In-sight Out: Challenges and Opportunities in Learning Mathematics Through Negotiating Egocentric and Allocentric Perspectives

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Abstract

Situated collaborative mathematics learning has been characterized as a negotiation between different perspectives that participants tacitly bring to bear in making sense of problematic situations. In this symposium, a collective of independent design-based research projects pool their insights on how participants in experimental pedagogical activities negotiated specifically across egocentric vs. allocentric perceptions of spatial displays, eventually leading to emergent insights that proved critical in grounding new concepts. We draw tentative inferences for a heuristic design framework oriented on supporting mathematics learning through encountering, surfacing, and reconciling perspectival tensions. Finally, we suggest how researchers may leverage analogous perspectival negotiations when investigating empirical learning data. As body-scale XR infuses classrooms, our insights could inform practices for expressing these experiences in normative forms.

Introduction

Gerofsky (2010) found that higher-achieving mathematics students experienced concepts by assuming a first-person perspective on symbolic displays. For example, students would orient themselves with a rising and falling graph line, like riding on a rollercoaster—they would "be" the graph. The finding encouraged educators to offer *all* students opportunities to experience mathematics. Still, full-body dynamic experiences must dialogue with the static symbolic register of normative practice. How should teachers support students in perceiving flat displays, such as graphs, equations, and tables, as signifying embodied interactions? What multimodal discursive practices would enable these semiotic coordinations? In turn, how should researchers approach the analysis and theorization of these coordinations? It's all, we believe, a question of perspective. We hope to present results from a set of design-based research studies where students learn by coordinating between intuitive and formal perspectives.

Abrahamson and Wilensky (2007) proposed an educational design framework—*learning axes and bridging tools*—geared to inform the engineering of activity resources for mathematics students to learn conceptual content through reconciling their own conflicted interpretations of situated problems. One "axis" that students must acknowledge and reconcile is *perspectival*, namely whether students are experiencing a situation egocentrically or allocentrically. An *egocentric* perspective is defined as experiencing entities in the perceptual field, including other agents, objects, and actions, with spatial reference to one's own body (Tversky & Hard, 2009).

An *allocentric* perspective refers to the representation of entities and their actions in relation to some objective point, regardless of the observer's subjective location (Filimon, 2015; Herbst et al., 2017). For example, walking on a street provides an egocentric perspective, while looking at a map of that street provides an allocentric perspective. In order to find their way, the individual must *negotiate intra*personally between the apparently conflicting views from these two perspectives, achieving a dynamic perceptual routine to *synergize* these views as complementary and establish a sense of grounded navigation (Benally et al., 2022). Similarly, *inter*personal perspectival conflict may prompt interlocutors toward reflecting on their own perceptual experience and articulating common ground that reconciles the perspectives to essentialize the activity's inherent conceptual content (Hegarty & Waller, 2004).

Our session pools together new findings from a set of distinct yet thematically affiliated pedagogical designs, where students learned mathematical content by recognizing and resolving ego-/allo- centric perspectival conflicts. We hope to advance the field's understanding of the roles that artifacts, tasks, and instructors can play in promoting synergistic reconciliation of conflicting egocentric and allocentric perspectives on a shared learning experience, ultimately informing the practice of mathematics education.

Table 1 details across the projects the targeted perspectival coordination that establishes conceptual understanding.

Table 1

Independent Projects & Conceptual Understanding

Project Name	Walking the Number Line	Grounding Geometry as Movement Discourse	Angling the Stars	A Micro-Phenomenological Study of Diagrammatic Reasoning
Concept	Integer operations (addition & subtraction)	Geometric reasoning & auxiliary lines	Angle measurement	Diagrammatic reasoning
Perspectival Coordination	Egocentric walking number line and allocentric paper number line	Egocentric and allocentric perspectives on evaluating a 90^0 degree angle	Egocentric experience of being an angle and allocentric experience of seeing an angle	Methodological coordination of egocentric narrative and allocentric representations

The 90-minute symposium will begin with a 6-minute introduction by the Chair. Following, each presenter will highlight the unique contribution of their respective paper (4 x 12 min.). Next, our Discussant, Dr. Susan Gerofksy, an internationally celebrated expert on embodied mathematics education, will comment on the papers (20 min.). Finally, we will receive questions and comments from the audience (14 min.). Below are further details on each of the papers.

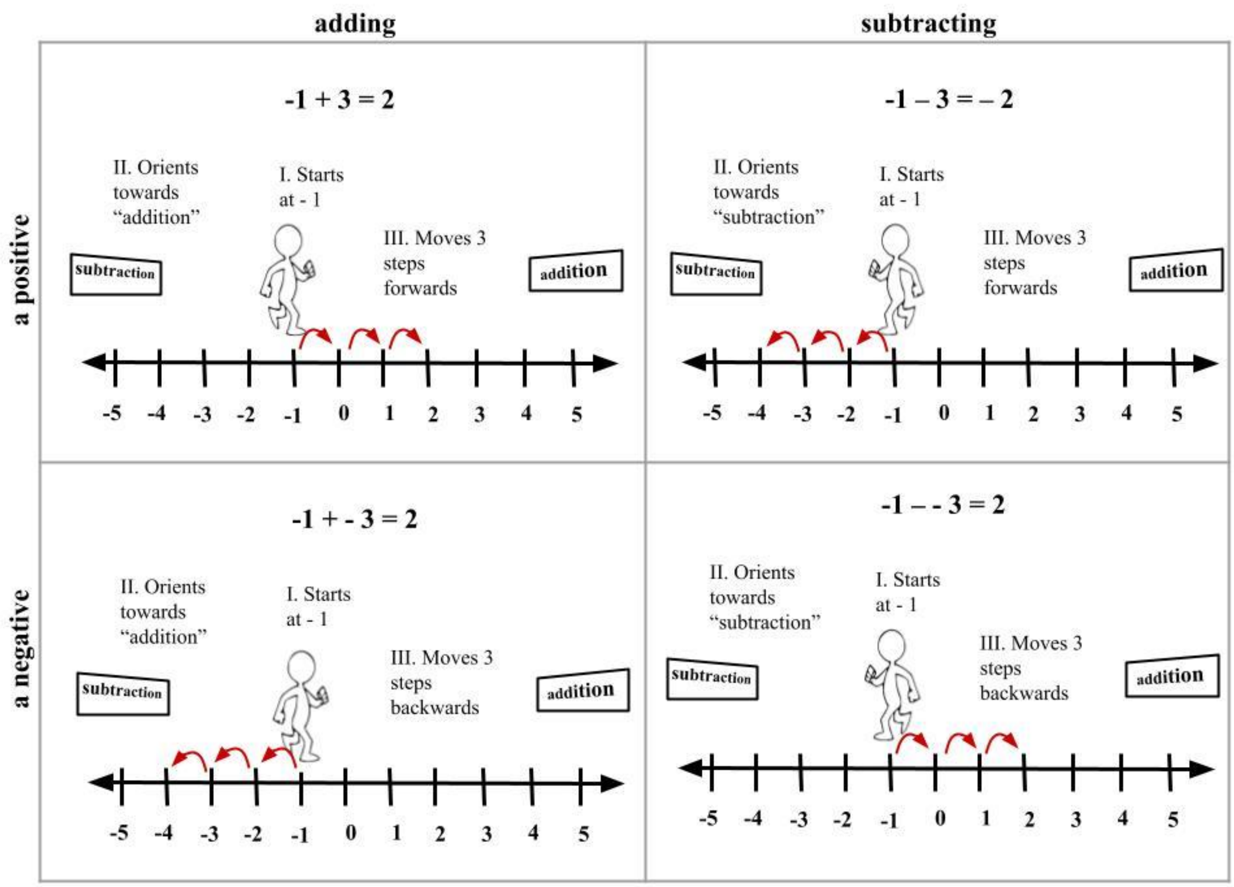
Walking the Number Line: Towards an Enactive Understanding of Integer Arithmetic

Jacqueline Anton

Students often struggle to add and subtract positive and negative integers (Bossé et al., 2016; Hawthorne et al., 2022). This paper reports on preliminary results from the implementation of an innovative educational design that utilizes the number-line (NL) as a cognitive resource for early integer arithmetic, centering on movement-based calculation activities. Nurnberger-Haag (2018) found that students developed stronger integer fluency when utilizing a walking number line over other methods. However, the researcher did not enable students to link these body-scale embodied experiences with typical desk-scale classroom tasks. This design seeks to enable students the opportunity of coordinating their egocentric experience on a walking NL with an allocentric experience using a desk-scale NL.

Figure 1

Walking Number Line (egocentric perspective)

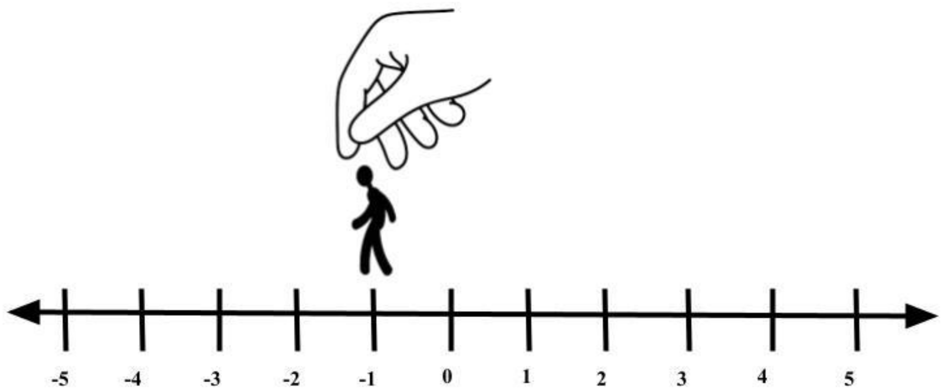


Note: Teachers' instruction (in text) and students' actions (the animating figure) regarding body-scale enactments of four basic addition or subtraction moves with positive or negative numbers on the walking NL.

First, students enact simple addition and subtraction problems on the body-scale NL (egocentric orientation; Figure 1). Next, they "walk" a figurine across a small, traditional NL (allocentric perspective; Figure 2).

Figure 2

Small Number Line (allocentric perspective)



Note: Student walking a figurine along a desk-scale miniature Walking Number Line

Embodied designs (such as the Walking NL) typically allow for students to enact problems from an egocentric perspective, whereas ultimately they are required to adopt an allocentric perspective in utilizing standard materials, such as paper and pencil (Abrahamson, 2014). Our design solution for this perspectival reconciliation was to offer students resources for *combining egocentric and allocentric elements*: we invited students to reenact the walking NL experience upon the paper NL by "marking" the action of a proxy agent who experiences the paper NL from an egocentric perspective (cf. Kirsh, 2010).

Fifteen Grade 7 students participated in this pilot study. The activity proved pedagogically advantageous in that it elicited students' implicit confusions surrounding the content of integer arithmetic. For example, multiple participants in this study, who had just excelled in solving problems on the body-scale NL, then faced challenges when they switched to the desk-top NL and reverted to previous erroneous strategies and "rules" they had learned in the classroom. However, when reminded that they should move the figurine as if it were their own body on the Walking NL, students were able to reconnect, reenact, and sustain this imaginatively mediated egocentric perspective, even as they were visually apprehending the figurine allocentrically; they thus arrived at correct solutions. Students' initial faltering, along with their eventual "tuning in" to the figurine's walking experience, suggest that they were proactively negotiating between two different spatial perspectives, thus achieving *perspectival mutuality* (i.e., using an alternate perspective to inform their own; Benally et al., 2022). Eventually, students may achieve *perspectival synergy* (a combination of two perspectives that is greater than each perspective alone; Benally et al., 2022). Perspectival synergy in this case would allow students to develop a linear, spatial–numerical mental number line (Mock et al., 2019), which would serve them in all tasks requiring integer arithmetic and fluency.

In summary, the constant availability of both the body-scale and desk-scale NL models created opportunities for students to coordinate across perceptual perspectives. This design allowed all students the opportunity to ground the abstract notion of a negative-integer arithmetic in concrete action (Varma & Schwartz, 2011).

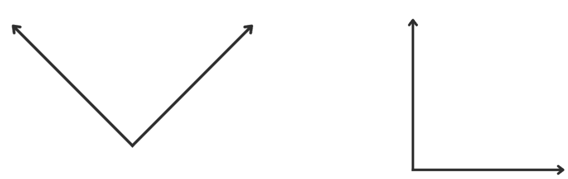
Grounding Geometry as Movement Discourse: The Case of Auxiliary Constructions in Balinese Dance

Ratih Ayu Apsari

Engaging in geometrical reasoning and proof often relies on generating auxiliary lines (Palatnik & Dreyfus, 2018). Teaching how to generate auxiliary lines, though, can be challenging, in part because students must envision figural elements that, by definition, are not yet available for perception (Fan et al., 2017; Herbst & Brach, 2006). Notwithstanding, humans share an innate capacity to spontaneously generate *attentional anchors*—perceptual forms that tighten one's grip on the environment (Abrahamson & Sánchez-García, 2016). Designed appropriately, we conjecture, geometry students' natural ability to produce attentional anchors as movement solutions could be tapped as a means of training them to construct auxiliary lines as geometry solutions. In this study we introduce a gridded floor mat and marking accessories, collectively called GRiD (Geometry Resources in Dance), designed to serve students as a frame of reference for both eliciting and objectifying their tacit attentional anchors as explicit auxiliary lines, as they engage in solving a Balinese dance task.

A 10-year-old Balinese dancer in training, Anna (pseudonym), participated in a task-based semi-structured clinical interview, in which she practiced a dance posture called Tapak Sirang Pada (TSP), where the feet, touching at the heels, formed a 90^0 angle. Qualitative analysis focused on Anna's multimodal justification of her posture.

When performing the posture, Anna *became* part of the angle incarnate, and saw the angle from a first-person (egocentric) perspective (Figure 1a; imagine looking down at your feet). The challenge is that, in traditional mathematics textbooks, a 90^0 angle is usually captured in a third-person (allocentric) perspective (Figure 1b). Therefore, Anna had to coordinate her perspectives.



a. Egocentric   b. Allocentric

Figure 1. Different perceptual perspectives and orientations of a 90^0 angle

Anna justified the positioning of her feet by focusing on the angle vertex, where her heels met. Using GRiD, she first gestured the imaginary point and lines that helped her determine she was forming a 90^0 angle (Figure 2a-c), and later placed three dot stickers on the tarp to mark her heels' junction and the tips of her toes, respectively (Figure 2d-e).

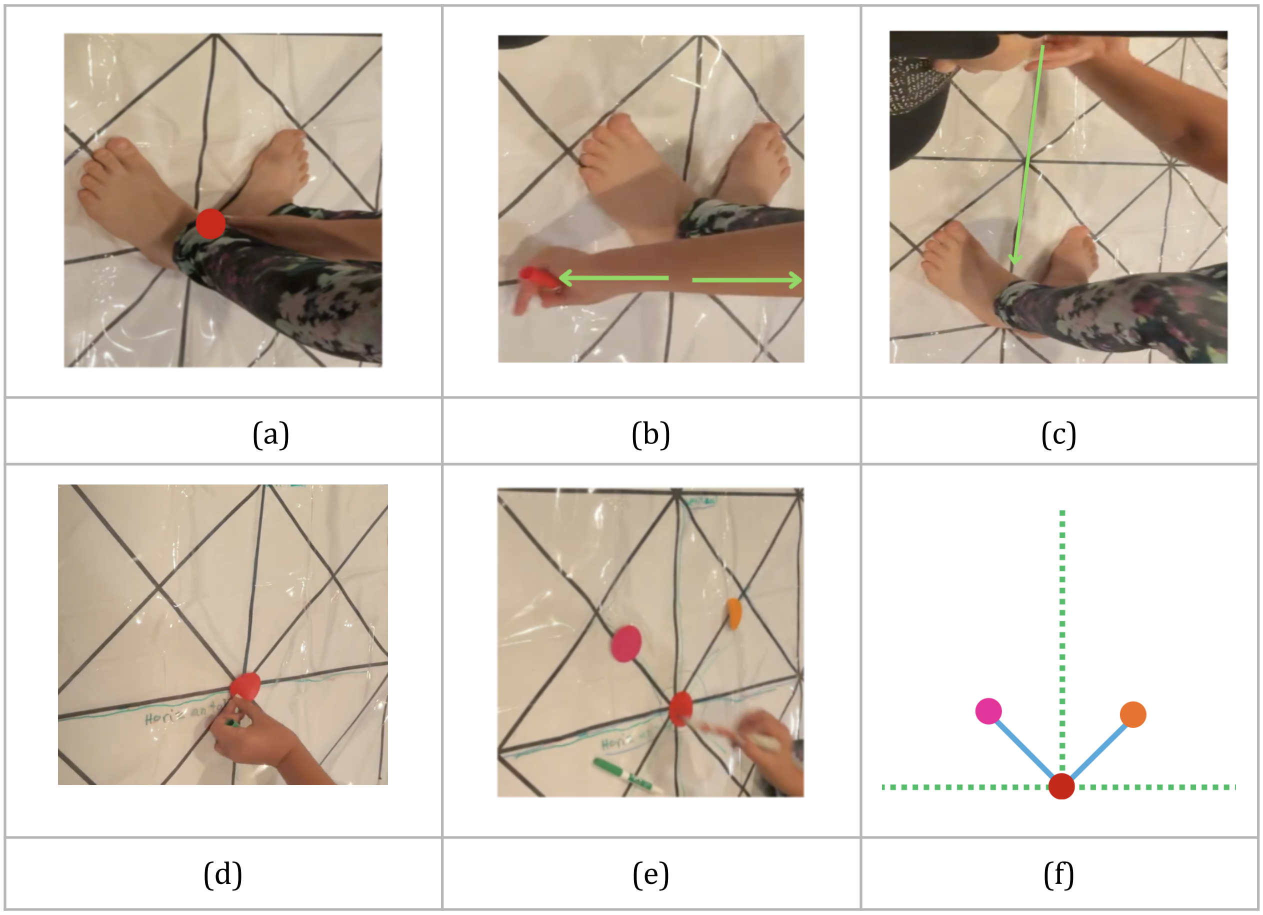


Figure 2. Anna's construction of auxiliary lines (green) to enact, evaluate, and explain (see Abrahamson et al., 2011) her heel-to-heel right-angled Balinese TSP dance posture. The blue lines mark the direction of her feet as diagonal to the green lines. The dot stickers mark the position of her heels (red) and toes (pink and orange).

First, Anna related metaphorically to the red dot as "*a place like a center, where everybody comes to meet. It is also where my heels click.*" Next, she ensured that her left and right feet constitute angle bisectors of the left and right 90^0 angles, so that the angle *between* her feet becomes the sum of twice 45^0: 90^0.

Once Anna had adjudicated the GRiD's auxiliary lines as effective attentional anchors for performing TSP, she was able to "transport" this geometrical construction when moving to different locations on the mat (see Figure 3). Anna exclaimed, "*This triangle thing always stays with my feet like whenever I move around!*" Anna was thus synergizing egocentric and allocentric perspectives (Benally et al., 2022)—her feet were forming a geometrically formal 90^0

angle (egocentric perspective), even as she could imagine the 90^0 angle markings (allocentric perspective) as affording TSP.

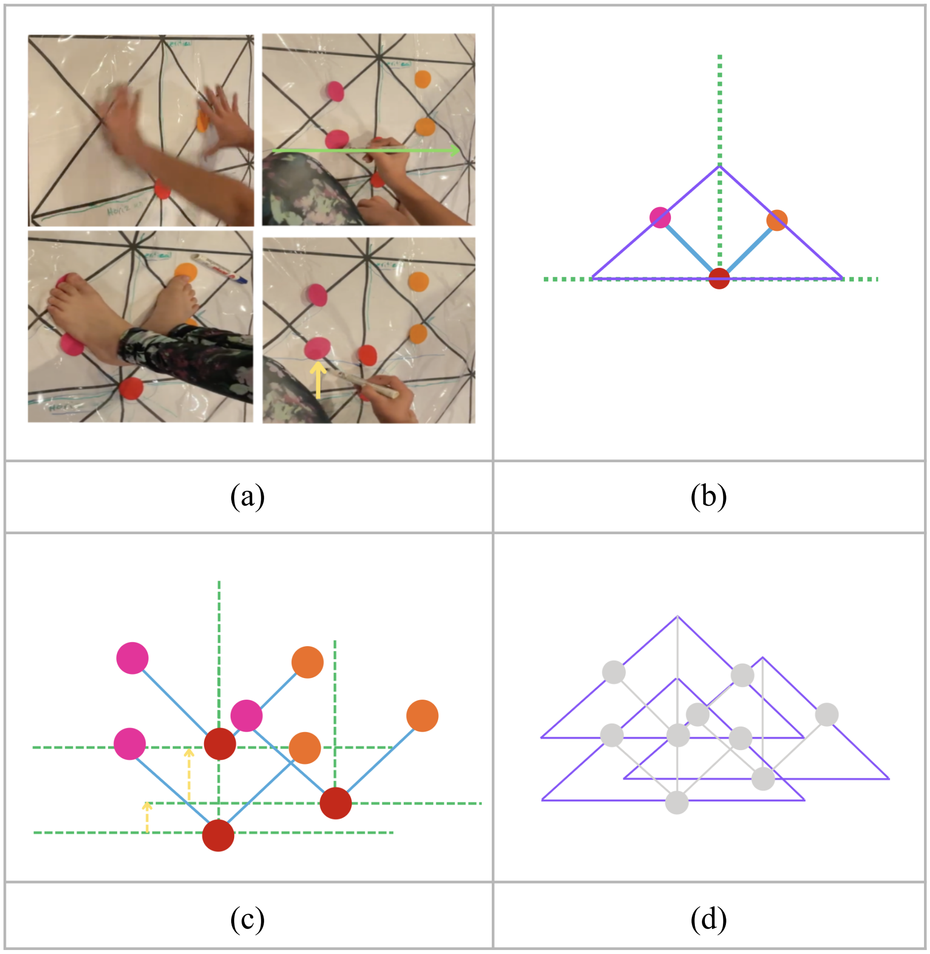


Figure 3. Anna uses available semiotic means (see Radford, 2006) to objectify her presymbolic attentional anchor for performing TSP in the form of geometrical constructions. The purple triangle (3b,d) is her attentional anchor to tackle a coordination of different elements (3c).

In summary, engaging in dance practice creates situated, authentic opportunities for students to negotiate allocentric and egocentric perspectives in order to objectify their movement-based attentional anchors in the form of auxiliary lines that construct mathematical reasoning.

Angling the stars: A Geometry Design Reconciling Indigenous and Colonial Perceptions

Jessica Benally

Mainstream mathematics is grounded in European epistemology and for Diné students, it is a colonial imposition that fails to recognize cultural knowledge and practices of this land. In particular, Indigenous epistemic patterns are not elicited, accepted, or recognized in classroom mathematics. The following research aims to rematriate (Tuck, 2011) mathematics curriculum by investigating Diné land-based ethnomathematics and designing an enactive, intersubjective activity in discourse with Western approaches. Diné geometrical reasoning encompasses the individual within the land and sky environment in practice and language and fosters egocentric perspectives (D'Ambrosio, 2001; Pinxten et al., 1983) while traditional pencil and paper classroom mathematics requires an allocentric perspective. The project builds an epistemologically pluralistic learning environment (Turkle & Papert, 1991) that restructurates common mathematical practices (Wilensky & Papert, 2010) as emulating and extending embodied immersion in the natural world (Dimmel & Milewski, 2019).

STARR (Students Tracking Angular Rotation Recorder) is a design-based instructional activity meant to tackle students' "absence of meaning" in the notion of an angle (Thompson, 2013). Implementing enactivist theory, we operationalize Thompson's (2008) cognitive analysis of the angle concept by grounding it in perceptual–motor enactment, particularly in the action of "opening" the angle's interior with the forearms (cf. Hardison, 2019, p. 361, on attentional motions; see Figure 1).

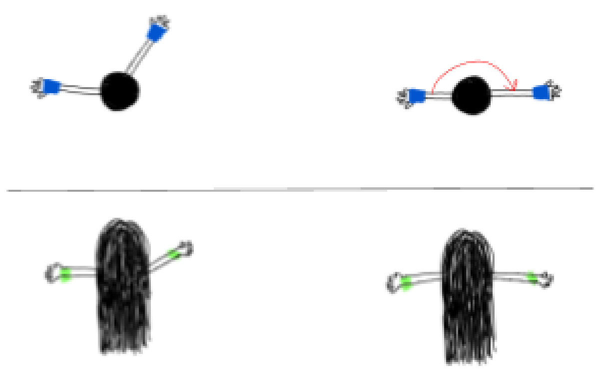


Figure 1. Top: the view above the student—the red arrow shows the full opening of their arms and the angle's interior. Bottom: back-of-the-head view of the student, showing the same opening of the arm and angle as the top. These two views are to be coordinated.

Dynamic Geometry Environments (DGE) offer design solutions for exploring angles interactively, however those technologies enable only single-person allocentric perspectives (Crompton, 2015; Smith et al., 2014). The embodied-enactive-egocentric approach elicits a perspective that is compatible with Diné epistemology, where being the angle has historically regulated ecological practices related to personal orientation and navigation. In contrast, the dualist Cartesian ontology that is inherent in Euclidean geometry portrays an allocentric orientation and perspective.

In STARR, students work together to create various angles that mimic the angles in a projected star constellation. Student A (see Figure 2, on the left) coordinates their egocentric and allocentric perspectives *inter*personally while creating the angle with their arms; Student B coordinates these perspectives *intra*personally while watching the movement of their peer. As such, students engage in mixed-media task-based collaborative experiences, through which they are to reconcile two complementary embodied perspectives on angle: (a) egocentric—"being" (Gerofsky, 2011; Student A as the angle vertex); and (b) allocentric—"seeing" the angle as it is portrayed in the constellation and as it is being enacted by another person (Benally et al., 2022; Student B).

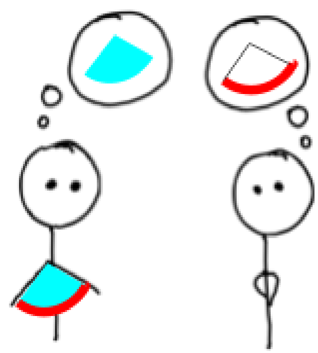


Figure 2. Student A (on the left) views their arms-angle egocentrically (angle interior shown in blue, angle arc shown in red), while Student B (on the right) views the same angle allocentrically.

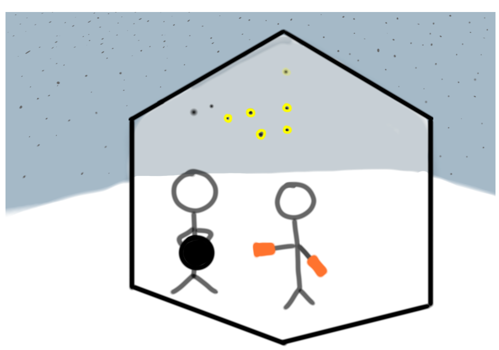


Figure 3: STARR planetarium. The student on the right is the Sensor who virtually walks the stars (egocentric perspective); the student on the left is the Navigator who guides the "Sensor (via allocentric perspective on the same constellation) using a digital compass. Students collaboratively negotiate their perspectives to achieve a complementary Navajo–Euclidean understanding of geometrical concepts, e.g., angle.

This project's astronomy design comprises both concrete and digital elements (see Figure 3) that recruit a variety of full-body multimodal interactions (i.e., visual, kinesthetic, proprioceptive, etc.). The angle interior is to emerge as students' attentional anchor (their perceptual solution for a motor-coordination problem; Abrahamson & Sánchez-Garcìa, 2016; see red curve in top of Figure 1). Through collaborative action, students objectify the interior space as a novel discursive ontology and can begin to perform operations on the interior of the angle (e.g., adding, equi-partitioning, measuring etc.). Project evaluation is investigating the co-constructed language used in the learning environment and whether the activity enables students to ground Euclidean geometric understanding in Diné epistemology.

Negotiating First-Person and Third-Person Perspectives in Micro-Phenomenological Research on Diagrammatic Reasoning

Julien Putz

For learning scientists, the phenomena they study in students are often the same cognitive processes that constitute their own research practices. This situation prompts us to reflexively apply our theoretical constructs to our own learning as researchers (Valenzuela-Moguillansky et al., 2021). When emerging research suggests that students gain conceptual insights by negotiating complementary perspectives on a phenomenon (Benally et al., 2022), then we may reflexively ask how analogous perspectival coordinations play out in our own scientific work. Here I reflect on the generative role that coordinations of first-person and third-person perspectives have played in a recent pilot study on diagrammatic reasoning (Putz, 2023).

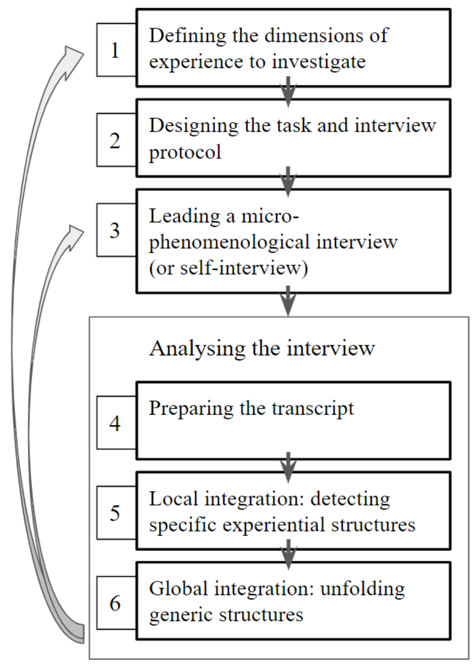


Figure 1. The micro-phenomenological research process (adapted from Petitmengin et al., 2019).

The study leveraged micro-phenomenology, an interview-based methodology for investigating the microdynamics of lived experience (Petitmengin, 2006; Figure 1). Volunteering graduate students (n=7) were invited to a session where they were asked to look at, and make sense of, a proof-without-words diagram (Figure 2, left) before participating in an interview about this experience. The interviewer helped the participant re-enact the experience and become aware of tacit aspects of their mental actions. The participant explored their own experiential landscape in the first-person, while the interviewer progressively built a descriptive map of this landscape from a third-person perspective. Such an 'I-You' perspectival coordination has been termed the *second-person* position in phenomenological research (Depraz et al., 2003).

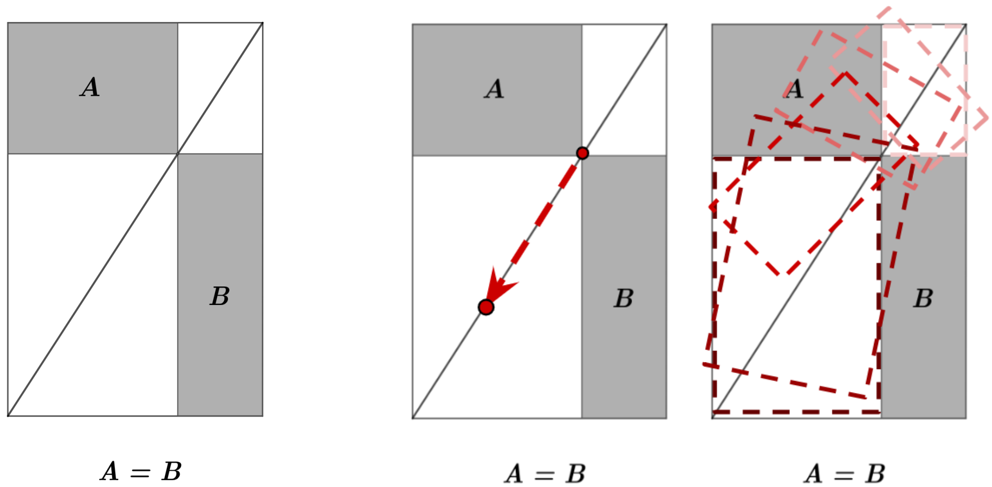


Figure 2. A sample proof-without-words diagram (left) and illustrations of how participants imaginatively moved parts of this diagram (middle, right). Anika (pseudonym) imaginatively moved a dot along a line (middle), while Mai (pseudonym) rotated and shrank one shape onto another (right).

One goal of the study was characterizing the experience of imaginatively moving part of a diagram (Figure 2). The analysis process (Petitmengin et al., 2019) consisted in identifying and representing, for each participant, the specific structure of such imaginative manipulations while iteratively unfolding the generic structure of this type of experience across all participants (Figure 3). While this analysis method relies on graphical displays and abstraction operations—necessitating the adoption of an allocentric perspective—the micro-phenomenologist always cycles back to the first-person perspective by staying close to the participant's concrete descriptions; the researcher may also experientially verify an emerging generic structure through *self-interviews* (Vermersch, 2007). These processes allow the researcher to perceive the flat representations as signifying dynamic first-person experiences.

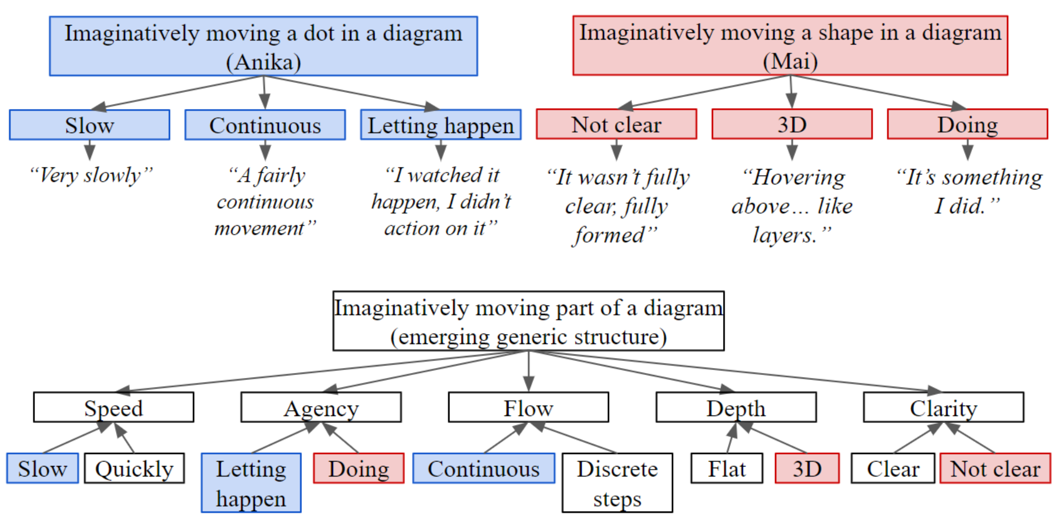


Figure 3. (Simplified) synchronic structure of imaginatively moving part of a diagram. Top: particular experiential structures from two singular experiences. Bottom: an emerging generic experiential structure of this type of experience.

Once an experiential structure has been identified, it can be put into dialogue with other research methods, including those relying on neurophysiological measures. In this study, experiential data were used to front-load analysis of complementary eye-tracking data collected during the participants' engagement with the diagram. This neurophenomenological approach (Varela, 1996; Berkovich-Ohana et al., 2020) is based on the idea that a comprehensive investigation of mental phenomena must reconcile two modes of orientation: a first-person perspective allowing for direct experiential contact with a phenomenon, and a third-person perspective facilitating the extraction of shared, stable features. Thus, for researchers, unlike for learners, perspectival coordinations are not merely tacit aspects of problem solving but can be explicitly utilized as central tools of methodological frameworks.

The perspectival coordinations presented here may be found in qualitative research more broadly. Indeed, qualitative analysis can be construed as the struggle to approach the first-person experience of participants from third-person observations and representations; learning scientists seek to "enter the child's mind" (Ginsburg, 1997).

References

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction, 2*(1), 1–16. https://doi.org/10.1016/j.ijcci.2014.07.002

Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Journal of the Learning Sciences*, *25*, 203–239. https://doi.org/10.1080/10508406.2016.1143370

Abrahamson, D., Trninic, D., Gutièrrez, J.F., Huth, J., & Lee R.G. (2011). Hooks and shifts: A dialectical study of mediated discovery. *Technology, Knowledge, and Learning*, *16,* 55-85.

Abrahamson, D. & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning 12*(1), 23–55.

Benally, J., Palatnik, A., Ryokai, K., & Abrahamson, D. (2022). Learning through negotiating conceptually generative perspectival complementarities: The case of geometry. *For the Learning of Mathematics. 42*(3), 34–41.

Berkovich-Ohana, A., Dor-Ziderman, Y., Trautwein, F.-M., Schweitzer, Y., Nave, O., Fulder, S., & Ataria, Y. (2020). The hitchhiker's guide to neurophenomenology – The case of studying self boundaries with meditators [Hypothesis and Theory]. Frontiers in Psychology, 11(1680). https://doi.org/10.3389/fpsyg.2020.01680

Bossé, M. J., Lynch-Davis, K., Adu-Gyamfi, K., & Chandler, K. (2016). Using integer manipulatives: Representational determinism. *International Journal for Mathematics Teaching and Learning, 17*(3), 1–20.

Crompton, H. (2015). Understanding angle and angle measure: A design-based research study using context aware ubiquitous learning. *International Journal for Technology in Mathematics Education*, *22*(1), 19–30.

D'Ambrosio, U. (2001). What is ethnomathematics and how can it help children in school? *Teaching Children Mathematics*, *7*(6), 308–310.

Depraz, N., Varela, F. J., & Vermersch, P. (2003). *On becoming aware: A pragmatics of experiencing*. John Benjamins Publishing.

Dimmel, J. & Milewski, A. (2019). Scale, perspective, and natural mathematical questions. *For the Learning of Mathematics, 39(3)*, 34–40.

Fan, L., Qi, C., Liu, X., Wang, Y., & Lin, M. (2017). Does a transformation approach improve students' ability in constructing auxiliary lines for solving geometric problems? An intervention-based study with two Chinese classrooms. *Educational Studies in Mathematics*, *96*(2), 229–248. https://doi.org/10.1007/s10649-017-9772-5

Filimon, F. (2015). Are all spatial reference frames egocentric? Reinterpreting evidence for allocentric, object-centered, or world-centered reference frames. *Frontiers in Human Neuroscience*, *9*, 648.

Gerofsky, S. (2011). Seeing the graph vs. being the graph: Gesture, engagement and awareness in school mathematics. In G. Stam & M. Ishino (Eds.), *Integrating gestures* (pp. 245–256). John Benjamins.

Gerofsky, S. (2010). Mathematical learning and gesture: Character viewpoint and observer viewpoint in students' gestured graphs of functions. *Gesture*, *10*(2–3), 321–343.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Hardison, H. L. (2019). Four attentional motions involved in the construction of angularity. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 360–369). University of Missouri.

Hawthorne, C., Philipp, R. A., Lamb, L. L., Bishop, J. P., Whitacre, I. & Schapelle, B. (2022). Reconceptualizing a mathematical domain on the basis of student reasoning: Considering teachers' perspectives about integers. *Journal of Mathematical Behavior, 65*, 1–15. https://doi.org/10.1016/j.jmathb.2021.100931

Hegarty, M., & Waller, D. (2004). A dissociation between mental rotation and perspective-taking spatial abilities. *Intelligence*, *32*(2), 175–191.

Herbst, P., & Brach, C. (2006). Proving and doing proofs in high school geometry classes: What is it that Is going on for students? *Cognition and Instruction*, *24*(1), 73–122. https://doi.org/10.1207/s1532690xci2401_2

Herbst, P., Fujita, T., Halverscheid, S., & Weiss, M. (2017). *The learning and teaching of geometry in secondary schools: A modeling perspective*. Routledge.

Kirsh, D. (2010). Thinking with the body. In S. Ohlsson & R. Catrambone (Eds.), *Proceedings of the 32nd Annual Meeting of the Cognitive Science Society* (pp. 2864–2869). Cognitive Science Society.

Nurnberger-Haag J. (2018). Take it away or walk the other way? Finding positive solutions for integer subtraction. In L. Bofferding & N. M. Wessman-Enzinger (Eds.), Exploring the integer addition and subtraction landscape: Perspectives on integer thinking (pp. 109–141). Springer.

Palatnik, A., & Dreyfus, T. (2018). Students' reasons for introducing auxiliary lines in proving situations. *Journal of Mathematical Behavior*, *55*, 100679. https://doi.org/10.1016/j.jmathb.2018.10.004

Petitmengin, C. (2006). Describing one's subjective experience in the second person: An interview method for the science of consciousness. *Phenomenology and the Cognitive Sciences, 5*(3), 229–269.

Petitmengin, C., Remillieux, A., & Valenzuela-Moguillansky, C. (2019). Discovering the structures of lived experience: Towards a micro-phenomenological analysis method. *Phenomenology and the Cognitive Sciences, 18*(4), 691–730.

Pinxten, R., van Dooren, I. & Harvey, F. (1983). *The anthropology of space: Explorations into the natural philosophy and semantics of the Navajo.* University of Pennsylvania Press.

Putz, J. (2023), *A Micro-Phenomenological Approach to the Study of Diagrammatic Reasoning*. Unpublished manuscript, UC Berkeley.
Radford, L. (2006). The anthropology of meaning. *Educational Studies in Mathematics, 61*, 39-65.
Smith, C. P., King, B., & Hoyte, J. (2014). Learning angles through movement: Critical actions for developing understanding in an embodied activity. *The Journal of Mathematical Behavior*, *36*, 95–108.
Thompson, P. W. (2013). In the absence of meaning… . In Leatham, K. (Ed.), *Vital directions for research in mathematics education* (pp. 57–93). New York, NY: Springer.
Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education* (Vol 1 [Plenary papers], pp. 31–49). PME.
Tuck, E. (2011). Rematriating curriculum studies. *Journal of Curriculum and Pedagogy*, *8*(1), 34–37.
Tversky, B., & Hard, B. M. (2009). Embodied and disembodied cognition: Spatial perspective-taking. *Cognition, 110*(1), 124–129. https://doi.org/10.1016/j.cognition.2008.10.008
Turkle, S., & Papert, S. (1991). Epistemological pluralism and the revaluation of the concrete. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 161–192). Ablex Publishing.
Valenzuela-Moguillansky, C., Demšar, E., & Riegler, A. (2021). An introduction to the enactive scientific study of experience. *Constructivist Foundations, 16*(2), 133-140.
Varela, F. J. (1996). Neurophenomenology: A methodological remedy for the hard problem. *Journal of Consciousness Studies, 3*(4), 330–349.
Varma, S., & Schwartz, D. L. (2011). The mental representation of integers: An abstract-to-concrete shift in the understanding of mathematical concepts. *Cognition, 121*(3), 363–385. https://doi.org/10.1016/j.cognition.2011.08.005
Vermersch, P. (2007). Bases de l'auto-explicitation. *Expliciter, 69*, 1-31.
Wilensky, U., & Papert, S. (2010). Restructurations: Reformulations of knowledge disciplines through new representational forms. In J. Clayson & I. Kallas (Eds.), *Proceedings of the Constructionism 2010 Conference (The 12th EuroLogo conference)*. Paris.