to “make the bars green” in front of the class while the document camera projected students’ interactions with the iPad to the class. Second, students were prompted with “your goal is to make the bars stay green the entire time in as many ways as your group can.” Students then explored the MIT-P 3 iPad activity in groups of 2-4 students while one teacher, two researchers, and one class volunteer floated from group to group, facilitating team discussions and asking probing questions. After students had sufficient time to interact with the design and articulate their reasoning, the researchers facilitated student group presentations where students in each group gave an interactive explanation of how they made the bars green. Finally, students participated in a short, written individual evaluation before they left the class, commonly referred to as an “exit ticket.”

After data collection was complete, a research team collaboratively engaged in reflective micro-ethnographic qualitative analysis in an attempt to interpret the students’ actions and utterances. (Nemirovsky, 2011; Schoenfeld, Smith, & Arcavi, 1991; Siegler & Crowley, 1991; Parnafes & diSessa, 2013).

Results and Discussion

This paper reports on results from a study that investigated classroom activities through the theoretical lens of visualization. In particular, the team analyzed cognitive mechanisms and discursive processes implicit in students' collaborative problem solving by attempting to determine how individual interlocutors visualized a shared referent in their perceptual field and how utterances about these visualizations mobilized the discussion in directions that the teacher evaluated as aligned with the activity's pedagogical objectives. Collaborative micro-ethnographic analyses of selected episodes led to the emergence of two types of relation between interlocutors' respective visualizations: conceptually complementary and conceptually contradictory. These categories were then used to code and evaluate the effect of the instructors' mediatory utterances, as they attempted to facilitate productive conversation.

The following three vignettes selected from our empirical data serve as focal points for examining how teachers work with students' visualizations. In particular, we will look at how practical moves can foster productive student argumentation in cases when student visualizations are conceptually contradictory or when students position their own visualizations as superior.

Twenty Seven and Nine: A Case of Coordinating Complementary Visualizations

We begin with a case of a group of four students in the exploration phase who were able to make the screen green under all the challenge settings (ratios 1:2, 1:3, 2:3, and 2:5), albeit they did so without necessarily attending to mathematical construction relations between the bars. After the group articulates a strategy for the first challenge of ratio 1:2, the researcher, RGL, asks the students to articulate a rule for Challenge II, a 1:3 ratio. Attempting to comply, two of the students—Daniel and Alberto—are working in Mode C (grid and numerals; all names here and throughout the thesis are pseudonyms).

Researcher Rosa Lee (RGL), Daniel (Dan), Alberto (Al):

RGL: Alright, what's it telling you? *[Alberto finds green at grid readings 9 and 27.]*

Dan: It's saying, 27 to 9. What? 9. *[To Alberto.]* I don't know what the heck…

Al: *[Without touching the green bars, Alberto taps on an interface feature that changes the grid's unit calibration. The grid readings rotate through several proportional ratios, e.g., 18:54, 36:108, and then back to 9:27]* *[Unintelligible]* So 27 and 9. It's like, um, 3, but, it's like, 3, 1, 3. It's like, it's um, 3 quarters. I don't know. It's like 1...

Dan: It's like 3 times 9 you get a…*[unintelligible] //*

Al: *[Counts from 9 to 18]* // One two three four five six seven nine. It right there. *[Counts from 18 to 27]* One two three four five six seven eight nine. Then it's right there.

Dan: See it's 9 times 3! And then there's 3 right there. *[Gestures 3 iterated hand waves.]*

Al: *[With finality.]* 9 times 3 twenty seven.

Figure 4. From left to right, students Alberto and Daniel are interacting with the MIT-P iPad on a 1:3 ratio. Alberto gestures as he says, “It’s like, um, 3, but, it’s like, 3, 1, 3.”

What begins as an interaction with differing visualizations develops into a delicate weaving of understandings. Alberto begins with an intuitive sense of rhythmic counting and “3 quarters.” Daniel, on the other hand, proposes a multiplicative visualization of “3 times 9.” Both students seem unsure until Alberto counts from Gridline 9 to 18 and 18 to 27. In this moment, both students verify and ground their respective visualizations in iterated addition and confirm a multiplicative strategy.

This excerpt speaks to the value of probing students to articulate their reasoning and seeing. Researcher RGL prompts the student group in an open forum to generalize how students make the bars green at heights 9 and 27. If it were not for Alberto’s verbal utterances, Daniel would not have jumped on the chance to speak about his own special way of viewing the same object—in this instance, multiplicatively. Alberto was struggling to articulate his additive visualization, while Daniel speculated on the validity of his own multiplicative visualization. One might argue that Alberto drops his intuitive sense of rhythmic counting. However, I argue that Alberto was given the opportunity to ground multiplicative reasoning in repeated addition.

Two Thirds or Three Fourths: A Case of Coordinating Conflicting Visualizations

Next, I discuss a group of students who articulated conceptually complementary as well as conceptually contradictory visualizations, which the researchers then attempted to steer toward pedagogical goals. The group of students is up at the front of the class, presenting their work. Their iPad screen is projected onto the board for all other students to see. At this point, all students have had a chance to explore the application in small groups and have been prompted either by the researchers (Andrea Negrete & Rosa Lee) or teacher (Mr. Lam) to explain how they “made green.” As Yesenia begins to present, the tablet is set at a 2:3 ratio, with left bar at Gridline 6 and right bar at 9. Yesenia has difficulty articulating her actions. She is helped by a student in the audience and a fellow group member, who occasionally interject with their own visualizations.

Figure 5. From left to right, students Yadira, Yesenia, and Carlos present their strategies for making green. Carlos is seated at a desk with a tablet that is projected onto the screen by a document camera.

Researcher Rosa Lee (RGL), Yesenia (Yes), Student (St), Carlos (Carl):

Yes: Um, so since it's 9 *[points at 9]*, and then, it, it's like, *[using two fingers to parse the Right Bar]//*

St: *[Speaks out from audience]* //Three.//

Yes: //**2 quarters** of, of 9, so it's like **3 quar**--*[holds up three fingers, then stops mid speech, places hand on table, frustrated sigh]*

St: *[Shouts out from the audience]* It's a **multiple of three**//

Yes: //Yeah.//

Carl: //It's like **three fourths**.

St: *[Shouts out from the audience, confidently]* **Three, six, nine**.//

Carl: *[Repeats what he said earlier to Yesenia.]* //**Three fourths**.

Yes: **Three f--** *[hesitates, then speaks to Carlos.]* Noo.

Carl: *[Quietly to Yesenia.]* Yeah! That's **three fourths**.

Yes: *[To Carlos, quietly.]* No, this, **two thirds** *[holds up two fingers, slaps hand on table.]* 'Cause it's **three**.

Carl: *[Raises his eyebrows.]*

RGL: Tell us more.

Yes: *[To the whole class, loudly.]* Oh, um, so we just, like, figured out that either way, that it's gonna be the same, we're gonna keep green if we keep it as, like, two, like if it's...if it's 18 it's gonna be **like two quarters** of of how much 18 is so like **two thirds**, I guess?

As Yesenia is explaining how their group made green, she becomes confused. A student in the audience shouts out his own view. Yesenia agrees to these contributions, until her fellow group member suggests “three fourths.” A short moment passes in which Carlos and Yesenia disagree over “three fourths,” with Carlos offering an opaque warrant for his claim.

Immediately after, AGN prompts Yesenia to reflect on her conflicting claims, two thirds and three fourths.

Researcher Andrea Negrete (AGN), Yesenia (Yes):

AGN: So, can you explain to us a little more about how you, how you found out it was like **two quarters or two thirds**?

Yes: Just by looking at 9, at like, when we have it at 9, and then, it’s at 6. So, so like, **3 times 2 equals 6**...basically.

AGN: 3 times 2 equals 6?

Yes: So it stays at 6, and then, when it’s at, when it’s at 9, so yeah. I guess.

AGN: Okay.

This excerpt is a case of attempts at coordinating many student visualizations toward the instructors’ pedagogical goals. I would argue for the ultimate utility of teachers’ discursive interventions that do not evaluate students’ visualizations as correct or incorrect but, rather, validate each visualization as meaningful and identify areas for elaboration. AGN was able to validate both of Yesenia’s visualizations, two thirds and three fourths, even though at face value clearly these constructions are conceptually contradictory so that only one of them could be correct. And yet for Yesenia, each of these conflicting mathematical inferences was grounded in at least partially meaningful visual constructions—they both made some sense to her as candidate responses.

Yet Yesenia was able eventually to coordinate two conceptually complementary visualizations: (a) her own budding visualization of the 9-bar as parsed into several equal sub-units; and (b) the “multiples of three” visualization that was originally contributed by a student in the audience. In so doing, Yesenia was selective in considering visualizations that complemented her own pre-symbolic visualizations of the referent at hand. In fact, she evaluated Carlos’s suggestion of “three fourths” as a conceptually contradictory visualization, so much so that the two students privately engaged in a brief argument.

The brief exchange between Carlos and Yesenia was only caught in the later analysis of the recorded video; the two students spoke much too quietly for the teacher and researchers to hear let alone steer and reconcile. However, I propose that rich conceptual understanding can emerge when a teacher is able to identify moments such as these, validate each students’ visualizations, especially if they are conceptually competing visualizations, and orchestrate students’ productive argumentation and coordination. It is the teacher’s role to pull student inferences into a common space for students to grasp, interpret, coordinate, and synthesize. In like vein, when a single student expresses conceptually contradictory constructions of a visual display, teachers would do well to appreciate that each visualization carries for the student some grain of truth. Rather than force the student to select one, the teacher should explore both.

What’s 7 times 3?: A Case of Validating Student Visualizations

Last, we return to the student group from the first vignette, Daniel, Alberto, and friends. The current vignette demonstrates a case of conceptually complementary visualizations that would not have been cast into the group’s discursive space if it were not for the teacher’s validation of students’ visualizations. Although different visualizations emerged spontaneously, strong instructional steering was still required so as to enable the group to nurture their diverse visualizations into mathematical meaning that are greater than the sum of its parts. I view the teacher’s discursive tactic as paradigmatic of interventional techniques for reconciling among visualizations that differ in their surface features yet share enough deep structure so as to warrant productive negotiation.

Quinelle is assisting Alberto in presenting the groups’ work to the whole classroom. Prior to this event, Quinelle’s interaction with the group during the explorative activity is worthy of noting. He had interacted with the tablet application only for a few minutes. For the most, he had observed Alberto, Daniel, and Michael as they worked on the problem. Thus, Quinelle’s experience with the embodied design was for the most vicarious.

Figure 6. From left to right, students Quinelle and Alberto manipulate the tablet as they are presenting how they make green at the front of the class.

Now at the presentation, Alberto is explaining a multiplicative strategy for the 1:2 ratio setting while demonstrating with the tablet. Mr. Lam proposes that another group member present to the class, and Alberto tries unsuccessfully to coax Michael to take a seat at the desk. Finally, Mr. Lam proposes a compromise: Alberto may present the strategy as long as Quinelle demonstrates it with the tablet. Quinelle then takes a seat and control of the device. RGL then continues to moderate the presentation, with the ratio soon set at 1:3 with the grid and numerals showing.

Researcher Rosa Lee (RGL), Researcher Andrea Negrete (AGN), Quinelle (Quin), Alberto (Al), Student (St):

RGL: Okay, so as you're moving them, why don't you tell us what you're thinking. How are you moving these? Say it out loud so everyone can hear.

Quin: Me?

RGL: Both of you!

Al: Well, it's um, 7...then... *[Al. is on Mode C, Challenge II. He makes green at 7:21]*

Quin: Wait. Hold on, let me try to think. *[To Alberto]* Move your fingers. Move your fingers. Let's see. *[Counts with his fingers.]*

Al: *[Looks up toward audience.]* Wait, what's 7 times 3?

RGL: What's 7 times 3 you guys?

St: 21?

RGL: Okay, 21.

Al: Oh yeah. So this one it's always times three 'cause it's on 7...// *[Backs away from the iPad over to the projected screen to point.]*

Quin: So, hold on hold on, wait. //

Al: //Right there then times three is //

Quin: //So whatever, whatever number you put it on and you times it and whatever answer you get you you put it on and it might turn green.

RGL: //Okay.//

AGN: //Wait wait, explain that one more time? Whatever number you get on on *which* bar and what number do you get on the other bar?

Al: Like, if it like it's on 7 times 3...it's...//

Quin: //Whatever answer you get (then/and) it would change green.

Figure 7. From left to right, student Quinelle and Alberto are simultaneously giving their own distinct explanations to the audience.

Both students' visualizations are multiplicative, in the sense that they attend to the multiplicative relation between 7 and 21. At the same time, these visualizations differ in a nuanced way. Alberto's visualization is "arithmetic–relational," in the sense that he performs a multiplicative comparison between two pre-existing quantities. That is, he views both the 7 and 21 as "givens," and he seeks to determine their factor. However Quinelle's visualization is "functional," in the sense that he sees the left bar as an "input" from which to calculate the right-bar "output." AGN was alert to this subtle distinction between these multiplicative visualizations and took measures to surface this subtlety. In so doing, AGN enabled both students as well as the entire classroom to validate and reconcile these conceptually complementary visualizations. In the absence of AGN's moderation the two students may have continued scuffle with each other for their own chance to shine, whereas in fact their scuffle was trivial. For instance, Quinelle asks Alberto twice to "hold on, wait," so that he can articulate his own reasoning.

Conclusion

This study examined classroom discourse from the perspective of student visualization as well as the effects of instructors' mediatory utterances as they attempt to facilitate productive conversation among students bearing different visualizations. Individual students learn mathematical content via generating inferences that integrate competing meanings into cohesive conceptual networks (Abrahamson, Lee, Negrete, Gutierrez, 2013). When groups of students come together and speak about common referents in the world, a diversity of visualizations arises, because each of the students applies their idiosyncratic perceptual resources to make sense of the ambiguous mathematical objects. Discourse around mathematical objects both elicits multiple individual visualizations and, ideally, ushers negotiation and reconciliation of these different visualizations into new meanings that do not reside in any individual visualization but rather emerges from their latent relationships. As we say, the whole is bigger than the sum of the parts. Thus in the mathematics classroom, ambiguity is not a curse but a virtue—ambiguity is where the conversation begins (Foster 2011; Mamolo 2010; Newman et al. 1989; Rowland 1999; Sfard 2002).

The teacher plays a major role in both cultivating a culture of tolerance for competing visualizations and fostering discursive norms for availing of perceptual conflict. In leading students to new ways of seeing, the skilled teacher exercises formative assessment while at the same time instantiating constructivist pedagogy. Moreover, by eliciting multiple visualizations from *individual* students, the teacher is also constantly on the watch for opportunities to usher , frame, and orchestrate productive negotiations *among* students.

Although this study was modest in scale and experimental design, I argue that it constitutes a generative case study (Clement, 2000) and, as such, warrant careful attention. In particular, this study points to promising directions for future research as well as implications for practicing teachers.

Implications for Teachers, Designers, and Future Research

How can teachers foster and weave visualizations together such that students do not discard their own intuitive senses nor reject other students' ways of seeing? Perhaps to do so, *teachers* should begin by understanding and validating *student* visualizations rather than sticking to their normative ways of seeing the world. Moreover, teachers should develop visualization-based formative-assessment skills as a means of moderating group work. These professional practices may foster both grounded content learning and effective argumentation practices.

Student and teachers should seek a common ground in which meanings are woven together. When faced with multiple visualizations, teachers, rather than selecting one they are most familiar with and acting on it, should leverage a plurality of visualizations to enrich student discourse and spur student argumentation. Similarly, students should be encouraged to validate each other's visualizations, rather than dropping interpretations they deem unintelligible and/or inferior. All participants to mathematical conversations should acknowledge and valorize all other visualizations as legitimate and worthy contributions. In this perspective, student diversity in a mathematics classroom is seen as a strength for building meaningful mathematics discussions. It takes a village. But it also takes a teacher.

In summary, this thesis is far from comprehensive. Rather, we presented and discussed conceptually complementary and contradictory visualizations of mathematical objects and examined the pedagogical potential inherent in these perceptual conflicts for fostering productive student argumentation. Moving forward, we continue asking ourselves:

- How do we foster these moments?
- How do we then identify them?
- How do we leverage the richness of diverse visualization toward our pedagogical goals?

In ending, we step back from the classroom scene to address those who create learning materials for this scene, namely educational designers. I argue that pedagogical materials are not created equal. Some materials are better fit for eliciting multiple visualizations conducive of productive negotiation (e.g., see Abrahamson et al., under review). Teachers and designers may wish to pursue not only multiple representations, but representations that each allows for multiple visualizations.

Acknowledgements

I would like to give my deepest gratitude and thanks to my advisor, Professor Dor Abrahamson, for his endless support and profound guidance; my second reader, Professor Alan Schoenfeld, for his thoughtful comments; Marcus Hung for his collaboration and support in this project; and last but not least, my most supportive and dependable research partner and friend, Andrea G. Negrete.

References

Abrahamson, D. (2009). Orchestrating semiotic leaps from tacit to cultural quantitative reasoning—the case of anticipating experimental outcomes of a quasi-binomial random generator. *Cognition and Instruction, 27*(3), 175-224.

Abrahamson, D., Bryant, M. J., Gutiérrez, J. F., Mookerjee, A. V., Souchkova, D., & Thacker, I. E. (2009). Figuring it out: mathematical learning as guided semiotic disambiguation of useful yet initially entangled intuitions. In S. L. Swars, D. W. Stinson & S. Lemons-Smith (Eds.), *Proceedings of the Thirty-First Annual Meeting of the North-American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 662-670). Atlanta, GA: Georgia State University.

Abrahamson, D., Gutiérrez, J. F., & Baddorf, A. K. (2012). Try to see it my way: the discursive function of idiosyncratic mathematical metaphor. *Mathematical Thinking and Learning, 14*(1), 55-80.

Abrahamson, D., Gutiérrez, J. F., Charoenying, T., Negrete, A. G., & Bumbacher, E. (2012). Fostering hooks and shifts: tutorial tactics for guided mathematical discovery. *Technology, Knowledge, and Learning, 17*(1-2), 61-86. doi: 10.1007/s10758-012-9192-7

Abrahamson, D., Lee, R. G., Negrete, A. G., & Gutiérrez, J. F. (submitted). Coordinating visualizations of polysemous action: values added for grounding proportion. In F. Rivera, H. Steinbring, & A. Arcavi (Eds.), Visualization as an epistemological learning tool [Special issue]. *ZDM: The international Journal on Mathematics Education*.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215-241.

Asterhan, C. S. C., & Schwarz, B. B. (2009). The role of argumentation and explanation in conceptual change: Indications from protocol analyses of peer-to-peer dialogue. *Cognitive Science, 33,* 373-399,.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates.

Foster, C. (2011). Productive ambiguity in the learning of mathematics. *For the Learning of Mathematics, 31*(2), 3-7.

Godino, J. D., Font, V., Wilhelmi, M. R., & Lurduy, O. (2011). Why is the learning of elementary arithmetic concepts difficult? Semiotic tools for understanding the nature of mathematical objects. In L. Radford, G. Schubring, & F. Seeger (Eds.), Signifying and meaning-making in mathematical thinking, teaching, and learning: semiotic perspectives [Special issue]. *Educational Studies in Mathematics, 77*(2), 247-265.

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96*(3), 603-633.

Isaacs, E. A., & Clark, H. H. (1987). References in conversations between experts and novices. *Journal of Experimental Psychology: General, 116*(26-37).

Mamolo, A. (2010). Polysemy of symbols: signs of ambiguity. *The Montana Mathematics Enthusiast, 7*(2&3), 247-261.

Moschkovich, J. N. (2008). "I went by twos, he went by one." *The Journal of the Learning Sciences, 17*(4), 551-587.

Nemirovsky, R. (2011). Episodic feelings and transfer of learning. *Journal of the Learning Sciences*, 20(2), 308-337.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: working for cognitive change in school* (Chapter 4). New York: Cambridge University Press.

Parnafes, O., & diSessa, A. A. (2013). Microgenetic learning analysis: a methodology for studying knowledge in transition. *Human Development*, 56(1), 5-37. doi: 10.1159/000342945

Rowland, T. (1999). Pronouns in mathematics talk: power, vagueness and generalisation. *For the Learning of Mathematics, 19*(2), 19-26.

Schoenfeld, A. H., Smith, J. P., & Arcavi, A. (1991). Learning: the microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55-175). Hillsdale, NJ: Erlbaum.

Siegler, R. S., & Crowley, K. (1991). The microgenetic method: a direct means for studying cognitive development. *American Psychologist*, 46(6), 606-620.

Sfard, A. (2002). The interplay of intimations and implementations: Generating new discourse with new symbolic tools. *Journal of the Learning Sciences, 11*(2&3), 319-357.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you - making sense of mathematics learning from a commognitive standpoint. *Journal of Learning Sciences, 16*(4), 567–615.

Stevens, R., & Hall, R. (1998). Disciplined perception: learning to see in technoscience. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: studies of teaching and learning* (pp. 107-149). New York: Cambridge University Press.

Appendices

Appendix A—Transcript for Vignette 1: Twenty-Seven and Nine: A Case of Coordinating Complementary Visualizations

Researcher Rosa Lee (RGL), Daniel (Dan), Alberto (Al):

RGL: Alright, what's it telling you? *[Alberto finds green at grid readings 9 and 27.]*

Dan: It's saying, 27 to 9. What? 9. *[To Alberto.]* I don't know what the heck…

Al: *[Without touching the green bars, Alberto taps on an interface feature that changes the grid's unit calibration. The grid readings rotate through several proportional ratios, e.g., 18:54, 36:108, and then back to 9:27]* *[Unintelligible]* So 27 and 9. It's like, um, 3, but, it's like, 3, 1, 3. It's like, it's um, 3 quarters. I don't know. It's like 1...

Dan: It's like 3 times 9 you get a…*[unintelligible] //*

Al: *[Counts from 9 to 18]* // One two three four five six seven nine. It right there. *[Counts from 18 to 27]* One two three four five six seven eight nine. Then it's right there.

Dan: See it’s 9 times 3! And then there’s 3 right there. [Gestures 3 iterated hand waves.]
Al: [With finality.] 9 times 3 twenty seven.
RGL: Okay. Can you do other numbers besides 9 and 27?
Al: Yeah. Yeah.

Appendix B-- Two Thirds or Three Fourths: A Case of Coordinating Conflicting Visualizations

Researcher Rosa Lee (RGL), Yesenia (Yes), Student (St), Carlos (Carl):

Yes: [To group member] Hold up hold up hold up. Okay. So there. [Yesenia makes green using 1:3 ratio on Challenge II, then switches into Challenge III and attempts to make green.]
AGN: Whoa whoa whoa. You’re—you’re doing a lot of things but you’re not talking.//
Yes: //Yeah, I know, ‘cause then, we’re trying, we’re trying to figure out which one we did. [Makes green on 2:3 ratio on Challenge III with Left Bar at 6 and Right Bar at 9.]
Yes: Okay. So on that one, we just did, for us, this is how we, we looked at it. Um, so since it’s 9 [points at 9], and then, it, it’s like, [using two fingers to parse the Right Bar]//
St: *[Speaks out from audience]* //Three.//
Yes: //**2 quarters** of, of 9, so it’s like **3 quar**--*[holds up three fingers, then stops mid speech, places hand on table, frustrated sigh]*
St: *[Shouts out from the audience]* It’s a **multiple of three**//
Yes: //Yeah.//
Carl: //It’s like **three fourths**.
St: *[Shouts out from the audience, confidently]* **Three, six, nine**.//
Carl: *[Repeats what he said earlier to Yesenia.]* //**Three fourths**.
Yes: **Three f--** *[hesitates, then speaks to Carlos.]* Noo.
Carl: *[Quietly to Yesenia.]* Yeah! That’s **three fourths**.
Yes: *[To Carlos, quietly.]* No, this, **two thirds** *[holds up two fingers, slaps hand on table.]* ’Cause it’s **three**.
Carl: *[Raises his eyebrows.]*
RGL: Tell us more.
Yes: *[To the whole class, loudly.]* Oh, um, so we just, like, figured out that either way, that it’s gonna be the same, we’re gonna keep green if we keep it as, like, two, like if it’s...if it’s 18 it’s gonna be like two quarters of of how much 18 is so like two thirds, I guess?
RGL: //Two quarters and two thirds.//
Yes: //Yeah. Yeah.
AGN: So, can you explain to us a little more about how you, how you found out it was like **two quarters or two thirds**?
Yes: Just by looking at 9, at like, when we have it at 9, and then, it’s at 6. So, so like, **3 times 2 equals 6**...basically.
AGN: 3 times 2 equals 6?
Yes: So it stays at 6, and then, when it’s at, when it’s at 9, so yeah. I guess.

AGN: Okay.

Appendix C—Transcript for Vignette 3: What's 7 times 3?: A Case of Validating Student Visualizations

Researcher Rosa Lee (RGL), Researcher Andrea Negrete (AGN), Teacher Mr. Lam (Mr. L), Quinelle (Quin), Alberto (Al), Student (St):

Al: *[A clears throat dramatically into the microphone.]*
RGL: Alright, yeah, make sure we hear you nice and loud. Okay.
Al: Okay guys. *[Playfully. Opens up iPad under document camera.]* What I noticed here in this iPad uh, 2--is it? Um, // //excuse me, class?
AGN: //Undivided attention!//
Mr. L: Alberto.
Al: *[More seriously.]* Yeah. Um, for number 1, you have to uh, always be half, because look, if it's not half, it it won't like, turn green. Like--if I put 30... //
St: //Can you move your head out of the way?//
Al: //SH! Oh-sorry! If I put 30, then the half of 30 is...? *[Waits dramatically.]*
St: Why don't you use an even number?
Al: *[Loud stage whisper]* 15! Oh! Yeah, it goes to 15. So it turns green. And if I, if I go to 40, half of 40 is? *[Dramatic pause.]*
St: 20.
Al: Wow! Wow. Right. There! You see, it keeps green.
St: I like your *[unintelligible]*.
Al: Thanks! Sandra gave it to me. And then for number, for number 2,
Mr. L: Alberto, why don't you let someone else explain.
Al: *[Alberto glares at Mr. Lam. Leaves desk resentfully.]*
[Alberto tries unsuccessfully to lead another teammate to the desk. Mr. Lam proposes that Quinelle controls the iPad while Alberto explains. Quinelle then takes a seat.]
RGL: Okay, so as you're moving them, why don't you tell us what you're thinking. How are you moving these? Say it out loud so everyone can hear.
Quin: Me?
RGL: Both of you!
Al: Well, it's um, 7...then... *[Al. is on Mode C, Challenge II. He makes green at 7:21]*
Quin: Wait. Hold on, let me try to think. *[To Alberto]* Move your fingers. Move your fingers. Let's see. *[Counts with his fingers.]*
Al: *[Looks up toward audience.]* Wait, what's 7 times 3?
RGL: What's 7 times 3 you guys?
St: 21?
RGL: Okay, 21.
Al: Oh yeah. So this one it's always times three 'cause it's on 7...// *[Backs away from the iPad over to the projected screen to point.]*
Quin: So, hold on hold on, wait. //
Al: //Right there then times three is //
Quin: //So whatever, whatever number you put it on and you times it and whatever answer you get you you put it on and it might turn green.

RGL: //Okay.//
AGN: //Wait wait, explain that one more time? Whatever number you get on on *which* bar and what number do you get on the other bar?
Al: Like, if it like it's on 7 times 3...it's...//
Quin: //Whatever answer you get (then/and) it would change green.