THEORY AND PRACTICE OF DESIGNING EMBODIED MATHEMATICS LEARNING

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Different approaches to embodied learning—conceptual learning of curricular content grounded in a new capacity for enacting forms of purposeful physical movement in interaction with the environment—have become increasingly central to mathematics-education research. This research forum provides participants with an up-to-date overview of diverse and complementary theoretical perspectives on embodied learning, principles derived from these perspectives governing the design of environments for learning various mathematical content, and demonstrations thereof. We speculate on promising directions for future embodied design research.

INTRODUCTION

Embodied mathematics learning is grounded in the human capacity to interact with the physical and social environment through purposeful movements, use of the senses, and creation and utilization of artifacts. As a field-wide paradigm, embodied learning stems from the embodied turn in the cognitive sciences, which maintains that perception and action are formatively constitutive of our thinking—cognition is inherently modal and situated activity (e.g., Chemero, 2013) that draws on the body's physical interaction with the world (Gibson, 1986). As such, cognition, including learning and knowing, emerges from activity of the perceptual and motor systems and, thus, in turn, is shaped by the body's physical properties and movement capacity (Glenberg, 2010). These ideas provided a powerful incentive for updating curricula design and resources for learning and teaching (Shapiro & Stolz, 2019).

There are several pedagogical precursors of the embodied approach to learning. Among them are Friedrich Fröbel's ideas on the importance of the child's activity in learning (Brosterman, 1997), Dewey's (1938) conceptualization of 'learning by doing' and reflective inquiry, and Maria Montessori's (1949) work based on the observation that "Watching [the child], one sees that he develops his mind by using his movements….[One cannot learn by] sitting down, moving nothing" (pp. 203–204). Similarly inspired early researchers of mathematics education advocated an educational approach capitalizing on the importance of physical space for learning: "We live in space, …. move in space, …. analyze space, to be better adapted" (Freudenthal, 1971, p. 418). This Research Forum (RF) also heeds Schoenfeld's (2016) statement that the variety of approaches to embodied learning (see Abrahamson et al., 2020) is becoming increasingly important for mathematics

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education. The RF provides an up-to-date overview of diverse yet by-and-large complementary theoretical perspectives on embodied learning, including their philosophical roots and influences from cognitive science. The exposition continues with design principles derived from the theory, then examples of environments designed by the contributors to this RF for learning various mathematical content. We conclude with perspectives and future directions for research on embodied learning in mathematics education.

THEORY OF EMBODIED LEARNING OF MATHEMATICS

A tree of embodied perspectives

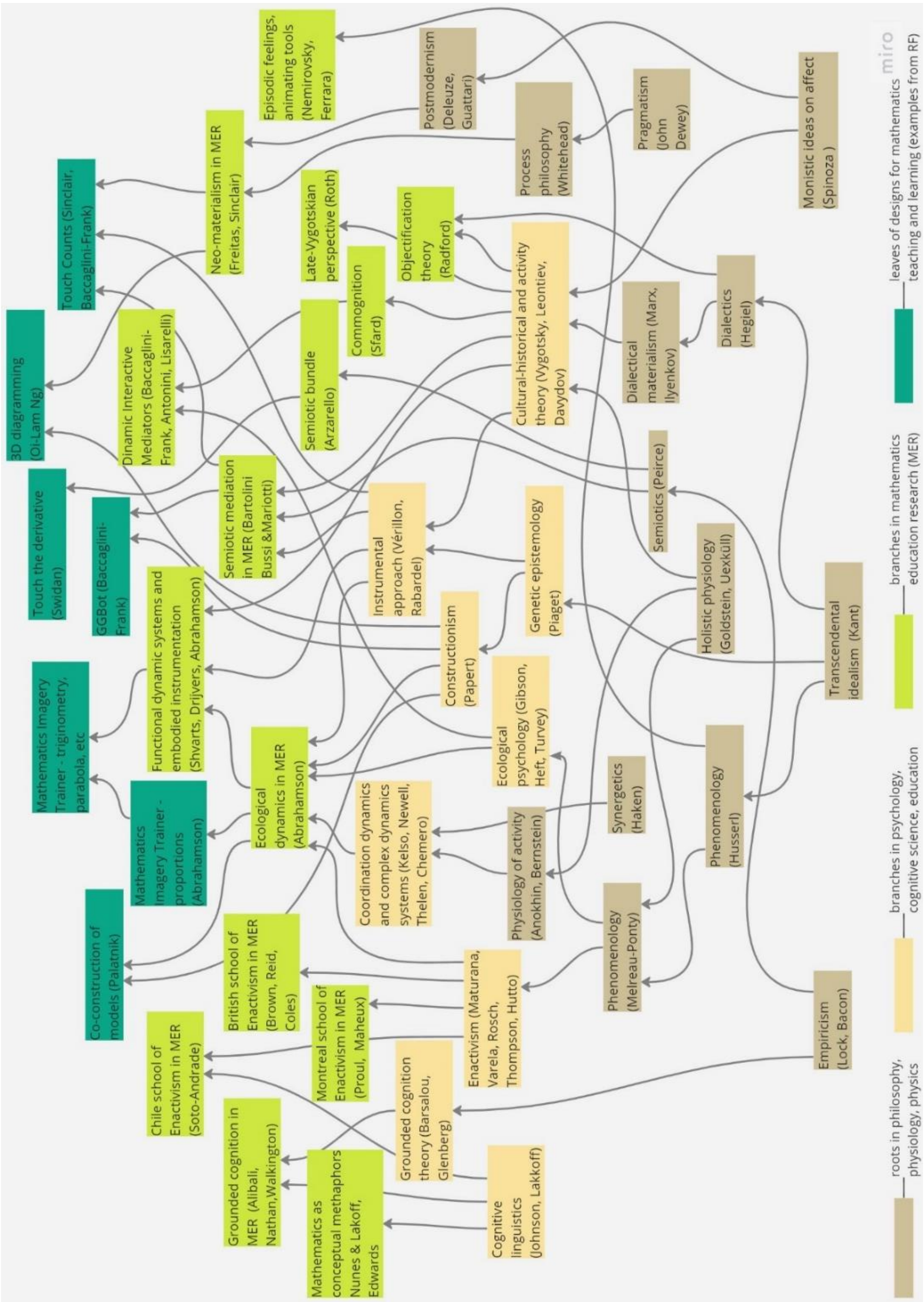
There is a great variety of theoretical approaches in mathematics education research that call for attention to bodily processes that enable mathematical thinking and learning, including, but not limited to motor performance, gestures, and eye movements as well as multimodal sensory experiences. The scope of these approaches is difficult to discern, as they are rooted in diverse ideas from philosophy, biology, physics, and branches of cognitive science and psychology, including cognitive linguistics, developmental psychology, the science of movement, and many other disciplines (see Figure 1 for a rough sketch of the tree of ideas).

Traditionally, researchers distinguish between conservative and radical approaches based on their stance towards mental representations and the separation between mind and body (Hutto & Abrahamson, 2022). For example, *grounded cognition* (Barsalou, 1999, 2021) assumes that a variety of experiences gained through eyes, ears, and other body parts is accumulated in mental representations called *perceptual symbol systems*. Grounding mathematical concepts on those experiences allows researchers to theorize the role of gestures and other forms of embodiment (Nathan & Alibali, 2021; Walkington et al., 2022). Another relatively "conservative" approach comes from cognitive linguistics: Lakoff and Johnson (1980) introduced the cognitive semantics theory *of conceptual metaphors* to explain how language propagates embodied experience into abstract concepts. Metaphorical mapping of mathematical concepts on bodily experiences, such as treating sets as containers, was extensively explored by Lakoff and Núñez (2000) and spread through math-ed scholarship (e.g., Sfard, 1994). "Mapping" between different contexts, e.g., written numbers, a number line, and physical objects, would form conceptual integration (Edwards, 2009). Once such mapping and grounding are assumed, educators might wish to induce mathematical understanding by inviting students to gesture out particular shapes hypothetically associated with mathematical content (e.g., Walkington et al., 2022).

Radical approaches abandon the idea of conceptual structures as ecologically independent epistemological entities preserved by the cognitive system as representations or schemes, instead embedding concepts into embodied experiences (Varela et al., 1991). A key root of those approaches is *complex dynamic systems* and *coordination dynamics* ideas (Bernstein, 1967; Kelso &

Schöner, 1988), which propose the non-linear and self-organizing nature of cognitive processes. Another intellectual grounding of embodied learning is *ecological psychology* (Gibson, 1986), which states that an individual's perception constantly evolves by actively detecting in the environment aspects relevant to interaction, i.e., *affordances*. While the interrelation between these approaches is hotly debated in cognitive science (Di Paolo et al., 2021), their combination is productive for theorizing mathematics learning (Abrahamson & Sánchez-García, 2016) and designing for learning within the embodied design framework (Abrahamson, 2014).

Figure 1: A tree of embodied perspectives.
https://embodieddesign.sites.uu.nl/tree/



Diving into the mathematics education research field, we may notice that some embodiment-oriented approaches bypass cognitive science; thus, the radical versus conservative landscape does not apply to their classification. For example, the *neo-materialist* approach (de Freitas & Sinclair, 2013) is inspired by Deleuze's ideas of a concept as an *assemblage,* where material artifacts, nature, and bodies get intertwined in their actualization. This approach calls for attention to materiality and how it invites students to interact. Grounded in Hegel's and Vygotsky's ideas, the *theory of objectification* (Radford, 2021) considers cognition to be sensuous and explores how students become aware of mathematics as they encounter material culture in practical collaborative activity with teachers and peers.

When theoretical tenets rest on disparate ontological and epistemological assumptions and share only family resemblance, still they may nevertheless undergird common pragmatic agendas. For example, researchers who conceptualize a cognitive mapping between sensorimotor experiences and mathematical ideas (Walkington et al., 2022) may embrace principles of *ecological dynamics* (Abrahamson & Sánchez-García, 2016). Another combination comes from joining enactivist ideas with metaphorical mapping between mathematical concepts and other interaction contexts (Díaz-Rojas et al., 2021). Yet another approach combines coordination dynamics and cultural–historical ideas into the notion of a *functional dynamic system* (Shvarts & Abrahamson, in press). In this monist approach, mathematical notions are reconsidered as direct extensions of the bodies via mathematical artifacts that come forth in the intercorporeal sensorimotor dynamics of teachers' and students' task-oriented embodied activity.

In the following sub-section, we present the key ideas essential for understanding mathematical learning as an embodied process: moving in a new way, perceiving in a new way, and naming in a new way. We do not offer a coherent theory but point to the ideas of many authors who, using different terms, refer to these aspects of learning. We strive to bridge these perspectives around the same phenomena.

Moving in a new way

Movement is the deepest aspect of human nature:

> … we are essentially and fundamentally animate beings. In more specifically dynamic terms, we are animate forms who are alive to and in the world, and who, in being alive to and in the world, make sense of it. We do so most fundamentally through movement, unfolding a kinetic aliveness that is in play throughout the course of our everyday lives from the time we are born to the time we die. (Sheets-Johnstone, 2011, p. 452)

When learning and developing, human bodies come to move in new ways: sucking, walking, writing, drawing. To grasp how understanding is grounded in movement, we must acknowledge that body movement already requires understanding the world. An alive movement is not a blind repetition of a pre-programmed sequence of motions. Instead, it is an emergent phenomenon that arises in a constant attempt to solve motor problems that the world places in front of an organism (Bernstein,

1967) as it strives for relative equilibrium (Merleau-Ponty, 2002) in a constantly transforming environment. Many levels of regulation act collaboratively in accomplishing a movement: Our body responds to gravity, our posture is built in relation to all body parts, and our hands encompass the spatial relations of the objects (Bernstein, 1967). All these levels find their way through constant probing and adjusting of sensory-motor processes by anticipating and receiving feedback from the environment. An exact repetition is never possible in this complexity. Yet, invariants are enabled by the synergetic character of our motor systems (Kelso & Schöner, 1988): a multiplicity of muscles assemble into a functional dynamic system that enables a relatively stable movement that repeatedly and efficiently solves motor problems (Bernstein, 1967). To get a grip on this theoretical idea, stand for a while on one leg and observe a complex play of the muscles of your supporting leg: never stable, in continuous co-adjustments, they enable your continuous still position. Try then to swing your free leg. You may notice how your posture became shakier for a while and then regained a relatively stable balance, continuously supporting your swings by iteratively modifying the position of your mass center.

Perceiving in a new way

Through "synergies of meaningful movements" not only the motion but the world itself is constituted for the subject (Sheets-Johnstone, 2011, p. 453). Just like the muscles' activation is organized into synergies, our sensors organize into meaningful perceptual experiences: we come to discern from the world what is relevant to our enactment. *Attentional anchors*—namely, perceptual structures (Gestalts) that arise to serve enactment or are given by educators to support new forms of action—act as a proxy between the world and the actor (Abrahamson & Sánchez-García, 2016). Notice, when swinging your leg, you likely stopped being focused on the muscles of your supporting foot but paid attention to the swing trajectory of your free leg— imbricating perceptual structure on the world.

A focus on the living body moving in the environment suggests that the environment is not independent of the perceiver but provides *affordances*—possibilities to act (Gibson, 1986). Perception of affordances is direct but not innate. On the contrary, it constantly develops as a learner discovers higher-order variables through differentiating relevant information. It is through movement in the environment that learners can come to perceive "aspects of stimulus information that persist despite movements of the perceiver (or are actually brought into existence by those movements), and that correspond to relatively permanent features of the objective situation. (Neisser, 1987, p.12)

Applying those ideas to Dynamic Geometry Environments, a figure can be conceptualized as a set of affordances that the solver comes to perceive through dragging. Further, we may expect that, through dragging, a learner will discover new mathematical *invariants* (Leung et al., 2013). Similarly, Mason (1989, 2008) characterizes learning by *what (focus)* is attended to and *how (form)* the objects are attended to. He distinguishes five different forms of attention: *holding the whole* without focusing on particularities or *discerning details* among the other elements of the attended object. From there, one may *recognize relationships* between discerned elements, *perceive properties* by actively searching for additional elements fitting the relationship, and, finally, *reason based on perceived properties*. Mason's theory has been applied by Palatnik (2022) to the analysis of embodied activities for learning spatial geometry—he identified correlations between students' physical interactions with artifacts and their attentional shifts to inherent properties and structures.

Finally, Marx introduced the idea of "sensuousness as practical activity" (Radford, 2021), allowing a further understanding of human perception as shaped by cultural practices within society. Accordingly, Vygotsky considers perception to be a higher-order function—a complex systemic social entity that cannot be reduced to sensation *per se* (Vygotsky, 1978). Human practices in specific domains, such as archeology, reveal *professional vision*: an ability to notice structures that are unnoticeable by a non-experienced eye (Goodwin, 1994). In like vein, Radford (2010) discusses the role of *theoretical perception* in mathematics: patterns that educated adults perceive, students must learn to distinguish. Teachers appropriate multimodal resources, such as gesture and rhythm, to highlight those patterns in the environments (Goodwin, 2018; Radford, 2010). To make sense of teachers' rich utterance, students actively scan the environment to establish concordant perception (Roth & Thom, 2009; Shvarts, 2018). This is why designing special *fields of promoted actions* might be beneficial for learning to move and perceive in a new way (Abrahamson & Trninic, 2015).

Naming in a new way

While new ways of moving and perceiving are the key parts of embodied mathematics learning, embedding those individual—often idiosyncratic—dynamic practices into cultural discourse and environments requires further theorization. In embodied design, learning might start by developing new sensorimotor coordination or individual exploration of an artifact that triggers new forms of perception and only later becomes gradually interconnected with more and more advanced cultural artifacts (Abrahamson et al., 2011) and scientific discourse (Flood, 2018; Mariotti, 2009). Importantly, artifacts and discourse are not self-evident for students. While mostly rooted in Vygotskian ideas, theories vary in explaining how artifacts and discourse come to be part of students' mathematical knowledge

Theory of Semiotic Mediation (TSM) argues that the relation between the artifact and the learners in the course of accomplishing a specific task is expressed by signs such as speech, gestures, symbols, and tools (Bartolini Bussi & Mariotti, 2008). On the one hand, when repetitively accomplishing a (well-designed) task with an artifact, a solver develops schemes (Vergnaud, 2009), which constitute a "hidden" psychological component associated with the visible actions and with the other signs produced. An artifact and a scheme constitute an instrument—a psychological construct used for further instrumental actions (Verillon & Rabardel, 1995). On the other hand, while solving the task and interacting with peers, a learner produces personal signs that are closely related both to the task and to the artifact, thus constituting personal meaning. The *semiotic potential* of the artifact with respect to the mathematical knowledge allows the teacher to conduct mathematical discussions involving a variety of signs (e.g., Mariotti, 2009). In these discussions, the signs produced by the students (artifact signs) are gradually transformed into shared signs. Such shared signs generalize the situated signs referring to personal meanings, and are transformed into mathematical signs (through successive didactical cycles) to the knowledge being taught.

Other theories avoid talking about meaning and scheme as independent cognitive constructs, thus removing representationalist ideas from mathematics education discourse. Following the *commognition* perspective (Sfard, 2008), sensorimotor processes comprise a mathematical object's realization tree, consisting of various inscriptions, such as formulas, sketches, definitions, and primary objects of the world interconnected. From a *functional dynamic systems perspective* (Shvarts et al., 2021), artifacts and discourse directly extend the dynamics of students' bodies once they are appropriated for solving the task. The students' ways of acting and naming come to correspond the cultural forms of acting and naming through joint-attention with a teacher, when a student's and a teacher's sensory-motor processes are coordinated in common environment (Shvarts & Abrahamson, in press).

Overall, cultural artifacts and discourse transform students' perception by making it mediated by the artifacts (Bartolini Bussi & Mariotti, 2008; Vygotsky, 1978) and allows them to decrement conceptual aspects and see mathematical properties (Leung et al., 2013; Mason, 2008). Soon, though, perception accommodates from *mediated* to *immediate*, as the artifacts seamlessly "stick" to the bodily system, giving rise to new forms of concrete experience (Shvarts et al., 2022). Thus, artifacts, including discourse, appear to be new tools for actions and lenses for perception: a student equipped with a ruler sees distances as measurable; or, equipped with a notion of exponential growth, she readily anticipates the shape of a graph depicting epidemic spread of a disease (cf. Vérillon & Rabardel, 1995, on utilization schemes).

DESIGN PRINCIPLES OF EMBODIED LEARNING

The design principles (DPs) underlying artifacts used in embodied learning are motivated by the perspective that human beings think with and through their body (Merleau-Ponty, 2002; Radford, 2021). This perspective has stimulated many design-researchers to create artifacts fostering bodily engagement in mathematics learning. This section, organized along five DP, outlines tenets and heuristics shared by all the RF authors.

DP1: Involve students' bodies in the learning process

This involvement can be achieved through perception-based design artifacts or action-based design. The first genre builds on learners' early mental capacity to draw logical inferences from the perceptual judgment of intensive quantities in source phenomena; the second genre builds on learners' perceptuomotor capacity to develop new kinesthetic routines for strategic embodied interaction (Abrahamson, 2014).

The dualistic view of cognition, which distinguishes between the mind and the body, considers our body movement (e.g., gestures and gaze) as a window through which we can make inferences about the learner's perception and thinking. On the contrary, the monistic view of cognition, which considers the unity of the body and mind, considers our body movement a constitutive part of our thinking, because the use of any kind of artifact creates a specific kind of interaction and, thus, a specific kind of thought. Hence, our second principle for embodied environment design:

DP2: Offer immediate sensorimotor interactions with artifacts

Perception is viewed as a cognitive structure that emerges first as the psychological means of enabling some stable form of coordinated motor engagement with the environment, then as the reified semiotic kernel of mathematical practice, including verbal and nonverbal language, gestures, extra-linguistic expression, and inscription (Abrahamson & Mechsner, 2022). These semiotic means give rise to learners becoming aware of mathematical knowledge embedded in the artifact (Bartolini Bussi & Mariotti, 2008; Radford, 2021). Hence, our third design principle is

DP3: Attend to the semiotic sensitivity of the design

Semiotic sensitivity considers the signs included in the artifacts, which can be produced by the educators or by the students through solving a task. Designers should deeply reflect on the genre of signs and their relationship with mathematical knowledge, which they expect to connect through the activities with the artifacts. They should also reflect on the ways students will interpret and endow with meanings the several genres of signs. For example, the designer should be sensitive to the colors used in the artifact and the potential associations between these and the target mathematical knowledge. Mathematical understanding happens when the learners are able to convert from one semiotic register (representation) to another and to treat within the same semiotic register (Duval, 2006). Thus, our fourth principle:

DP4: Include a variety of semiotic registers and artifacts that potentiate mathematical perception and discourse

Gallese and Lakoff (2005) claim that sensory-motor system of the brain is multimodal and theorize language as multimodal as it includes speech, gestures, drawings. Thus, to understand cognitive processes we should analyse all the modalities (Arzarello & Robutti, 2010, Abrahamson et al, 2020). The embodied-design approach puts forward that learning new concepts begins with discovering new ways of acting in the environment (Abrahamson & Bakker, 2016). While sprouting from enactment, embodied activities further inevitably require socially scaffolded "languaging" to enter cultural discourse (Flood, 2018). Multimodality lies at the core of learning. Thus, our fifth design principle is:

DP5: Foster multimodal engagement and "languaging".

DESIGN OF EMBODIED LEARNING FOR THE DIFFERENT MATHEMATICAL TOPICS

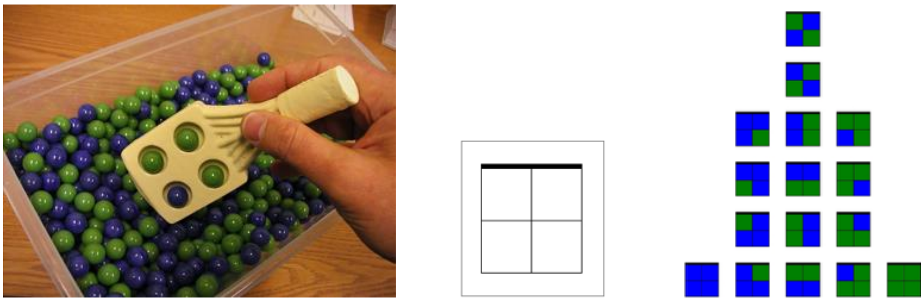
Embodied design: A framework in search of a theory

Originally emanating from a researcher's practical experience as a mathematics tutor and resource innovator, *embodied design* (Abrahamson, 2014) gradually sprouted theoretical roots. These roots wandered among different grounds, at first testing "conservative" forms of embodiment, such as the *cognitive semantics theory of conceptual metaphor* (Lakoff & Núñez, 2000), *conceptual symbol systems* (Barsalou, 1999), and various intellectual foundations of gesture studies, such as *gesture as simulated action* (Hostetter & Alibali, 2008). However, micro-ethnographic analyses of data collected in empirical evaluations of embodied-design activities were raising challenges to the basic epistemological and ontological assumptions of those theories. Moreover, Dor Abrahamson was affiliated with Uri Wilensky's Center for Connected Learning and Computer-Based Modeling, where he had developed computationally enabled projects to foster youth understanding of natural and social phenomena from a complexity perspective, and so, Abrahamson had espoused the methodologies of dynamic systems theory as his *modus operandi* in modeling the ontogenetic emergence of conceptual understanding in socio-cultural contexts (Tancredi, Abdu, et al., 2022). Still wandering, embodied design drew on movement science (Abrahamson & Mechsner, 2022), evolutionary biology (Abrahamson, 2021), dance scholarship (Abrahamson & Shulman, 2019), and contemplative practice (Morgan & Abrahamson, 2016). Increasingly radicalized, though, embodied design turned to *ecological psychology* (Gibson, 1986), *coordination dynamics* (Kelso, 2000), and their amalgamated *ecological dynamics* (Araújo et al., 2020), and then further "left" to enactivism (Hutto & Myin, 2013), which led to collaborative scholarship (Abrahamson & Sánchez–García, 2016). While embodied design is fairly grounded now, it keeps searching for nutrients that will increase the integration, coherence, and generalizability of its theoretical models. Currently the work looks to elaborate on Vygotsky's cultural–historical theory of cognitive development using phenomenological and complexity tools (Shvarts & Abrahamson, 2019, in press). Below we exemplify two genres of activities emanating from the research program, *perception-based* and *action-based* embodied design (Abrahamson, 2014).

Perception-based genre of embodied design: The Seeing Chance project

Humans have innate or early-developed perceptual sensitivity to ecologically adaptive exemplars of *intensive quantities*, that is, quantities that are scientifically defined as *a/b*, for example, velocity (distance / time), aspect ratio (height / width), or likelihood (favorable events / possible events) (e.g., Xu & Garcia, 2008). We can perform judgments on these *perceptually privileged intensive quantities* (Abrahamson, 2012), such as determining the representativeness of color samples from a mixed-color population, just as long as the quantities are presented asymbolically (cf. Zhu & Gigerenzer, 2006). Yet such tacit perceptual capacity by no means implies explicit mathematical knowledge. To the extent that educators wish to leverage students' tacit capacity as a cognitive grounding for conceptual understanding, further design resources and mediation techniques are required to coordinate the natural and cultural. Abrahamson (2012c) sought to leverage tacit judgments of likelihood as an epistemic grounding for the notion of probability.

Figure 2: Selected materials from the Seeing Chance design for the binomial. Project page: https://edrl.berkeley.edu/projects/seeing-chance/



a. b. c.

The *marbles scooper* (Fig. 2a) is a random generator approximating a binomial experiment. Bearing 4 concavities, the scooper draws samples of 4 marbles from a tub containing equal numbers of green and blue marbles. Young students correctly predict that the most likely outcome is a scoop with 2 green marbles and 2 blue marbles, the least likely scoops are of uniform color, etc. However, students do not attend to the specific *order* (pattern) of the marbles sample, only to the green-to-blue *ratio*, which they compare to the green-to-blue ratio in the tub. Students then use a set of empty 2-by-2 iconic formats of the scooper (Fig. 2b) and green and blue crayons to create "all the things we could get when we scoop." The tutor guides them to create not just five cards (e.g., the bottom row of Fig. 2c) but all 16 permutations. Once the entire sample space is completed and configured, students spontaneously realize how to perceive it as a model of the source phenomenon that enables them to argue for their initial judgment. This negotiated learning sequence has been called *product before process* (Abrahamson, 2012b), where students perform a *semiotic leap* (Abrahamson, 2009) via *abductive reasoning* (Abrahamson, 2012a) that bridges the natural–cultural gap.

Action-based genre of embodied design: The Mathematics Imagery Trainer

Varela et al. (1991) submit that “(1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided” (p. 173). The Mathematics Imagery Trainer is an *instrumented field of promoted action* designed to elicit recurrent sensorimotor patterns from which emerge cognitive structures grounding targeted concepts, such as proportion (Abrahamson, 2019).

Figure 3 features a Mathematics Imagery Trainer for proportion. The student’s task is to place the cursors at locations that make the screen green. Unknown to the student, the screen is green when the respective heights of the two cursors over the screen base relate at a set ratio, here 1:2. Through exploration (Fig. 3a), the student happens to make the screen green (Fig. 3b). Asked to move her hands continuously, keeping the screen green, she keeps the interval between the hands fixed, thus violating the ratio (Fig. 3c). With time, she realizes that the interval should change with the height.

Figure 3. The Mathematics Imagery Trainer for Proportion—the Parallels motor-control problem: schematic depiction of four key events in learning to enact the conceptual choreography. https://edrl.berkeley.edu/projects/kinemathics/

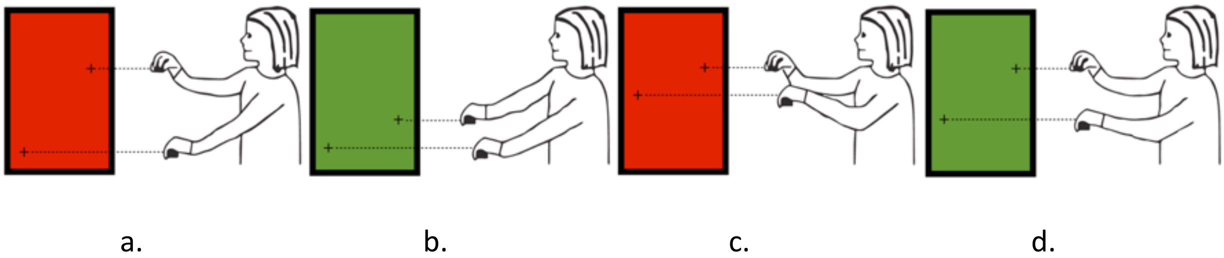
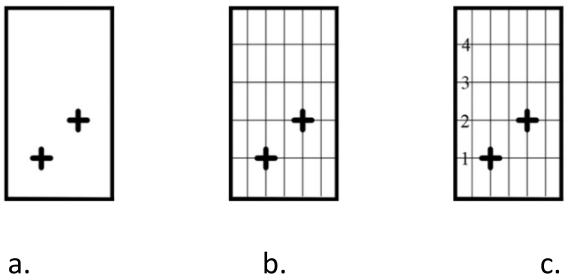


Figure 4. Three schematic interface modes of a Mathematics Imagery Trainer for Proportion—the Parallels motor-control problem



Having worked in continuous space (Fig. 4a), we then introduce a grid on the screen (Fig. 4b). Students appropriate this artifact as a pragmatic-cum-semiotic resource, enhancing the enactment, explanation, or evaluation of their method, shifting into math discourse. Once we introduce numbers (Fig. 4c), students draw on their multiplicative knowledge to re-describe their actions. So doing, they are able to coordinate logically between their various strategies (Abrahamson et al., 2011, 2014). At the same time, classroom observations reveal that appropriations of a grid or numbers might be problematic for some students (Alberto et al., 2022).

Instead, children might prefer to use self-invented artifacts, such as rulers or dice, for marking the length and measuring the ratio. As Palatnik and Abrahamson (2018) found, in the absence of a grid, students may utilize the cursor icons both for measuring and to develop rhythmic forms of movement that facilitate task performance.

Moving action-based embodied design to more advanced mathematical topics: parabola, trigonometric functions and statistics

The domain of functional relations appeared to be a natural area for further exploration of the action-based design genre. Graphs are the most usual way to represent functions, however, perceiving a Cartesian plane requires special forms of perception (Krichevets et al., 2014; Radford, 2010), avoiding focusing on irrelevant aspects of a graph (Arcavi, 2003). The action-based embodied design promotes the development of new forms of motor action and, thus, new forms of perception (Abrahamson, 2019). For example, a mathematical perception of a parabola would include seeing the curve as the set of points that are equidistant from a directrix and a focus. In our action-based embodied activities for parabola, students play with a triangle's vertex and discover that triangle is green when it is isosceles, and its vertex is equidistant from a line and a point (Shvarts & Abrahamson, 2019, Figure 5, a, b). In further mathematization, they interconnect the trace of the isosceles triangle's vertex with a quadratic equation (Figure 5 c).

Figure 5. Action-based embodied activities for learning parabola. The triangle turns green when AC=AB. The letters and the parabola were not visible to the students.

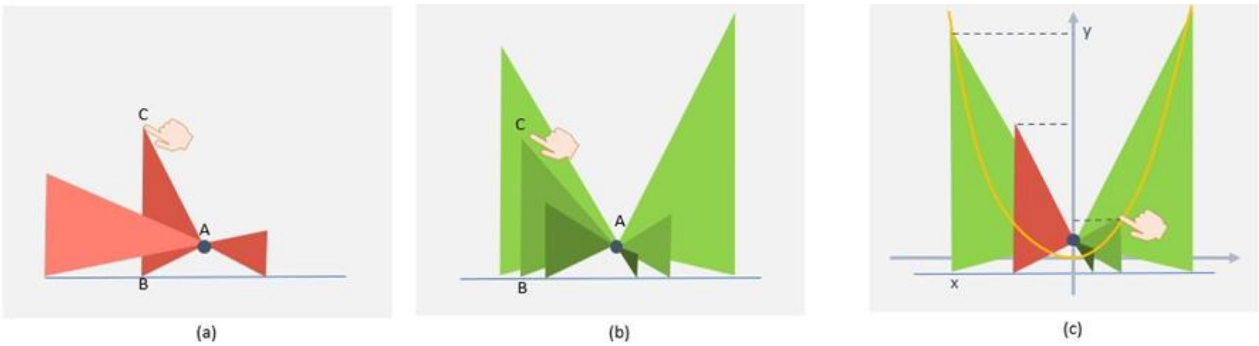
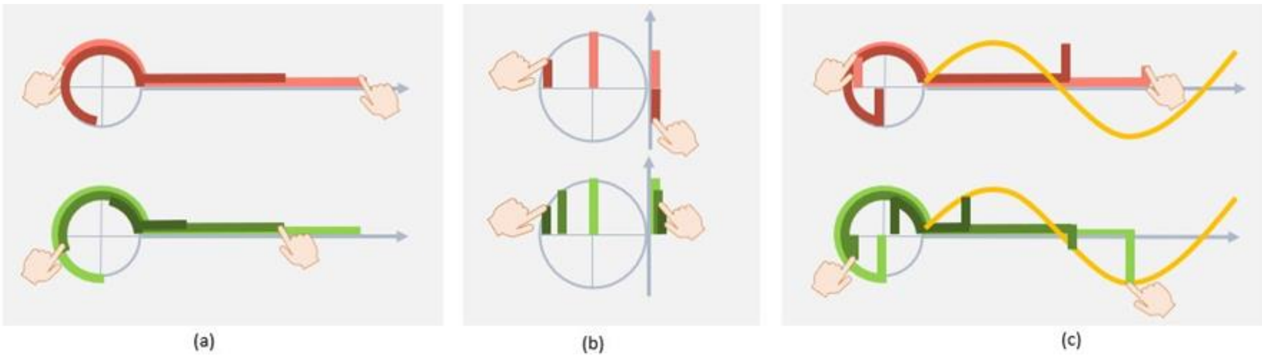


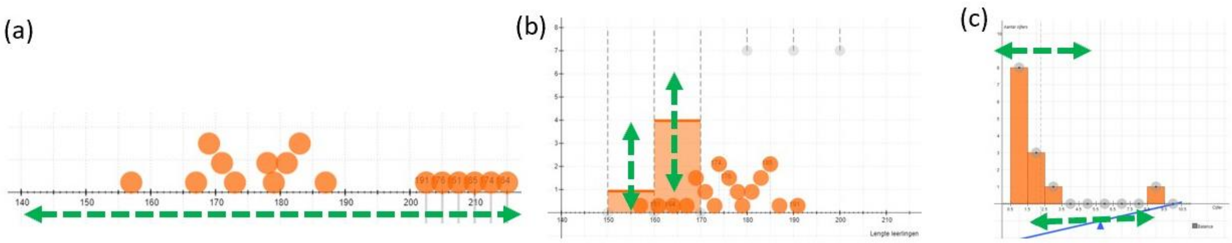
Figure 6. Action-based embodied activities for learning trigonometry.



Learning trigonometric functions is another difficult topic as students struggle coordinating visual inscriptions that present trigonometric functions—triangle, unit circle, and graph (Presmeg, 2008). After an unsuccessful attempt to create a field of promoted action for *connecting* different inscriptions (Alberto et al., 2019), we prompted students to *reinvent* a sine graph based on the interaction with the unit circle. Supported by continuous feedback, students would establish new sensory-motor coordinations: at first, they would uncover the correspondence between the length of an arc on a unit circle (Figure 6a), and further the correspondence between the sine value on the unit circle and *y*-coordinate on the graph (Figure 6b). Combining these two coordinations (Figure 6c), they draw a line that matches the sine graph, reflect on its construction, and embed their sensory-motor findings into the algebraic notations.

Contrary to the instrumented field of promoted actions where the designers introduce mathematical artifacts, we melted the target artifacts (Shvarts & Alberto, 2021), i.e., deliberately removed a sine graph and parabola from the digital environments. Instead, we created conditions in which students could reinvent those artifacts themselves. Based on our 4-year design research results, the same principle of melting holds for any support the system might provide for the students. For example, if a segment connects the points on a unit circle and grid, the students talk about keeping this segment horizontal and miss the correspondence of sine values across visualizations.

Figure 7. Embodied activities to study histograms. Green arrows represent the degrees of freedom for possible movements and are not visible to the students.



As empirical analysis with eye-tracking shows, cultural perception of histograms requires specific sensory-motor strategies (Boels et al., 2019). We created the fields of promoted actions that would solicit the actions that match cultural sensory-motor strategies for building a histogram. Instead of providing continuous feedback, as in the action-based genre, we constrained the students' actions leaving only one degree of freedom, and presented the motor problem as meaningful word problems.

The students would reinvent histograms to represent the heights of all children in a class by solving motor problems of (1) moving the balls along the x-axis to represent the height of individual children (Figure 7a) and (2) moving the bars along the y-axis to represent how many children fall into a particular interval of height (Figure 7b). Note, similarly to the designs on parabola and trigonometry, the target mathematical artifact—a histogram—was melted (see Boels et al., in press for more details).

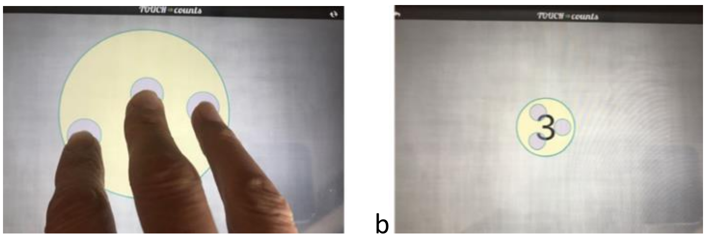
Further, the students explore the mean on a histogram by manipulating a balance line (Figure 7c), thus promoting the development of a new form of perception based on their experience of balancing. This design builds on students' natural capacity to estimate weight based on 2D representations, thus resembling the perception-based design (Abrahamson, 2014). However, rendering students' embodied intuitions from other modalities needs further theorization.

Sensing number sense: the case of Fingu and TouchCounts

The importance of using fingers in the development of number sense has been studied within a growing body of mathematics education literature (e.g., Baccaglini-Frank & Maracci, 2015; Coles & Sinclair, 2018); much of such literature attends to issues of embodiment in learning mathematics, acknowledging that sensorimotor activity such as touching, moving and seeing are essential components of mathematical thinking processes (e.g., Radford 2021). Moreover, many of these studies have focused on children's learning with multi-touch apps.

In particular, Baccaglini-Frank and Maracci (2015) and Baccaglini-Frank et al. (2020) have tried to associate childrens' actions (especially gestures) in certain multi-touch apps with their number sense abilities being elicited. In doing so, we have been pursuing the hypothesis that apps that use multi-touch capabilities may uniquely influence children's mathematical understandings and strategy development concerning number sense, also shared by other researchers (e.g., Tucker & Johnson, 2022). An interesting construct in terms of embodiment in this context concerns conceptually congruent gestures (Tucker & Johnson, 2022), which involve actions with fingers matching the quantity. For example, a group of three fingers placed "all at once" (Figure 8a) creates in TouchCounts a herd of 3 (Figure 8b).

Figure 8: a) Placing three fingers all-at-once on the screen in the Operating World of TouchCounts; b) the herd of 3 that is formed after the fingers are lifted (from Baccaglini-Frank et al., 2020, p. 785)



a b

In both our studies, we use Vergnaud's notion of scheme (2009), focusing especially on operational invariants: the implicit knowledge which structures the whole scheme, driving the identification of the situation and its relevant aspects, and allowing to select suitable goals and inferring the rules for generating appropriate sequences of actions for achieving those goals. While being aware that the notion of a scheme might be seen as being in opposition to embodied cognition approaches, we followed other colleagues. We tried to gain from the two perspectives useful analytical tools to describe crucial cognitive aspects of students' interactions with digital artifacts.

In the former study, we found children's interactions with the app Fingu and LadyBug Count to elicit mostly number sense abilities related to cardinality, that is, abilities elicited in answering the question "How many?" via one-to-one correspondence between physical sets of objects or between or between objects counted and spoken numbers and understanding that the last number spoken in a counting sequence names the quantity for that set. On the other hand, in the latter study, we used the two "worlds" offered in TouchCounts, showing how certain tasks we had designed could elicit fundamental number sense abilities also related to ordinality, which comprises associating number symbols to number words, knowing the number symbols sequence, knowing the number words sequence, and knowing what number comes before or after a certain one (Baccaglini-Frank et al., 2020). The openness of the microworlds in TouchCounts offers the possibility of proposing many interesting and different tasks, as well as "play situations", that can be addressed and solved in countless ways, each with potentially different gestures and schemes.

Experience the dynamics and touch the derivative

The learning environment includes an AR prototype that collects real-time data regarding a dynamic phenomenon (a ball on an inclined plane) during a physical experiment. The sensors collect the data, analyze them, and instantly display their mathematical representations to the students on designated headsets (Fig. 9 right).

The students will thus be able to observe both the real-world experiment and the mathematization of the dynamic object immediately and in real time. When looking through the AR device, the students not only perceive the real experiment, as shown in Figures 9 and 10, but also an elaborated mathematization, which is described below. Figure 9 presents the Hooke's law experiment, which examines the relationship between the mass and elongation of a spring. Figure 10- the Galileo experiment, which examines the relationship between time and distance that a cube travels as it slides down along an inclined plane.

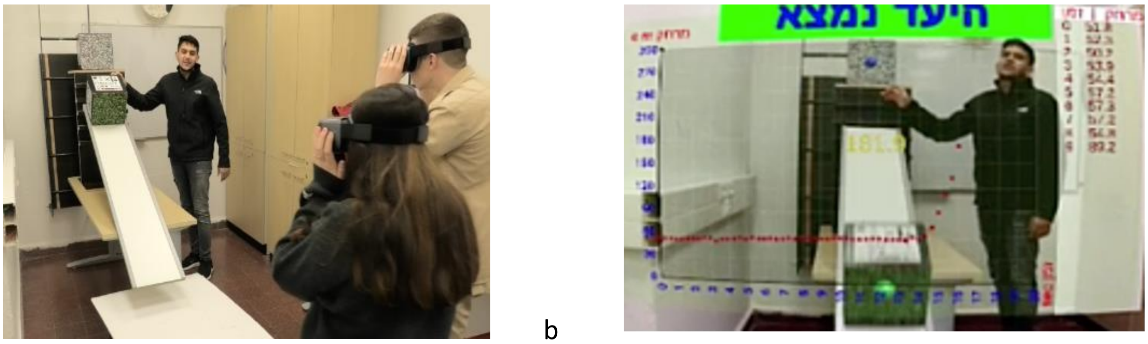
Figure 9. Students collected data by the headset (left side), the display students see, which includes a graph of the mass-length function, order pairs of mass of the cube, and the length of the spring (right side).

The following mathematization-decisions have been made for the development of the AR design. Concerning the Hooke's law experiment, the AR device provides a coordinate system with the weight of the mass on the x-axis and the length of the spring on the y-axis. Whenever a certain weight is added, the elongation of the spring gets larger; the AR device detects the new length and plots it corresponding to the added weight. Adding weight step by step, a diagram is built up showing the relation between the weight and the length of the spring. This diagram overlays the real experiment (Fig. 9, left) and develops with it. A table of values can also be displayed.

For the Galileo experiment, the AR device detects the position of the cube when sliding down a ramp. The AR device provides a scatter plot that plots the time (x-axis) versus the distance (y-axis) at the same time. The distance at a given moment is between the original position of the cube and the current distance along the ramp (see Fig. 10b). The AR device also displays a table of values showing the corresponding time and distance.

These design decisions, accordingly, allow us to enhance real experiments and body involvement with robust mathematical representations like scatter plots and tables of values. For Hooke's law experiment, students can interact with the physical model by carrying the cubes, feeling their weight, hanging the cubes to the spring, observing the elongation of the spring, and interpreting the relation between the weight and the length of the spring by observing the evolving scatter plot. The AR device also shows the length in numbers with a yellow ellipse (Fig. 9, right), which helps the students to better relate the length of the spring to the graph plotted in the diagram.

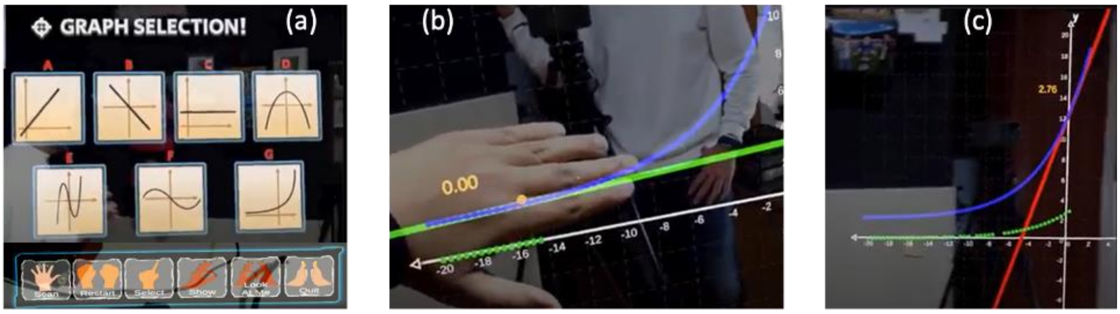
Figure 10. (a) Galileo experiment: a cube travels as it slides down along an inclined plane. (b) The graph and table of values of the distance-time function of the cube movement that one of the students sees through his headset.



a b

One of the specific aspects of the mathematization in both experiments that contributes to the development of conceptual understanding is the simultaneous presence of different representation modes for the same phenomenon. It is also worth mentioning that the data presented on the headsets are affected by the students' location and the angle the students are looking at the physical objects.

Figure 11. a) The software interface; b) A green tangent line and the derivative graph are created simultaneously with the hand's movement; c) If the student is not close enough to the graph, the tangent line becomes red



An additional embodied learning environment described here is designed with the support of an AR headset called Magic Leap: it is a pair of glasses through which the user can see reality and the augmented objects. In the default interface of the application, the student can choose among seven elementary types of functions (upper part of Figure 11a). In the lower part of the interface, the gestures useful to interact with the technology are recalled jointly with their function (Figure 11a). When selecting one of the functions, a Cartesian system with numbered axes appears, showing the selected function in blue color (Figure 11b). The learner is asked to move his hand along the graph: simultaneously with the hand's movement, a green tangent line appears, and the derivative curve is sketched point by point in the same color (Figure 11b). Moreover, a yellow number denoting the value of the slope of the tangent line is displayed. The student should be close enough to the function graph so that the tangent line is displayed. Otherwise, the line becomes red instead of green, and the derivative graph is not created (Figure 11c).

The design principles underlying this software were inspired by Alberto et al. (2019) work in which students can create the graph of a function by coordinating their hands' movement along a specific constraint. Two a priori hypotheses underlie this learning environment. First, since the environment juxtaposes the function graph, its derivative function graph, and the hand movement, students should come up with conjectures about the mathematical relationship between the two graphs. Second, body involvement should help students find that relationship by 'feeling' the slope behavior as they move and control their hands.

Constructing and diagramming geometry designs

Johnston-Wilder and Mason (2005) pointed out that "A central feature of geometry is learning to 'see', that is, to discern geometrical objects and relationships, and to become aware of relationships as properties that objects may or may not satisfy" (p.4). However, what constitutes learning to 'see'? If learning is "education of perception" (Goldstone et al., 2010), how do we design the learning environments that "take the natural affordances of our long-tuned perceptual systems, which are at their core spatial and dynamic, and retask them for new purposes" (ibid p. 280)?

Echoing Freudenthal (1971), how do we reclaim learning geometry as a living experience for students? Several lines of DBR attempt to answer these questions.

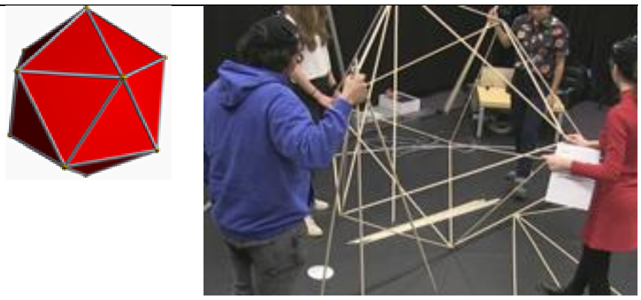
Co-construction of human-scale tangible models

Consider a group of students first building a human-scale model of a geometric body while referring to its two-dimensional diagram and textual description and then using this model to explore the properties of the polyhedron (Figure 12). Several variations of this relatively simple goal-oriented problem-solving design provided various insights into the multifacet nature of students learning to 'see' polyhedra properties (Benally et al., 2022; Palatnik & Abrahamson, 2022; Palatnik, 2022).

Figure 12: The example of collaborative co-construction geometry activity. Left and center—the instruction; Right—participants constructing a model

Using the kit provided to you, your team has to construct a three-dimensional model of the following geometric solid, a polyhedron.

The polyhedron has the following properties: All the faces are congruent; The same number of edges converges at each vertex



The construction problem has several perceptual-material-social characteristics that influence students' actions. When constructing a model, each student has a unique perspective on the task at hand due to the natural constraints of human perception. Moreover, participants must simultaneously consider the dimensions and material of the model (human-size, tangible) and the instruction (hand-held, 2D diagram, and text printed on paper). Constructing the model jointly, students are also constrained by the actions of others and the physical features of the model. To succeed, they must coordinate their actions and make their multimodal referents to the properties of the emerging structure mutually intelligible.

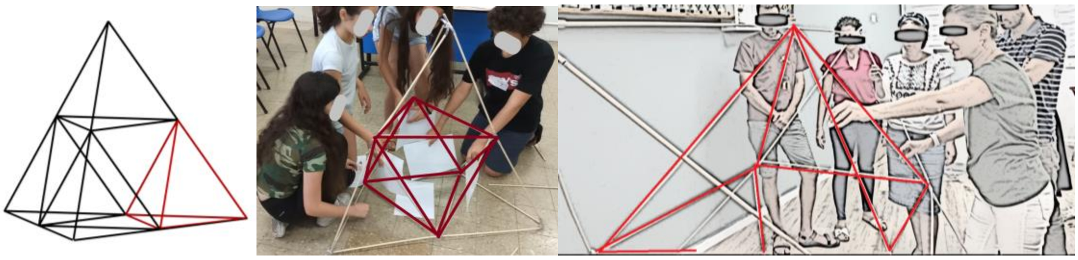
Palatnik and Abrahamson (2022) presented theoretical foundations and empirical arguments for a set of embodied spatial-geometry curricular resources for middle school. In the spirit of Felix Klein's statement, "A model—whether it be executed and looked at, or only vividly presented—is not a means for this geometry, but the thing itself" (Klein 1893, p. 42, cited in Halverscheid, 2019), they conjectured that tasks in which students construct 3D objects are more than "working with manipulatives". In these activities, students use their natural capacities of multimodal perception and collaborative action. We hypothesized that in an attempt to improve and coordinate collaborative actions aimed at building a model, the participants would come to recognize differences in the ways they perceive the model and attune toward each other's perspectives. The need to achieve a common goal will force each participant to explicate reflections on tacit perceptual mechanisms. The student's actions become more complex as the model grows and becomes more composite. In turn, a need to collaborate and communicate these efforts forces participants to engage iteratively in more and more complex forms of reasoning and expression.

Figure 13: An action leads to a shift of attention due to a change of perspective: tacit symmetry becomes visible

Initial designs of the collaborative co-constructive activities were explored in the ongoing DBR project. Palatnik (2022) applied the analytical apparatus of Mason's shifts of attention theory to investigate why and how using physical models of different scales can facilitate learning of (spatial) geometry. The study demonstrated that students' collaborative physical actions and multimodal perception triggered shifts in the focus and structures of attention that, in turn, led to a problem-solving breakthrough. In particular, having tilted the structure onto a vertex, the students perceived an icosahedron as tripartite: two opposing "bases" and a connecting "belt" (Figure 13). Benally et al. (2022) reported that models of different scales landed students different affordances for exploration, for noticing invariant scale-free features of a geometric object and influencing students' collaboration dynamic.

In a similar setting, to solve the problem of comparing the volume of pyramids (Figure 14 left), the students generated the auxiliary problem of defining the shape between four small tetrahedrons (Figure 14 center). Students repurposed sheets of paper lying on the desk as polyhedron faces (c.f. self-invented artifacts in Alberto et al., 2022). They also used their ability as a group to cover all the faces simultaneously. Then, students continued by counting the faces and defining: "The polyhedron between four triangular (pyramids) has eight identical faces. Each face is an equilateral triangle." When a similar activity was conducted during PD, discovering that a simple rotation of the model facilitates seeing its structural features became an insight for teachers: "Now I can not unsee two pyramids with a common square base" (Figure 14 right).

Figure 14: Shaping a void through collaborative action—rendering a contested object into an articulated and inspectable form (red lines are for readers' convenience)

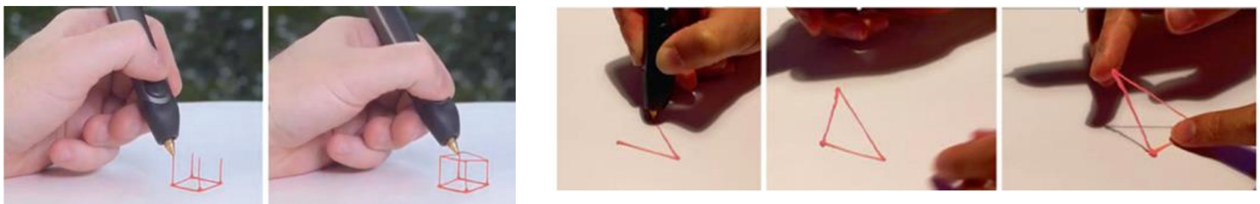


To summarize: co-construction and exploration of tangible models is a robust activity architecture for learning through the surfacing and negotiating learners' perspectives on situated phenomena; students' perception is active and enculturated through participating in the social enactment of the practice of construction; students' critical insights in problem solving can be characterized as shifts in perceptuomotor attention leading to the refinement of geometric argumentation while students' realization of available 3D medium and social setting affordances catalyzes these shifts.

Learning as making: the case of 3D pens

Grounded in Papert's Constructionism (Papert & Harel, 1991), several lines of research (e.g., Ng & Ferrara, 2020; Palatnik, 2023) are premised around the use of 3D printing (specifically, the "3D Pens") as a form of embodied technology for "learning-by-making." 3D printing can extend 2D products to the 3D environment; unlike traditional manipulatives preselected by teachers and which are usually fixed in size and how it is made, 3D printing can provide students with opportunities to generate 3D models flexibly, which Ng and Ye (2022) termed "embodied making" with 3D Pens.

Figure 15. Drawing a cube with a 3D Pen. Creating and manipulating a triangle drawn by a 3D Pen



In this light, the work of Ng and colleagues' research considers the potential transformations that 3D diagramming can induce in mathematics teaching and learning (Ng & Sinclair, 2018; Ng et al., 2018; 2020). Given our interests in embodied mathematics learning, 3D Pens afford increased immediacy and sensory interactions with mathematical representations that are lacking in screen-based tools. This work has shown the affordances of 3D diagramming as a practice of mathematical diagramming in engendering students' geometrical thinking.

The first unique feature of 3D diagramming is the ability to draw in 3D, which overcomes the limitations of paper and pencil and improves the visualization of 3D geometrical objects. For example, one way to draw a cube (Fig. 15 left), is firstly to draw four straight “segments” on a surface to form a square, then four vertical “segments” that join the four vertices of the square, and four more “segments” in the air, while drawing an identical square parallel to the base. It is noted that in the process of drawing such a 3D object, one can visualize vertices, segments, and planes and observe the relations among the 0D, 1D, and 2D objects (Ng & Ferrara, 2020). Besides, 3D diagramming simulates the very process of gesturing; as the hand moves with the 3D Pen, a 3D model is generated. This unique nature of generating 3D models by one’s hands affords some interesting hand movement that could not be possible on flat surfaces. Moreover, 3D diagramming supports additional tactile experience by affording students the modality to touch, turn, flip and rotate 2D models drawn. Diagrams that would have been drawn using paper and pencil, such as a triangle, can be recreated and become physical objects that can be held, moved, and turned when drawn by 3D Pens (Fig. 15 right). This suggests the dual nature of embodied making as creating both a diagram and a physical, hands-on manipulative.

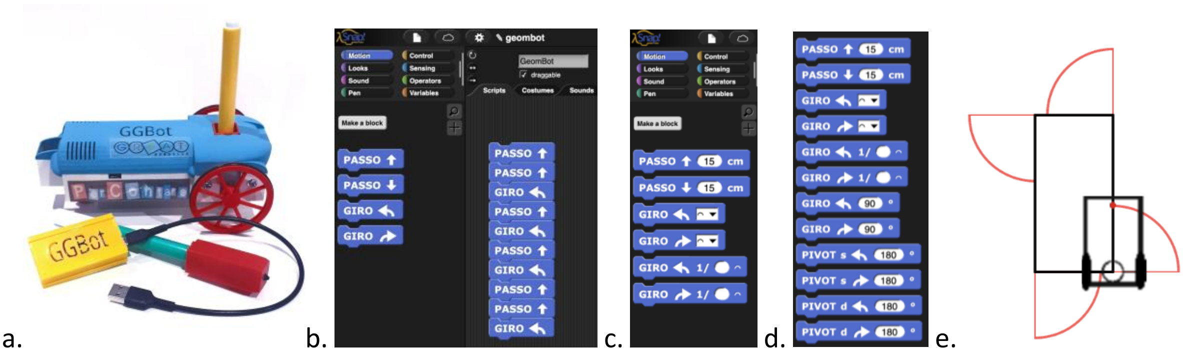
GGBot

The GGBot (short for “GREATGeometryBot”) builds on the convergence of physical and digital affordances, combining the well-known strengths and opportunities offered by Papert’s original robotic drawing-turtle (more recently developed into robotic toys like the “Bee-bot”) and LOGO programming with those of the block-based programming language SNAP! (Baccaglini-Frank et al., 2020). The GGBot can hold a marker between its wheels (Fig. 16a). When the marker is placed and the GGBot executes a code, the marker draws out its path as it moves on a sheet of paper on the floor. A second marker can be placed at the front, on the GGBot’s “nose”, to leave a trace of its movement when it changes direction. These design features were implemented so that GGBot’s traces can provide situated signs (Fig. 16e) that can be elaborated, through appropriate tasks and mathematical discussions, into geometrical notions, such as segment, vertex, angle, rotation, and polygon.

Commands are given to the GGbot through SNAP!, a Scratch based interface that was customarily designed (Fig. 16 b, c, d), and they can be gradually added based on the teacher’s needs (Fig. 16c, d). The way in which codes are given to the GGBot is quite different from other robotic toys like the Bee-bot, because the blocks on the screen represent commands (in the machine’s language) that can be put together into codes (Fig. 16 b) that can be transmitted to the GGbot via a wifi module. These graphical blocks are virtual objects that “live” on a screen (touch-screen of an interactive whiteboard, tablet, or computer screen); they are concrete enough to be accessible to and shared by the whole class and by each student. Moreover, they constitute another set of artifact signs that contribute to the complex network of signs emerging during activities and that can be put in relation with mathematical signs. Below is an example of the semiotic potential of a figure-to-code task with respect to some of the geometrical notions listed above

(Baccaglini-Frank & Mariotti, 2022). A figure-to-code task consists in giving students the name of a figure and asking them to use the blocks to produce a code, so that when sent, the GGbot draws the nominated figure. For example: “Make a code so that the GGBot draws a square”.

Figure 16: a) GGBot with a marker placed in its posterior holder; b) initial command set and example of code; c, d) commands with parameters for more advanced programming; e) drawing made by the virtual GGBot given the code in b)



Students need to envision the “square shape” as a contour, a path along its border that corresponds to the GGBot’s trace mark as it moves along such a path. Then, such a contour/path must be seen as a sequence of steps, leading to the realization of a code for the GGBot. A spontaneous approach consists in acting: imagining to be the GGBot and walking along the border of the figure (“acting the path”) and associating possible commands to the movements carried out (see Papert’s notion of “body syntonic” learning experiences (1980)). While it is straightforward to identify the four segments constituting the sides of the square and relate them to the step ↑ commands, which are translations, it might be more challenging to decide how to connect these four steps. This requires mastering the complex meaning of the Turn ↱ command (whether it is to the right or to the left), which is a “turn on the spot” (rotation) without translation. In this case, the traces left by the GGBot are thick dots left as the robot changes direction before taking another step forward. A consistent interpretation, leading to completing the drawing, also needs to put these points in relation with one another as the vertices of the square and as centers of the rotations of the external angles of that polygon. So, an essential feature of the semiotic potential of this artifact is its building on the relationship between the GGBot’s global movement and its breaking into steps and turning points and the geometrical meaning of a polygon at a global and analytical level. Moreover, especially if the student “acts the path”, a relationship needs to be conceived between the decomposition and transfer of complex continuous movement (of a child, with many joints working) to only two discrete components - steps and turns - corresponding to blocks of code that will determine the GGBot’s movement and the trace marks left on the paper. The GGBot also has a completely digital version (Fig. 16e) (https://sprintingkiwi.github.io/virtual- geombot-snap), that is more similar to the LOGO turtle (though still in the block-based SNAP! environment)

FUTURE DIRECTIONS

We, the authors of this Research Forum, are excited by the prospects of working together to curate, taxonomize, explicate, counsel, debate, and promote embodiment perspectives on the design and facilitation of mathematics education. Together with our students and collaborators, our collective work ahead falls into six categories: theory, practice, design, dissemination, and academics, as we elaborate below.

Theory

Whereas our epistemological roots vary, we all look to model and foster mathematical learning as a process of mediated negotiation between, on the one hand, biologically endowed sensorimotor capacity for developing perceptuomotor skill and, on the other hand, culturally evolved artifacts, both instrumental and semiotical. These two "hands"—the biological and cultural—are pre-historically "bimanual," in the sense that evolutionary processes have selected for our species' natural as well as cultural adaptation—the human ecology is artifactual through and through. Yet, the field of educational research and design is still figuring out the balance between "bottom-up" processes of discovery learning vis-à-vis "top-down" processes of semiotic mediation (cf. Cole & Wertsch, 1996).

What would be the ideal guided reinvention of mathematical concepts? That is, what comes first? Can "top-down" processes give rise to "bottom-up" symbol grounding? We believe that symbolic meaning must be grounded in sensorimotor experience (Harnad, 1990). At the same time, the field might pay more attention to prolepsis (Stone & Wertsch, 1984), a multimodal conversational technique of casting forward into the discursive space a yet-ungrounded structure as a mutually consensual target of sense-making. Micro-analysis of guided mathematical ontogenesis in task-based manipulation suggests that an effective proleptic methodology is to modify a student's perceptual orientation toward the environment by shifting their attention toward elements in the shared domain of scrutiny that would afford a tighter sensorimotor grip (Shvarts & Abrahamson, 2019, in press). This dyadic "dance" of attention, which serves humans in coordinating joint action, deserves further research. We hope to sustain our dialogue on prolepsis as a research focus for refining our theoretical alignments and contentions. Conducting empirical research studies on this central yet complex phenomenon could enable us to move forward collectively.

Practice

In a longitudinal study, Kosmas and Zaphiris (2023) have documented the instructional gains of introducing embodied learning into classrooms. Liu and Takeuchi (2023) argue for the diversification potential of embodied design specifically for racialized and minoritized students, while Tancredi, Chen, et al.

(2022) apply embodied design to building resources for students with atypical sensorial and motor capacity (see also Lambert et al., 2022), and Shvarts and van Helden (2021) demonstrate the digital reach of embodied design to remote students.

Integrating embodied design into school hinges on adapting and casting these resources as promoting curricular objectives. Yet, successfully interleaving embodied activities into instructional routines stands to change the classroom's epistemic climate, that is, students' implicit sense of what forms of discursive contributions mark legitimate ways of thinking, knowing, and problem-solving (Feucht, 2010). Are students allowed to share their idiosyncratic metaphors (Abrahamson et al., 2012), "be" the graph (Gerofsky, 2011), or invent their own diagrams (diSessa et al., 1991)? As institutional discourse changes around what it means to know a math concept, curriculum experts would need to update assessments, and teacher educators would need to create professional development modules that adapt professional practice.

Design

The brave new "flipped classroom" world is increasingly migrating core instruction from classrooms to homes, only enhanced by the pandemic. Moreover, personal devices (e.g., phones) render digital content universally accessible. Consequently, online learning plays central roles in students' conceptual development. In this context, embodied-activity applications could serve as productive resources, given that they offer dynamically engaging inquiry-based learning experiences.

Multimodality is slowly returning to interaction-based digital educational resources, thanks to engineering breakthroughs in virtual and mechatronic technology. As such, future embodied design may rely more heavily on sensory modalities such as the haptic, tactile, kinesthetic, vestibular, and somatic that by-and-large have been elided from interaction learning due to the technical constraints of early computational platforms. Eventually, as we achieve simulated augmentations of multimodality, we will return digitally to Fröbel's Gift #1, the yarn ball (Abrahamson et al., in press).

Dissemination

Authoring the RF has created a community of practice with much shared common research interests and outreach ambitions. We see value in coordinating enterprises that go beyond individual publications or annual workshops. As a collective, we may stand to mobilize greater resources, garner greater attention, and reach greater audiences. One potentially productive project would be to build a website that curates for the general public resources and information on embodied mathematics learning.

Academics

Change begins at home. As academicians, we need to be the change we want to see at large by creating in our own university departments graduate-level courses, specializations, and even programs dedicated to design-based research on embodied mathematics education. These developments could be supported through establishing special interest groups (SIGs) in annual conferences, such as those run by the Association for Computing Machinery (ACM), the International Society for the Learning Sciences (ISLS), and various regional networks, for example, the American Educational Research Association (AERA). At the same time, our academic activities should remain in dialogue with the wider community of research and practice.

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