A MICRO-PHENOMENOLOGICAL APPROACH TO THE STUDY OF DIAGRAMMATIC REASONING

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This pilot study investigated the experiential micro-dynamics of diagrammatic reasoning. Four graduate students were asked first to individually examine a proof-without-words diagram and then to participate in an interview about this experience. Leveraging the micro-phenomenological interview method, participants were guided through accessing, exploring, and describing tacit aspects of their experiential landscape. Analysis of these interview data consisted of: (a) characterizing, for each participant, the specific structures of their experience; while (b) iteratively discerning generic structures across participants. Two generic structures emerged as bearing across participants: (1) enacting a particular manipulation in one part of the diagram, repeated elsewhere in contracted form; and (2) imaginatively moving part of a diagram. Developing fine-grained characterizations of diagrammatic reasoning, I argue, may inform enactive cognitive theory as well as educational design.

INTRODUCTION: ZOOMING IN ON DIAGRAMMATIC REASONING

Diagrams are ubiquitous epistemic devices within mathematical practices, both in classrooms and among research mathematicians. Peirce (1933) defined a *diagram* as a sign "which is predominantly an icon of relations and is aided to be so by convention" (para. 418). He characterized *diagrammatic reasoning* in terms of constructing diagrams, experimenting on them, and observing the results of the experimentation. He held that "the best thinking, especially on mathematical subjects, is done by experimenting in the imagination upon a diagram or other scheme" (1976, p. 122). While subsequent work has corroborated the central roles diagrams play in mathematical cognition (e.g., Bakker & Hoffmann, 2005), researchers have yet to develop fine-grained accounts of the perceptual, imaginative, and attentional skills involved in diagrammatic reasoning. Here I present results from a pilot study investigating the micro-dynamics of how people perceive, and make sense of, *proofs-without-words* (PWWs; Nelsen, 1993). As I report, the study revealed covert mental actions within experiences of diagrammatic reasoning, notably shifts of attention and imaginative manipulations of diagrams.

METHODOLOGY: SURFACING TACIT ASPECTS OF DIAGRAMMATIC REASONING

Working individually, four volunteering graduate students, with either a degree or teaching credentials in mathematics, were asked to inspect a PWW (Figure 1a, 1c, 2a) presented on a computer screen, and then to participate in an interview about this experience. The interview followed the approach of *micro-phenomenology* (MP; Petitmengin, 2006), a research program for investigating the micro-dynamics of lived experience, where the interviewer induces from participants a vivid episodic memory to bring forth tacit aspects of an experience. Participants were guided to evoke their engagement with the diagram, to shift their attention from the content of this experience (i.e., what they perceived) to its structure (i.e., how they perceived), and to precisely describe this structure. After transcribing the interviews and evaluating their reliability, they were analyzed using the MP method (Petitmengin et al., 2019): I first examined the *diachronic* unfolding of each described experience, dividing it into temporal phases, before exploring *synchronic* characteristics of specific moments. While identifying and representing, for each participant, the specific structures of their experience, I iteratively uncovered generic structures across all participants.

FINDINGS: FROM PARTICULAR TO GENERIC EXPERIENTIAL STRUCTURES

Diachronic Structures: Repeating a manipulation in contracted form

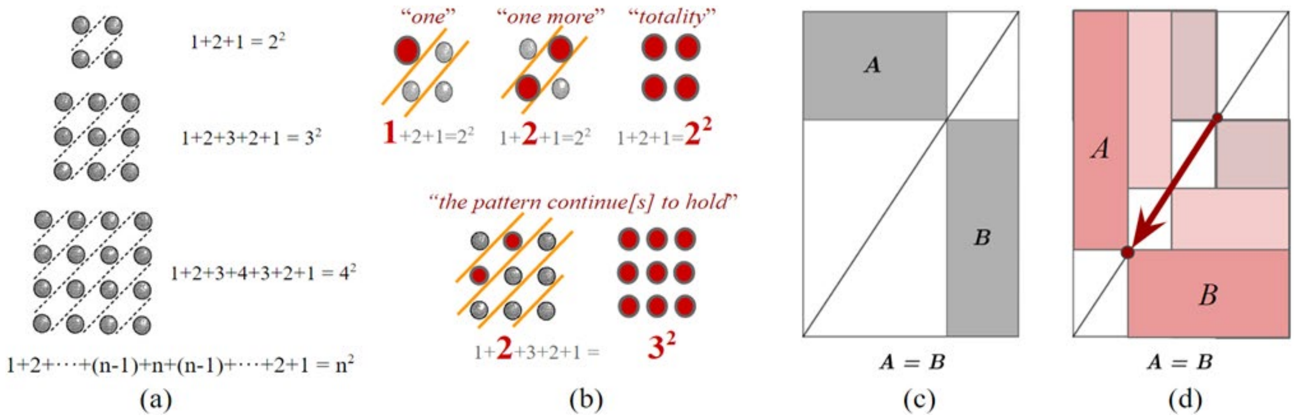

Figure 1: Proofs-without-words (a, c), with illustrations of how Philip shifted his attention to foreground different circles (b), and how Anika imaginatively moved a dot along a line (d).

When looking at the PWW in Figure 1a, Philip (all names are pseudonyms) inspected the equations on the right side of the diagram and noted the increasing–decreasing pattern in the summed terms. Attending to the top arrangement of circles, he "recognized that [it] … is broken up into sections":

> Philip: I saw this single circle … in the top left (Figure 1b, upper left). I felt a sense of concentration, my focus narrowing in. In that moment, I was mentally linking that circle as 'one.' After that, my eyes naturally lead … down diagonally [across the] dividing line. I'm taking in the [middle] section as a unit (Figure 1b, upper middle). I'm recognizing that there are two [circles], and that corresponds to the '2' in the equation on the right. There's like a scooping of the next section. My eyes have moved from … the top left, then sort of flowing down and zooming out. … My awareness travels down that flow, and then comes up at the bottom-right corner, at the last circle there.

After relating the diagonal cross-slices of circles to the summed terms in the equations, Philip turned to the "n-squared" term in the bottom equation. He again looked up at the top figure and was puzzled for a moment. He "unfocused … a little bit, allowing … more into focus," and "recognized how it's the totality of the square of all circles that is n-squared" (Figure 1b, upper right). It was "kind of like scooping up the whole thing." He then "looked down at the next [part] … and saw how the pattern continued to hold" (Figure 1b, lower). Philip now had "a sense of getting it," which marked the end of his engagement with the PWW. The specific diachronic structure of Philip's experience spans five distinct phases: (1) appreciating the figures; (2) parsing the equations; (3) "scooping" parts of the first figure; (4) taking in the whole square; and (5) a sense of getting it, each divided into further subphases.

After conducting similar analyses for the other interviewees (Anika, Jade, Mai), I looked for generic commonalities among the specific diachronic structures. A noteworthy convergence concerns Philip's act of extending an attentional shift from one part of the diagram to another: after "allowing more into focus" and perceiving the first collection of circles as a square, Philip saw that this "pattern continued to hold" for the second collection below (Figure 1b, lower). Similarly, Jade and Anika carried perceptual insight from one place to another while engaging with Figures 2a and 1c, respectively:

> Jade: I kind of envision this copy of the top triangle … being kind of rotated and moved onto the bottom triangle (Figure 2b). Then I checked … to see if the bottom part was a rotated version of the top (Figure 2c). I was able to … quickly see: … that's the same thing here.
>
> Anika: Then it was an immediate reaction: these [white triangles at the bottom left] are exactly equal. … As soon as I did that, I moved up to [the other white triangles] and was like, okay, the same thing is happening here.

Each of these descriptions follows an analogous diachronic structure: First, participants enact a process in one part of the diagram. Next, they re-enact this process elsewhere in reduced form; they note that the process can be repeated but do not feel the need to do so in full. We may regard this as a case of what Radford (2008) calls *contraction*—condensing a procedure to a more compact form.

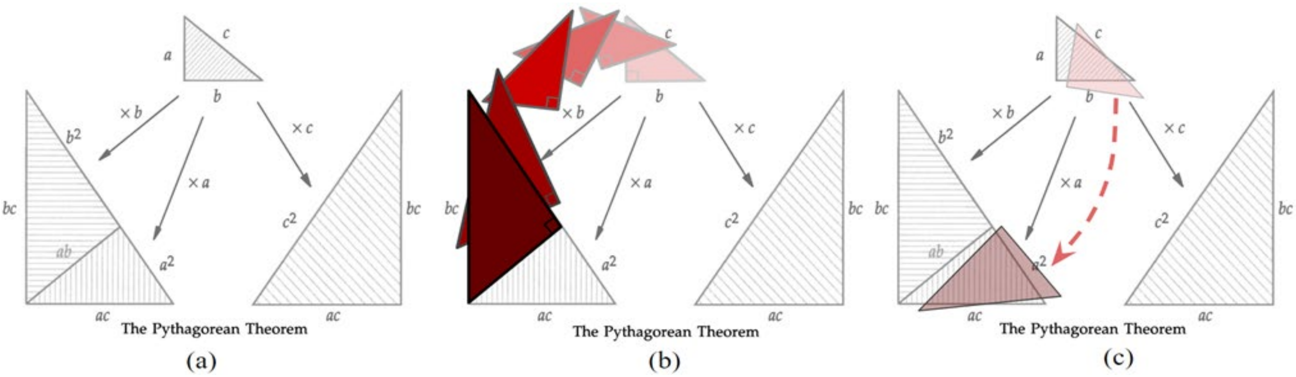

Figure 2: Proof-without-words (a), with an illustration of how Jade imaginatively rotated, dilated, and shifted one triangle onto another (b), and how she repeated this action in contracted form (c).

Synchronic structures: Imaginatively moving part of a diagram

After initially perceiving the PWW in Figure 1c as static, Anika animated it in her imagination, thus envisaging the grey rectangles' juncture point as moving down along the diagonal line (Figure 1d). She described this imagined sliding as "slow" and "continuous"; she let the movement happen, as opposed to deliberately controlling it; while she focused primarily on the lower part of A, other parts of the figure were also changing in her visual periphery. These descriptions reveal the specific synchronic structure of this phase, which is represented as a *semantic network* (Figure 3, upper).

Other participants also described experiences of imaginatively moving parts of their diagrams. While looking at the PWW in Figure 1c, Mai was trying "to rotate and shrink the [bottom white] rectangle… to match the other [top white] rectangle. … But it wasn't fully clear, fully formed." By analyzing all such moments across participants, I identified more general codes and structures (Figure 3, lower). Arrows indicate abstraction operations from the MP analysis process: the utterance "very slowly" is *classified* as an instance in the category 'slow'; the categories 'slow' and 'quickly' are *generalized* to 'speed'; and 'speed,' 'flow,' 'agency,' 'depth,' etc. are *aggregated* into the type of experience studied.

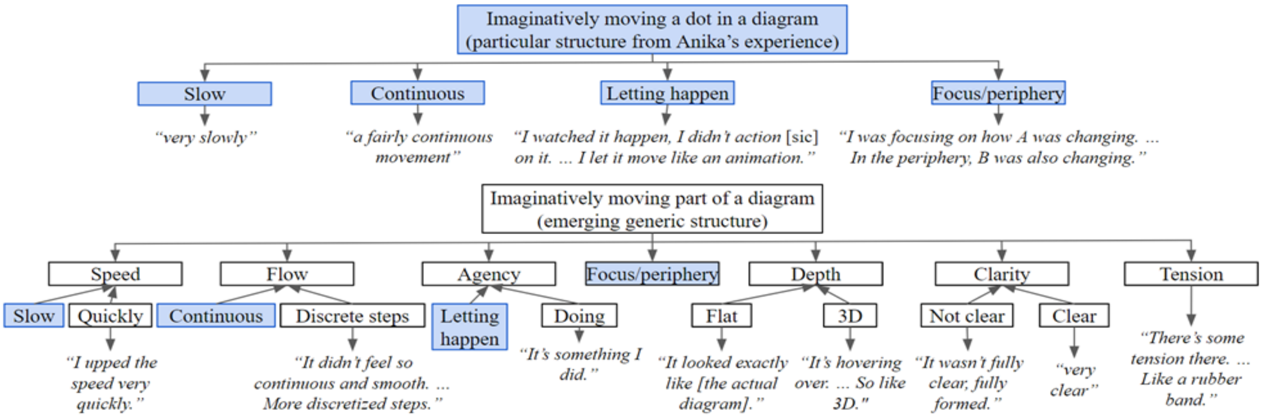

Figure 3: Synchronic structure of imaginatively moving part of a diagram. Upper: the particular structure from Anika's experience. Lower: an emerging generic structure of this type of experience.

CONCLUSION: TOWARDS ENACTIVIST THEORY AND DESIGN

The study leveraged MP to elicit precise descriptions of how graduate students make sense of PWWs. Analysis of these descriptions revealed potentially generic experiential structures. Each participant described the enactment of a particular manipulation in one part of the diagram, repeated elsewhere in contracted form. Most participants further reported imaginatively moving parts of a diagram. This type of experience may be characterized by speed ('slowly' or 'quickly'), agency ('doing' or 'letting happen'), flow ('continuous' or 'discrete steps'), and depth ('flat' or '3D'). Other experiences noted by interviewees include subvocalization and a narrowing/widening of attention. Through subsequent interviews, these experiential structures will be validated and refined. This work may elucidate how seemingly stationary reasoning processes centrally involve covert actions, thus contributing to an elaboration of the enactivist view on mathematical cognition (Varela et al., 1991). Finally, an MP approach may enrich classroom discourse by foregrounding pre-reflective aspects of mathematical sensemaking, and it may enhance the *embodied design* procedure (Abrahamson, 2014) by systematizing the process through which designers phenomenalize target mathematical notions.

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