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Like Plato's allegorical cave-dwellers, students of three-dimensional geometry seldom get to handle the real thing, working instead with two-dimensional silhouettes. Such historical sensory deprivation may partially explain students' generally poor conceptual understanding of this core content and alienation from the field. Operating from a perspective of embodied learning, our design-based research study invited middle-school students to collaboratively construct and investigate voluminous objects. We present qualitative analyses of empirical results from implementing two experimental geometry activities. For both cases, we characterize students' critical insights as shifts in perceptuomotor attention leading to refinement of geometric argumentation. We implicate students' realization of an available 3D medium affordances catalyzing these shifts. The findings contribute to a socio-material elaboration of embodied learning for school geometry.

Keywords: Spatial geometry, Embodiment, Manipulatives, Visualization

Introduction

Historically, the mathematical discipline of geometry originated from mundane practice—situated, embodied know-how serving personal, social, and professional contexts, such as carpentry, agriculture, and navigation. However, while humans live and act in 3D space and engage voluminous objects as part of this naturalistic spatial comportment, geometry scholarship—the reification and scrutiny of these objects' structural properties—has historically depended on 2D material media and their consequent "flat" perspectives (Alsina, 2010). This dependence has a price. For example, a recent study by Fujita et al. (2020) documents elementary- and middle-school students' poor performance on spatial geometry problems, especially those requiring multiple reasoning steps.

Scholars from different disciplines have claimed that if students began studying geometry with their embodied sensibilities, we could preempt their poor engagement and low performance in the discipline (Freudenthal 1971; Thompson, 2013). Pedagogical advocacy to work with "the thing itself" harks back to Enlightenment (Rousseau, 1755/1979; Froebel, 1885/2005) and modernity (Montessori, 1949/1967), bolstered by Gaspard Monge and Felix Klein's approach to the development of intuition about complex structures through the construction of concrete models (Mueller, 2001). Still, after over 60 years of empirical research on the use of manipulatives in mathematical classrooms, their cognitive effects and optimal utilization have yet to be established (Bartolini Bussi et al., 2010).

We argue that realizing a material transformation in (spatial) geometry instruction, from 2D to 3D, demands a paradigmatic epistemological shift from idealistic to realistic views on geometry; from a representationalism model that separates perception, cognition, and action to an embodied model.

Cognitive argument: from a perception-cognition-action model to embodied cognition

Per Plato, she who seeks mathematical knowledge should strive to obtain mental representations as close as possible to the ideal (non-physical) forms. Millennia later, Platonic metaphysics persevere through cognitive-science theoretical models that box the mind inside the skull as an amodal ethereal switchboard between its earthly perception input and action output (Hurley, 2010). Over the recent decades, however, cumulative data from various fields (neurobiology, robotics, kinesiology) is casting doubt on this classical model (e.g., Willems & Francken, 2012).

The embodied turn in cognitive science rejects the hierarchical mind–body separation and stresses that perception and action are formatively constitutive of our thinking—cognition is modal and situated activity (e.g., Chemero, 2013). The mind's function is not to *represent* the environment precisely but to *engage* with it dynamically vis-à-vis socio–biological task demands and emergent contextual contingencies. The environment offers opportunities for potential action—*affordances* (Gibson, 1986)—that the agent interactively discerns and incorporates. When we engage the world with fellow humans, we coordinate with them perceptual orientations in relation to shared situations, from early development (Tomasello, 2019) and through to professional practice (Goodwin, 2018).

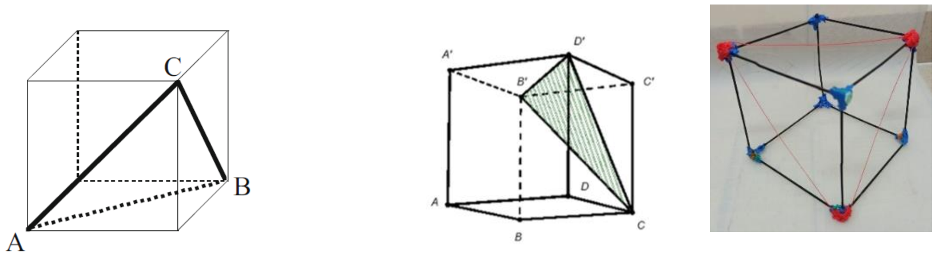
Imported to mathematics instruction, tenets from the embodied paradigm of the cognitive sciences suggest that learning new concepts begins with discovering new ways to act in the environment, using new instruments to perform tasks on discovered affordances (Abrahamson & Bakker, 2016). Working with the things themselves, students develop a capacity to act efficiently, describe the world mathematically to coordinate collaborative actions, iteratively encounter more complex problems, and ultimately modify the environments to solve emergent problems (Abrahamson et al., 2020).

Spatial geometry challenges and possible theoretical and practical solutions

Roth and collaborators demonstrated that geometry knowledge may emerge as enacted exploration of concrete instructional resources. Their examination of primary-school students' classification of geometric objects fuses material phenomenology and phenomenological sociology to view geometry as cultural–historical motivated sensuous labor (e.g. Bautista & Roth, 2012). Yet, whereas overtly embodied routines, such as gesturing, manipulating objects, and applying mechanical construction tools, are considered essential for young students' geometric reasoning and problem solving (Kaur & Sinclair, 2014), these resources almost disappear from older schoolchildren's instructional activities. Perhaps most acutely, mainstream spatial geometry education skips "the real thing," immediately requiring of students to visualize 3D objects given their 2D representations (Widder et al., 2019). Consequently, students struggle "to overcome the perceptual appearance (or 'look') of the given diagram" (Fujita et al., 2020, p. 235). For instance, for one of their survey items (see Figure 1, on the left), just 17 % of the 5th-grade students, 34% of 7th-grade students, and 52% of 9th-grade students marked the correct answer (percentages rounded). In light of their results, Fujita et al. (2020) call to revise primary and secondary school curriculum to provide students with more opportunities to develop both spatial skills and geometric knowledge for productive argumentation. However, the researchers do not indicate what types of tasks could possibly serve as context for realizing this call.

Investigating a 3D DGE (dynamic graphic environment), Mithalal and Balacheff (2019) explored conditions in which construction tasks stimulate students' transition from working with drawings and iconic visualizations to perceiving geometric properties of figures and non-iconic visual displays.

They claim that this transition depends on students’ ability to perform certain figural operations (dimensional and instrumental deconstruction) in tasks designed to hone this ability.



In a cube, can you identify the shape ABC? Choose your answer from (a) – (e).

(a) Right-angled triangle
(b) Isosceles triangle.
(c) Right-angled isosceles triangle
(d) Equilateral triangle
(e) Scalene triangle

ABCDA’B’C’D’ is a cube. Answer the true/ false questions and explain your reasoning.

1. CB’D’ is a right-angled triangle. The right angle is ___?
2. B’D’ is the shortest side of the triangle CB’D’.
3. Triangle CB’D’ has an obtuse angle.
4. In triangle CB’D’, all angles are equal.

Figure 1: Tasks involving 2D representations of 3D geometrical shapes

With Fujita et al. (2020) and Mithalal and Balacheff (2019), we acknowledge the key cognitive role of sensory perception in understanding spatial geometrical forms. Yet, *we conceptualize sensory perception as necessarily serving and emerging from goal-oriented action*. We thus seek to investigate how students ground geometry concepts in action-oriented perception. Our study accordingly evaluates a set of enactive tasks designed for high-school students to develop geometrical perceptions through multimodal action-based interactions with concrete material.

Constructing tangible models as embodied design for spatial geometry learning

Supported by the embodied perspective, we are looking to capture and theorize conceptually significant shifts in students’ perceptuomotor attention towards voluminous geometrical solids. The two vignettes, both from video-recorded data gathered in Jerusalem, Israel[1], illustrate how each activity’s unique sensorimotor affordances enabled students to engage in conceptually generative collaborative enactment and argumentation. The first vignette (V1) presents a task (Figure 1, right) designed to stimulate perceptual coordination of 2D diagrams and their 3D counterparts: students use a “3D pen” to construct a voluminous cube from its “flat” image, then they manipulate the model to investigate its properties per the problem instructions. The vignette exemplifies a “first step out of Plato’s cave,” where students tentatively realize what they can, may, and should do with a 3D model. In the second vignette (V2), students assemble a very large multi-unit geometrical form and then identify the “hidden” form that emerged in between the units. This vignette exemplifies students’ zealous unshackling of any remaining “geometry-is-flat” predilections.

[1] Qualitative analysis of additional cases is currently underway to evaluate for generalizability of these case studies

V1. Stepping out of the cave: Learning a 3D model's affordances for spatial problem solving

The teaching experiment presented in this vignette is a part of a wider educational research project evaluating students' experiences with 3D pen sketching while solving spatial geometry problems (for details, see Rosenski & Palatnik, 2021). Three 10th grade students (T, G, & M) faced the task shown in Figure 1 (right) for approximately 20 minutes. Male student G drew the triangle inside the cube with a 3D pen. Next, over approximately 8 minutes, the students discussed the problem but made no significant progress. They were inclined to believe that CB'D' is a right-angled triangle.

T: No, but you know that this is a cube, and this (cube face) is a square, and then, this is 90 degrees, then this is 90 degrees, and then it (diagonal) bisects the angle, so it is 45(degrees).

G: The question is, if we rotate (two faces, where the edge is an axis) [gestures "rotation" with two palms as faces], would it be the same angle? Could it be?

M: To them, it (the angle) in the picture also looks like that (90 degrees).

T: In the picture, it is just from a different angle, if you turn it like this [adjusts the model], you can see that this [points] is the right angle if this is stretched [pulls up the slightly sagging plastic diagonal of the top face of the cube.]

Figure 2: Investigating 2D representations of 3D geometrical shapes

During these eight minutes, students left the model standing between them on G's writing surface (see Figure 2). Remarkably, the students made inferences and conjectures about the task without taking the model in hand, only lightly touching it and making minor adjustments. Most of these adjustments reoriented the 3D model vis-à-vis the given 2D diagram.

In this phase of problem solving, the students were reluctant to make inferences based on the appearance of the 3D model. They approached the 3D model to correlate it with a cultural form that they are used to—a 2D diagram. In particular, the students sought to "flatten" the 3D model such that it would be seen as identical to the 2D diagram, where CB'D' presents an apparent right angle (see Figure 2-left, where T twists her body). However, a retinal image of a 3D object is not similar to a 2D projection. As Gibson (1986) has argued, we notice optical invariances of the object under the movement of the source of light, movement of the observer, movement of an observer's head, and manipulations and local transformations of the object itself. Naturalistic interaction could possibly untether the students from "paper math." Soon after, indeed, the students utilized these affordances of their 3D model, discerned it invariant features by handling it, and made correct inferences.

Several factors prepared the A-ha moment. For instance, G was unsatisfied with the claim that CB'D' is a right-angled triangle. He argued: "If these two angles look like the same shape, if they are both right angles and a triangle has a total of 180 degrees, then the triangle cannot exist"; "This is impossible. If you rotate it each way to make it (one of the angles) look like a right angle, then you can rotate it in a different way (to make other angle look like a right angle)". Yet his teammates, still "deep in the cave," had still to be convinced.

At the 14th minute of tackling the task, we witness a shift in students' interaction with a model. T took the model in hand (Figure 2-right) for the first time, rotated it for five seconds, and 20 seconds later said, "Now I get it." In the next minute, T tried to formulate her vision, accompanying her explanation with more than 5 complex model rotations (i.e., more than 2 axes involved). The core of her argument was in line with what G had previously argued: from different angles, the triangle appears isosceles, and yet its angle "can't be 90 for all of them." During this minute, G's attention was on the rotating model, enabling him to see it under these transformations. In contrast, M focused only on the 2D diagram and tried several times to draw her peers' attention to it: "Look at the picture!" It took another minute for G to formulate an answer and a valid argument.

T: So you think all the angles are 60 degrees? Is it an equilateral triangle?
G: Yeah, look, all the sides are of the same length [traces with finger three sides in succession] if you look at it. Is that true?
T: Mmmm…I don't know.
G: Look [adjusts the model], all the sides (of the triangle) are exactly a diagonal [traces a diagonal of the upper face], the diagonal of a square. All the squares are the same. And that means, if all of the sides are equal [points on a different face], all of the angles are equal, and they're all 60 degrees.
T: Ok... [types into the response form] We think that all sides are equal; therefore, the angles are also equal... equal to 60. It's an equilateral triangle.

To summarize, the task was difficult for the students, even with a 3D pen and a model. In line with Fujita et al. (2020), the task demanded that students harmonize their spatial reasoning skills with domain-specific knowledge of planar geometry (properties of squares and triangles). It took students time to utilize an available 3D medium and realize its affordances (a spectrum of perspectives on the equilateral triangle for the team members). M's decision to stick to a 2D drawing may explain her low contribution to the final effort. In contrast, when T physically rotated the model, her actions apparently spurred and supported her spatial reasoning and were visible to G. He, in turn, observed these *physical* rotations, which allowed him to refine his previous arguments based on *mental* rotations and apply the corresponding geometric knowledge to the new 3D situation.

V2. Further steps: Constructing enactive argumentation—gesture, action, medium

In the second teaching experiment, part of "Geometry In… and Out" (Benally et al., 2021), four 7th grade students constructed voluminous solids (Figure 3) then worked on the following questions: "Comparing the volumes of the large and small tetrahedra that you built, how many times the volume of the large tetrahedron is greater? Explain your answer. Several small tetrahedra compose the large tetrahedron. Can you describe a three-dimensional shape between them? Can you construct it?"

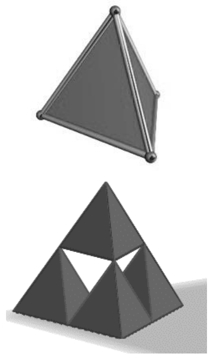
Once the group had constructed the first small pyramid, Yali placed it on his head (Figure 4a). Nami, using Yali as a stand, gestured on him that this polyhedron is called "arba-on" ("arba" is four in Hebrew). The palms of her hands present the polyhedron's faces. Then, removing the model off Yali's head, Nami gestured similarly, though with her forearms, to present the same four faces (Figure 4b).

It took the students approximately 11 minutes of collaborative work to construct a large model and begin answering the items. Their plan for estimating the large tetrahedron's volume was to decompose it into its component parts. They easily recognized four small tetrahedra: "three at the base, and one at the top." However, the students were not sure about the shape of a three-dimensional hollow between the tetrahedra (the octahedron outlined in red, for your convenience, in Figure 4c).

1) Your team has to construct a three-dimensional model of the following geometrical solid using a construction kit. The solid has the following properties:

- All the faces are congruent equilateral triangles.
- The same number of edges converge at each vertex.

2) The polyhedron you've constructed is called a tetrahedron. Construct a similar polyhedron whose edges are 2 times larger than the original one. You can use the image below for construction.



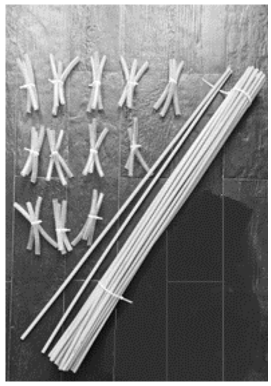
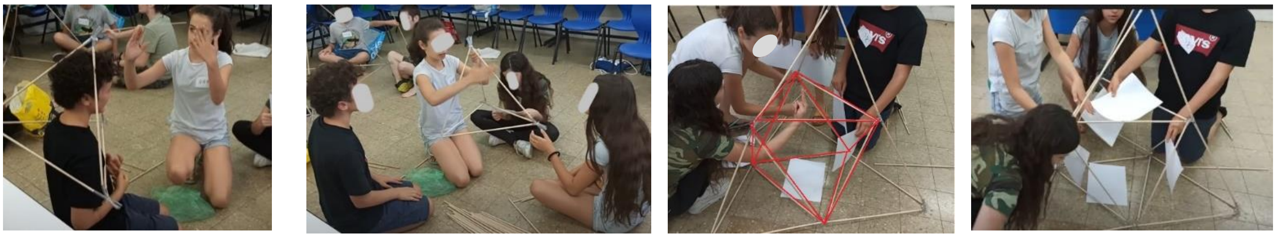


Figure 3: The tetrahedron construction task and materials.



a b c d

Figure 4: Different affordances of the available media as manifested in students' actions

Tami, Nami, and Gali suggested that the hollow is also shaped as a tetrahedron. Yali disagreed and offered to count the faces of the "empty space." He rotated the large model, hoping to render it more familiar, yet that action proved unhelpful. Tami remonstrated, "You just can't *see* this (tetrahedron)." To support her claim, she grabbed two sheets of paper lying on the desk and applied them successively as the polyhedron's faces, expecting these to total at four. Immediately, Yali appropriated Tami's strategy, just to disprove her. Summoning more paper sheets and distributing them over more group members, he marshaled an "octopus of hands" to simultaneously cover all the polyhedron's faces. The introduction of these auxiliary objects helped students to solidify the shape (Figure 4d), count the faces, and eventually write the following definition: "The polyhedron between four triangular (pyramids) has eight identical faces. Each face is an equilateral triangle."

This vignette illustrated the emergence of students' enactive argumentation through collaborative semiotic evolution of gestures into concrete media. Students' hands, semi-constructed models, and even repurposed found objects became instrumental in shaping a void—rendering a contested obscure object into an articulated, unequivocal, and publicly inspectable form. The hidden octahedron was born as a "prospective indexical" (Goodwin, 2018) then came forth through pointing, formative gestures, and construction media. Thus, the hollow solid inhabiting the larger structure was substantiated, reified. Eventually, once the form was delegated from imagination to media, the students could allocate cognitive resources to enumerate its facets and name the new geometric object.

Discussion and Conclusion

This article aimed to present theoretical foundations and empirical arguments for a set of embodied spatial-geometry curricular resources for middle school. We submitted that historical dependence of geometry education on 2D media is implicitly rooted in an idealistic Platonic tradition and the representational cognitivist model. We conjectured that tasks in which students construct 3D objects are more than "working with manipulatives"—they let students use their natural capacities of multimodal perception and collaborative action. We supported our argument through qualitative analysis of two vignettes exemplifying embodied design for spatial geometry learning.

We demonstrated how a group of middle-school students grounded geometric concepts of solids, their cross-sections, faces, and edges in goal-oriented, situated activity of constructing concrete 3D models. Our first vignette demonstrated that coordination of traditional and novel medium is not easy for students. They tentatively experiment with their new degrees of modal freedom—looking at objects, pointing at them, touching them, lifting and rotating them. New affordances catalyze shifts of students' attention to relevant features supporting their geometric reasoning.

Mithalal and Balacheff (2019) considered the possibility of a continuous evolution from iconic to non-iconic visualization, where the figural operation of instrumental deconstruction would play a cohesive role. Our second vignette provides an empirical basis for this assumption. The transition from iconic visualization to non-iconic visualization was carried out by introducing tangible auxiliary elements (paper sheets in the form of polyhedron faces) into the 3D model. Thus, the students performed a naive instrumental deconstruction of shape, a mereological deconstruction of a 3-dimensional shape into four tetrahedrons and an unfamiliar shape, and finally a dimensional deconstruction of a 3-dimensional shape, which focused their attention on the 2-dimensional faces of the octahedron (see Palatnik & Sigler, 2021, for a theoretical discussion on the introduction of an auxiliary element as a shift in attention). This argumentation by action was later transformed into a normative formulation of properties—the formal definition of a geometric solid.

Constructing and manipulating tangible models creates opportunities for students to harmonize spatial skills and rigorous geometric argumentation as well as bridge iconic and non-iconic visualization. Yet, it is challenging to step out of Plato's cave after a lifetime of unwitting incarceration. Middle school geometry should organize students' engagement with 3D objects as one would any artifact—with untampered senses; with gross and fine motor actions; with all the tacit, evolutionarily endowed naturalistic sensibilities for orienting in the environment. Once students realize how to work with three-dimensional objects in mathematical activities, they can tap their know-how to build valid arguments grounded in enactive experience.

References

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. *Cognitive Research: Principles and Implications, 1*(1), 1–13.

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning [Original Research]. *Frontiers in Education, 5*(147).

Alsina, C. (2010). Three-dimensional citizens do not deserve a flatlanders' education. In Z. Usiskin, (Eds.), *Future curricular trends in school algebra and geometry* (pp. 147-154). Information Age.

Bartolini Bussi, M. G., Taimina, D., & Isoda, M. (2010). Concrete models and dynamic instruments as early technology tools in classrooms at the dawn of ICMI.. *ZDM, 42*(1), 19–31.

Bautista, A., & Roth, W. M. (2012). Conceptualizing sound as a form of incarnate mathematical consciousness. *Educational Studies in Mathematics, 79*(1), 41–59.

Benally, J., Palatnik, A., Ryokai, K. & Abrahamson, D. (2021). Charting our embodied territories: Learning geometry as negotiating perspectival complementarities. *Manuscript under review.*

Chemero, A. (2013). Radical embodied cognitive science. *Rev. of Gen. Psychology*, 17(2), 145-150.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics, 3(*3/4), 413–435.

Froebel, F. (2005). *The education of man* (W. N. Hailmann, Trans.). Dover Publications. (pub. 1885).

Fujita, T., Kondo, Y., Kumakura, H., Kunimune, S., & Jones, K. (2020). Spatial reasoning skills about 2D representations of 3D geometrical shapes in grades 4 to 9. *Mathematics Education. Research Journal, 32*, 235–255.

Gibson, J. J. (1986). *The ecological approach to visual perception*. Psychology Press.

Goodwin, C. (2018). *Co-operative action*. Cambridge University Press.

Hurley, S. (2010). The varieties of externalism. In R. Menary (Ed.), *The extended mind* (pp. 101–154). MIT Press.

Kaur, H., & Sinclair, N. (2014). Young children's thinking about various types of triangles in a dynamic geometry environment. *Proceedings of PME 38 and PME-NA 36, 3*, 409–417.

Mithalal, J., & Balacheff, N. (2019). The instrumental deconstruction as a link between drawing and geometrical figure. *Educational Studies in Mathematics*, *100*(2), 161–176.

Montessori, M. (1967). *The absorbent mind*. Holt, Rinehart, and Winston. (pub. 1949).

Palatnik, A. & Sigler, A. (2021, July). *Introduction of an auxiliary element as a shift of attention*. [Paper presentation]. ICME14, Shanghai, PRC.

Rosenski, D. & Palatnik, A. (2021). Secondary students' experience using 3D pen in spatial geometry: affective states while problem solving. *Paper submitted to CERME12, TWG4.*

Rousseau, J.-J. (1979). *Emile or on education* (A. Bloom, Trans.). Perseus, Basic Books. (pub. 1755)

Thompson, P. W. (2013). In the absence of meaning…. In K. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 57–94). Springer.

Tomasello, M. (2019). *Becoming human: A theory of ontogeny*. Harvard University Press.

Widder, M., Berman, A., & Koichu, B. (2019). An a priori measure of visual difficulty of 2-D sketches depicting 3-D Objects. *Journal for Research in Mathematics Education, 50*(5), 489–528.