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Embodied Design: Bringing Forth Mathematical Perceptions

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ABSTRACT Embodied design is a proactive educational research program that promotes and investigates humans' universal capacity to understand STEM concepts. The program's empirical work is centered on design-based research projects that contribute theory to the Learning Sciences through the practice of building, implementing, and evaluating experimental pedagogical architectures that inform instructional practice. Using both historical and emerging technologies, embodied-design activities are typically two-stepped: (1) draw on students' evolutionary inclination for purposeful sensorimotor engagement with the natural environment; and only then (2) introduce heritage symbolic artifacts that students initially adopt to enhance the enactment, evaluation, or explanation of their intuitive judgments and actions, yet, in so doing, find themselves adopting normative disciplinary forms, language, representations, and solution procedures. Embodied-design researchers apply mixed methods — from ethnomethodological conversation analysis through to multimodal learning analytics and cross-Recurrent Quantification Analysis — in analyzing empirical data of learning process, including records of students' motor actions, sensory behavior, and multimodal utterance in conversation with peers and instructors. Several decades of projects across numerous mathematical content domains have increasingly implicated *perception* — a hypothetical Psychology construct believed to govern sensorimotor and cognitive behavior — as pivotal in explaining students' capacity to first solve challenging motor-control coordination problems and then bridge through to discursive articulation of their movement strategy. As they attempt to operate the educational technology according to an unknown interaction regimen, new information patterns, e.g., an imaginary line connecting their hands, come forth spontaneously into students' perceptual experience as their cognitive means of managing the enactment of the activity's targeted movement forms. These emergent, proto-mathematical, multimodal, dynamical ontologies are then languaged and entified into consciousness, grounding the meaning of conceptual terminology and procedural routines. The embodied-design framework has been applied in building technologies for students of intersectional diversity, including populations of minoritized epistemic — linguistic practices and atypical neural, cognitive, and sensorial capacity.

Keywords: Attentional anchor, Enactivism; Mathematics Imagery Trainer; Movement; Technology.

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1. Introduction to Embodied Design

1.1. Objectives, disciplinary foundations, inspirations, and ethical positionality

Embodied design (Abrahamson, 2009, 2012, 2014, 2015) is a research quest to understand what it means to learn a mathematical concept. We ask the ontological question, what is a mathematical concept that we can understand it, and we ask the epistemological question, what is the mind that it can understand a mathematical concept? Operating from a broad reading of the cognitive sciences, we address these grand questions through examining how people teach and learn together in activities centered on technological artifacts we build and develop. In our theorizing, design craft, and data analyses, we are chiefly informed by the embodied paradigm shift in the cognitive sciences, which has foregrounded the formative role of situated sensorimotor interaction in the phenomenology of conceptual reasoning (Newen et al., 2018).

The disciplinary affiliation of embodied design is the Learning Sciences, a field of study developed in the 1980's by cognitive scientists wishing to apply their theories and methods to empirical investigations of educational practice. A premise of the Learning Sciences is that researchers should assume a transformative orientation toward educational problems — they should not only document, diagnose, and denounce these problems (e.g., the "misconceptions" genre) but dismantle and ameliorate the phenomena by way of building and evaluating theory-based alternatives. This quest to engineer better educational practices was called *design experiments* (Collins, 1992). With time, the name evolved into *design-based research* (Cobb et al., 2013) or, variably, just *design research* (Bakker, 2018). In its ethical foundations to improve extant cultural practices, design-based research aligns well with revisionist readings of foundational tenets driving Lev Vygotsky's cultural — historical psychology: Culture is taken not as a *status quo* but in its very essence as a system in flux that necessarily requires continuous adaptation to avail of envisioned opportunities and counter emergent contingencies (Stetsenko, 2017). Embodied design work is always conducted as design-based research studies (Abrahamson, 2015).

Embodied design is inspired by educational visionaries, from Friedrich Fröbel, Maria Montessori, Caleb Gattegno, and Hans Freudenthal through to Seymour Papert, Mitch Resnick, Uri Wilensky, and Ricardo Nemirovsky, whose pedagogical artifacts creatively utilize technology to empower young learners. Originating in the University of California Berkeley at the Embodied Design Research Laboratory, embodied design is now pursued by collaborators and colleagues worldwide (Abrahamson et al., 2020). While most embodied design projects to date have addressed mathematical concepts, its framework caters more broadly to STEM domains (Abrahamson and Lindgren, 2014). The embodied-design framework has been applied to a range of concepts (Alberto et al., 2021) to serve students of intersectional diversity, including populations of minoritized epistemic — linguistic practice (Benally et al., 2022), atypical neural, cognitive, and sensorial capacity (Lambert et al., 2022; Tancredi et al., 2021), and at remote locations (Shvarts and van Helden, 2021).

1.2. Grounding conceptual meaning in perceptual phenomenology

Embodied design is a design-based research effort that includes a design framework mobilizing its research agenda. The embodied-design framework informs the creation of learning environments, where students construct the meaning of mathematical concepts and procedures. The embodied-design research agenda is to understand how students construct mathematical meanings in these environments.

By "construct" we draw on the theories of genetic epistemology (a.k.a., "constructivism," Piaget, 1971), radical constructivism (Steffe and Kieren, 1994), and enactivist cognition (Varela et al., 1991) to stipulate students' active role in making sense of the world through goal-oriented embodied engagement. As Piaget (1971) writes,

> Knowing does not really imply making a copy of reality but, rather, reacting to it and transforming it (either apparently or effectively) in such a way as to include it functionally in the transformation systems with which these acts are linked (p. 6).

By "meaning," in turn, we refer to a presymbolic notion (Radford, 2014) — a phenomenological orientation toward engaging the world purposefully that lends a sense of understanding for a mathematical sign, such as the notation "+" symbolizing the arithmetic operation of addition. The meaning of "+" might be experienced as bringing the hands toward each other to accumulate substance, whether one actually enacts this movement form or imagines doing so. This bimanual "image making" (Pirie and Kieren, 1989) or "concept image" (Tall and Vinner, 1981) associated with the notation "+" is experienced as a non-linguistic dynamic bodily feeling of acting on the world — the embodied experience grounds the mathematical symbol in sensorimotor phenomenology (Harnad, 1990). Put colloquially, the meaning of a mathematical concept is not inside the signs we read or write — it's what we experience when we first sense that we got its core idea, it clicked for us, we grasp it, we have a grip on it, we own it, we can improvise on it. But embodied design maintains that we can develop a new grip on the world even before we appreciate that it will become mathematically meaningful (Bartolini Bussi and Mariotti, 2008; Nathan, 2012; Vogelstein et al., 2019). This makes sense developmentally — we learn to add stuff with our hands long before we know the word "add" (L. B. Resnick, 1992); years before doing so lends meaning to the arithmetic idea of addition (Silverman, 2021).

Embodied designs are necessary, because mainstream education may not occasion opportunities for students to develop canonical dynamical image perceptions as the core proto-mathematical meanings grounding their conceptual understanding. The research program of embodied designs is motivated by a concern for students' general "absence of meaning" (Thompson, 2013) for mathematical concepts, which we diagnose as the absence of enactive capacity to understand the concepts. Embodied designs create the socio-material conditions for students to learn a mathematical concept by developing capacity to enact the movement form that later becomes the meaning of the targeted concept's inscriptional markings. For example, what might be a presymbolic enactment of "proportion" that would be analogous to the bimanual enactment of "addition" discussed above? Most people are absent an enactive meaning for "proportion" — they are hard pressed to enact the concept, to gesture it. How do you grasp a proportion and mobilize it? What might be a dynamical invariant that you enact and maintain as you move in proportion to think through it, talk about it, teach it? And how about a parabola? A sine function on the unit circle? As we now explain, to develop enactive capacity is to develop new ways of attending perceptually to the world for organizing the enactment of movement forms.

1.3. Perception: The cognitive pivot from phenomenology to language

According to Varela et al. (1991), "the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (p. 173).

Embodied design emulates the enactive maxim by presenting students with motor-control problems whose dynamic solution requires discovering sensorimotor patterns from whence emerge proto-mathematical cognitive structures. For example, a cognitive structure grounding the mathematical concept of proportionality emerges from solving a motor-control problem whose solution is raising the hands simultaneously at different constant speeds above a common surface. Try this. Place both hands palms down on your desk, and now raise them both at once, perhaps with the right hand moving double as fast as the left. Appreciate the strangeness of this movement form and the challenge of enacting it. How are you accomplishing this task? What sensory modality are you attending to? What are your criteria for maintaining the dynamical form? What have you figured out? We submit that learning to enact movement forms is where meaning is potentiated for mathematical concepts. Still, what exactly is the role of perception in performing this bimanual movement? Why do developmental psychologists and enactivist philosophers implicate our natural perceptual faculty as soliciting the mental construction of new cognitive structures from recurrent sensorimotor behavior? And how could doing all this become mathematics?

Empirical research in the movement sciences has demonstrated that the human capacity to enact challenging bimanual movements, such as lifting the hands at different speeds, is achieved by developing new perceptual orientations towards the activity situation (Mechsner, 2004). In the absence of appropriate perceptual orientation, a task may appear daunting, even insurmountable, and yet once the perceptual orientation has been established — through exploration, guidance, or some mix thereof — the impossible task becomes manageable. What more, one is often able to articulate how one is orienting perceptually toward a situation, such as when we teach a novice how to parallel-park, flip an omelet, or finesse a crochet stitch. Teachers are particularly good at explicating their expertise (Newell and Ranganathan, 2010; Shulman, 1986), and doing so often involves highlighting for the novice within a shared situation certain embedded forms that the expert discerns but the novice does not (Flood et al, 2020; Goodwin, 1994).

A given bimanual movement form may be perceptually guided in a variety of ways. For example, to raise your hands at different speeds, you might attend to the vertical spatial gap between the hands and keep increasing that gap; you might ensure that the right hand is always double as high above the surface as the left hand; or that the left hand is always half as high as the right; and so on. That is, the phenomenology of performing a bimanual movement form may vary, and each variation instantiates a different mathematical model of the movement form. Calling movement forms "polysemous," Abrahamson et al. (2014) demonstrated that coordinating among these models may lend new conceptual insight.

Note the ontological difference between the movement form as described by a third person, for example, "Dor is raising his hands such that his right hand is moving twice as fast as his left hand," and the individual's first-person experience, for example, "The vertical gap between my lower left hand and my upper right hand should always be equal to the height of my left hand over the surface." Embodied designers are interested in foregrounding the first-person experience — soliciting, characterizing, and documenting its variability across students — because we believe that talking and gesturing about these experiences can improve both research and practice (Abrahamson et al., 2022).

Embodied design begins from our species' universal capacities for thriving in natural and cultural ecologies. Being the biological organisms that we are, we are evolutionarily inclined to solve the existential problem of learning to perform new movements, whether walking, waltzing, or weaving, by discovering task-effective sensorimotor patterns — the how of attending to a situation. This natural neural proclivity to develop new perceptual orientation toward the environment as a means of operating on it can be solicited in fields of promoted action (Reed and Bril, 1996), social interventions that foster the development of culturally valued movement forms. Once novices figure out how to move in a new way, they can be encouraged to verbalize how they are perceiving the situation, which, under appropriate pedagogical settings, may lead to normative disciplinary discourse, including performing various inscriptional routines. Thus, cognitive structures that enable perception to guide action emerge as ontologies grounding mathematical concepts. It is in this sense that embodied design enables students to construct mathematical meaning from perception.

Having explained the rationale and theoretical underpinnings of embodied design, we now turn to discussing findings from research studies that evaluated activities built according to the framework. At the center of these activities is a type of pedagogical interaction architecture called the Mathematics Imagery Trainer.

2. The Mathematics Imagery Trainer

2.1. Rationale and build

The Mathematics Imagery Trainer (hence, the Trainer) is an activity architecture designed to serve as an instrumented field of promoted action (Abrahamson and Trninic, 2015) — a technological apparatus for administering embodied-interaction activities, in which students learn to participate in the physical enactment of an epistemic practice. Students are invited to solve a motor-control problem, where they manipulate virtual objects in an attempt to change the state of the environment, for example to cause a red screen to become green and then stay green as they keep moving the objects. That is, students learn to move in a particular way that is coded into the Trainer's digital feedback regimen, for example to lift their hands at the speed ratio of 1:2, where the right hand rises double as fast as the left (see Fig. 1). Learning to move in this new dynamical form is challenging, because the feedback regimen frustrates students' existing repertory of sensorimotor schemes for interacting with the environment. For example, they may try to raise their hands at the same speed, only to be repeatedly countenanced (red screen), so that they must readjust their hands' positions. To assimilate the feedback regimen of the obdurate environment, students must accommodate their schemes (Abrahamson et al., 2016). They learn to move in a new way — an ecologically coupled way (Abrahamson and Sánchez–García, 2016).

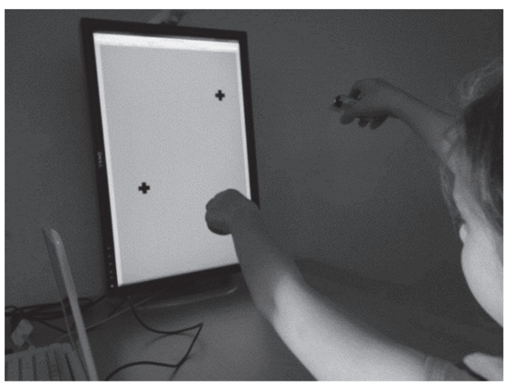

Fig. 1. The Mathematics Imagery Trainer for Proportion set at 1:2. A child is manipulating two virtual objects. The right-hand object is twice as high above the bottom of the screen as compared to the left object. This spatial configuration of the two objects relative to each other satisfies the task of making the screen green. To move her hands in constant green, the child would need to keep this ratio. She will learn to move in a new way by attending to a new information structure.

The new movement forms that students learn to perform have been designed as "conceptual micro-choreographies" (Abrahamson and Sánchez–García, 2016), in the sense that these dynamic forms bear semiotic potential (Bussi and Mariotti, 2008) to become mathematically meaningful through quantitative modeling. The semiotic consolidation of movement as mathematics is then ushered by making available to students a variety of symbolic artifacts (Sfard, 2002), such as a grid of lines laminated onto the activity space (see Fig. 2). Students recognize in these new resources potential instruments for enhancing the enactment, explanation, or evaluation of their effective movement strategy. Consider students who'd been raising the two virtual objects simultaneously while attending to the vertical gap between the objects (Fig. 2b). They say, "The higher the hands go, the bigger the distance between them" (Abrahamson et al., 2011). When the grid is flashed onto the screen (Fig. 2c), the students initially attempt to replicate this same strategy for enacting the movement form that had been satisfying the task conditions. Yet, as they raise their hands now, a horizontal line affords a convenient specified location to "park" one of the virtual objects, while the other hand searches for its complementary location that makes the screen green. As such, the sensorimotor pattern that had solved the motor-control problem of making the screen green becomes distributed over the environment, so that the students find themselves drawn into a new sensorimotor pattern, where the hands are moving sequentially, ratcheting up the lines. They say, "For every 1 line I go up on the left, I go up 2 lines on the right" (Abrahamson et al., 2011). Thus, as they engage the utilities that they discern in the new accessories to improve their grip on the world, students transition into enacting new movement forms that incorporate the symbolic artifacts. In so doing, the students appropriate quantitative frames of reference, so that their utterance takes on the linguistic forms of normative disciplinary discourse (Abrahamson and Bakker, 2016).

2.2. Attentional anchors

How does an invisible spatial interval between two virtual objects suddenly avail itself as a perceptual means of managing a challenging motor-control task? The embodied-

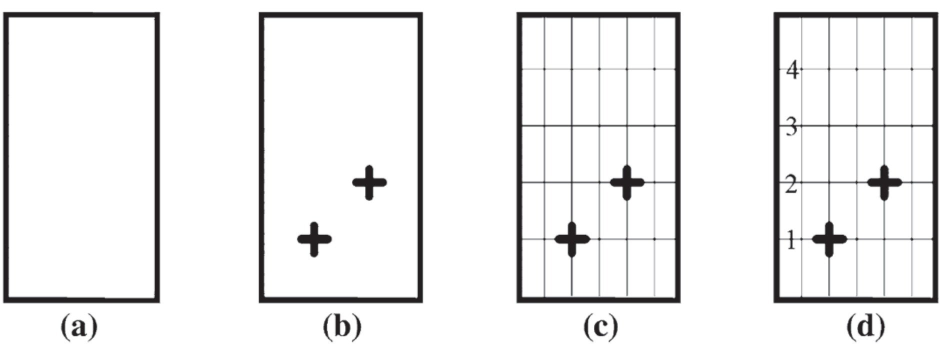

Fig. 2. From movement to mathematics — interpolating symbolic artifacts into students' activity space brings about transitions in acting, thinking, and speaking: (a) when the screen shows no virtual objects, students focus on their hands; (b) introducing virtual objects draw students' attention to the screen, where they explore for movement forms that sustain the favorable feedback; (c) supplementing a grid changes the activity space from continuous to discrete — students incorporate the lines as a frame of reference and develop a unitized movement form; and (d) further supplementing numerals solicits students' arithmetic skills, enabling them to calculate and predict right–left locations satisfying the feedback regimen.

design learning process depends on this intriguing perceptual phenomenon — if students didn't "mind the gap" between the objects, they could not develop a new cognitive structure that would enable them to enact the movement form that solves the problem; they could not articulate their solution; and they could not then transition to mathematical models. Yet how should we theorize this figment of perception for coordinating the motor actions of two independent limbs?

It turns out from Movement Sciences that: (a) the sensory and motor faculties are neurally intertwined and mutually constraining — we constantly grope for a better grip on the world by moving to sense, sensing to move (Fiebelkorn and Kastner, 2019); (b) sensorimotor activity is a complex system in flux, with new dynamic stabilities self-organizing adaptively to changing environmental contingencies (Chow et al., 2007; Kostrubiec et al., 2012); and (c) the mind relentlessly yet tacitly searches for, and generates new Gestalt structures to serve as perceptual means of organizing dexterous manipulation (Mechsner, 2004). That is, *we are evolutionarily inclined to complement our raw sensation of the phenomenal world with imaginary auxiliary constructions that facilitate its manipulation*. These Gestalts are the cognitive structures that emerge by and for task-driven, explorative actions, enabling action to be perceptually guided.

Enactivist philosophers call these emergent cognitive resources attentional anchors. *Attentional anchors* are perceptual orientations toward the environment that come forth through exploration and guidance as our means of accomplishing the sensorimotor enactment of complex movement forms (Hutto and Sánchez–García, 2015). Attentional anchors are information structures that we groom forth from the lived environment as affording our task-effective action; once detected, we thereafter iteratively adjust our actions to maintain our perceptual hold of those structures that, reflexively, enable us to act on the world (Abrahamson and Sánchez–García, 2016).

The very type of emergent structures that let us ride a bicycle, pole-vault, juggle props, or play a viola arpeggio could serve us in getting a grip on mathematics (Abrahamson, 2021; Hutto, 2019), albeit it takes an appropriate learning environment (Abrahamson and Sánchez–García, 2016; Hutto et al., 2015). It is thus, we believe, that theories of embodied cognition may inform the practice of mathematics education (Fugate et al., 2019; Shapiro and Stolz, 2019). We now take a closer look at practice.

2.3. Learning with the Trainer: from movement to mathematics

Drawing on research conducted by Utrecht University researchers of embodied design (Bongers, 2020; Bongers et al., 2018; Duijzer et al., 2017), this section elaborates on Trainer learning trajectories.

The activity begins by presenting the student with a bimanual motor-control problem. Here the student is working on an Orthogonal Proportion task. She is guided to manipulate the orthogonal dimensions of a rectangle, which initially is red (see Fig. 3a):

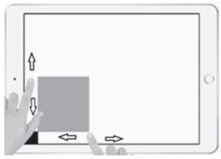

Fig. 3a. A Mathematics Imagery Trainer tablet activity. Initially, the manipulated geometrical figure, a rectangle, is colored red, because its selected dimensions do not comply with the yet-unknown specifications.

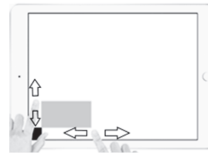

Fig. 3b. Reconfigured at a 1:2 height-to-width ratio, the rectangle turns green. Next, both hands must move simultaneously to keep the rectangle green while changing its dimensions.

Her left-hand (LH) index finger slides the rectangle's top-left vertex up/down along the *y*-axis to change its height, and her right-hand (RH) index finger slides the rectangle's bottom-right vertex right/left along the *x*-axis to change its width. The student is tasked first to make the rectangle green and, once that is accomplished, to keep moving the two vertices at the same time whilst keeping the rectangle green. The rectangle is green when the quotient of its height/width measured values is some yet-unknown constant number, for example 5 (see Fig. 3b). As such, once a green rectangle is generated, moving forward its dimensions must be adjusted simultaneously to maintain the rectangle continuously in its particular preset green aspect ratio.

In the course of solving Orthogonal Proportion problems, study participants typically develop some new Gestalt to coordinate moving their LH–RH fingers simultaneously at different rates along orthogonal paths. For example, Lars (see Fig. 4a) worked on a variant problem, where he was tasked to move cursors along orthogonal axes in the absence of a rectangle. When Lars achieved fluent movement in green, he was asked to explain his method. Lars said he was attending to an imaginary diagonal line connecting the cursors. The color blots in the images are post-production data-visualization overlays marking the location of Lars's foveal eye gaze. Soon after (see Fig. 4b), Lars demonstrated how he moves the diagonal line to the right.

Fig. 4a. Lars, a 14 years-old low-tracked prevocational-education student, gestures an imaginary diagonal line he perceives as connecting his LH and RH points of contact on the axes.

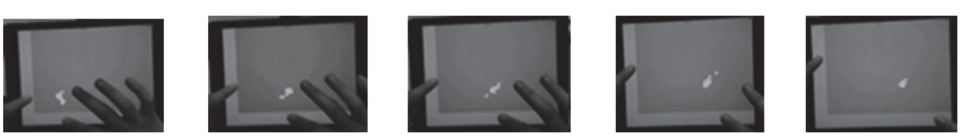

Fig. 4b. Lars uses his emergent attentional anchor to guide proportional bimanual coordination: He moves sideways the imaginary diagonal subtended between his fingertips.

The eye-gaze markers indicate that he is no longer foveating on his fingers but, rather, near the center of the LH–RH diagonal lines. As you scan the five photographs in Figure 4b, note the successive locations of the eye-gaze marker: Curiously, Lars's gaze path, as he imagines the successive LH–RH diagonals, runs along a different diagonal line — a diagonal trajectory from the origin (on the bottom left) and up to the right that describes a y = .5x function. This emergent foveal trajectory is a secondary attentional anchor. Lars's diagonal solution was quite typical. Yet, across participants, we found evidence for a variety of attentional anchors, such as gazing at the imaginary top-right corner closing a rectangle composed of two axial segments subtending between each fingertip and the origin and two lines from the fingertips to the imaginary point (see Duijzer et al., 2017, for the array of attentional anchors recurring across participants).

Once students have achieved a pre-specified criterion of minimal performance level, the activity proceeds with the teacher — who may be either a human or a virtual pedagogical avatar (Abdullah et al., 2017) — introducing onto the activity space supplementary resources designed to steer students to develop quantitative re-articulations of their movement forms. For example, Fig. 5 shows the presentation of a grid (Fig. 5a) and then numbers (Fig. 5b) onto the tablet interface.

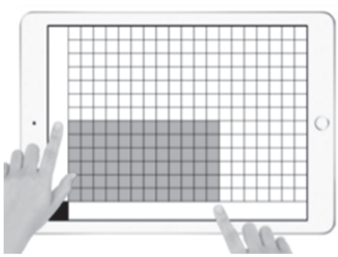

Fig. 5a. A grid is overlaid onto the movement space. The continuous space thus becomes discrete, affording the enumerative quantification of, and reference to uniform spatial intervals.

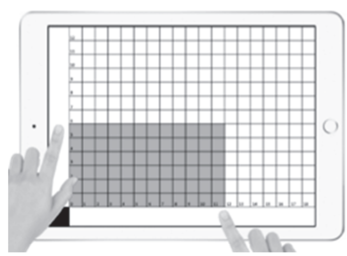

Fig. 5b. Numerals are supplemented onto the grid. Strategies of iterative manual incrementation are substituted by explicit arithmetic functions enabling multiplicative prediction of green rectangles.

Undirected, students count grid lines or units corresponding to their actions and, thus, are able to: (1) describe their strategy quantitatively; (2) draw on their arithmetic skills; (3) confirm the veracity of their strategy; (4) determine with greater precision the location and trajectory of the attentional anchor; (5) enact the movement form correctly independent of the color feedback; and (6) predict properties of yet-unenacted geometrical shapes satisfying the interaction regimen.

Students are now equipped with quantitative rules derived from the tablet activity, so that, given a new "green" geometric shape, they are able to calculate a set of additional "green" shapes. The lesson activity now disengages from the tablet and turns to paper. Fig. 6 demonstrates a paper-and-pen activity, where the geometrical form presented to the students "materializes" the imaginary diagonal attentional anchor, which they had generated imaginatively on the tablet as their means of solving the tablet interaction problem of moving in green. Students are asked to use the pen to show what would be other "green" triangles. As students engage the paper-and-pen offline tasks (see Fig. 6a), they no longer have recourse to immediate real-time interactive feedback on the quality of their performance. Nevertheless, the students now have a formalized rule for generating additional instances of the new equivalence class, which has yet to receive a mathematical name. Figures 6b–d demonstrate students' creative technical strategies, using available resources, for creating new lines running between the *y*-axis and *x*-axis parallel to the hypotenuse of the given triangle.

Fig 6a. A sheet of paper showing a starter shape is placed directly on the tablet screen.

Fig. 6b. Anna places an available sheet of paper alongside the triangle's hypotenuse.

Fig. 6c. Anna slowly slides the page away, keeping it parallel to the hypotenuse.

Fig. 6d. Using the sheet of paper as a straightedge, Anna draws a parallel line.

In Fig. 7, Bongers et al. (2018) deftly illustrate the study participants' typical quantitative strategies for generating further "green" diagonals on paper. Both the with the virtual grid. With that, the tablet-based perceptual strategy of handling animaginary Gestalt has materialized as a paper-based geometrical strategy of generating a set of "green" diagonals. The lines' mutual affinity — what makes them satisfy the tablet-based task. Yet, now on paper, the lines' setness in turn draws also on new perceptual criteria — their salient parallelism and the similitude of the triangles they configure.

Fig. 7a. A participant gauges a vertical span, transports it upwards to form an equivalent concatenated span, and marks its reach.

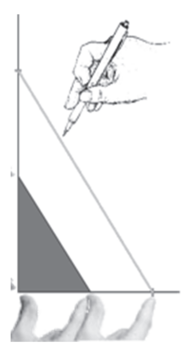

Fig. 7b. The participant next performs analogous actions along the horizontal span.

Fig. 7c. The participant draws units alongside the triangle legs, then extends 3 and 2 units, respectively, along the vertical and horizontal legs.

A set of triangles thus produced through rule-base iterated co-expansion of the legs (e.g., 3-per-2 in Fig. 7) is named as bearing the mathematical quality of "proportionality."

With that, we have demonstrated the evolution of a mathematical concept grounded in an attentional anchor: (1) from a personally experienced *ad hoc* imaginary percept that emerges spontaneously to organize the sensorimotor enactment of a movement solution in an assigned motor-control problem; through guided discourse, (2) into a publicly evoked qualitative ontology in the form of co-speech hand gestures adumbrating the imaginary line for the interlocutor; then becoming (3) a quantitative ontology pinned onto a frame of reference that enhances performance, calculation, and prediction; next, denoted (4) as the contour of a sheet of paper that indexes the prospective location and form of a linear inscription; and then materialized (5) as an actually inscribed line on paper, along with diagrammatic and symbolic labels and multimodal quantitative explanations for the diachronic and contextual meanings of this line. As such, we wish to detail the cascade of semiotic actions by which subjectively experienced perceptual structures that come forth to facilitate motor action are endorsed into mathematical discourse that imbues and articulates the structures with conceptual meaning by implicating their quantitative invariance.

3. Closing Words

The embodied-design research program speculates that "If you can't move it, you don't get it." That is, one's understanding of a mathematical concept begins at the point where one can enact a movement form that, per experts, instantiates the concept. Yet to enact a new movement form, one must attend in a new way to the environment, including one's body. That is, to perform a conceptual choreography, we must detect in the environment an information structure whose maintenance facilitates, enhances, and regulates our grip on the world (Abrahamson, 2021; Abrahamson and Sánchez–García, 2016). In turn, our mimetic capacity to reflect on our own actions (Donald, 1991; Piaget, 1971) enables us to surface these tacit forms in multimodal language and formalized inscription (Donald, 2010; Malafouris, 2013). The Mathematics Imagery Trainer constitutes an instrumented field of promoted action guiding this micro-genesis of movement into mathematics.

Trainer studies have generated empirical data enabling researchers to investigate, corroborate, and extend with unprecedented precision longstanding tentative tenets from seminal theories of cognitive development, including Varela's enactivist cognition (Hutto et al., 2015), Piaget's reflecting abstraction (Abrahamson et al., 2016), Vygotsky's zone of proximal development (Shvarts and Abrahamson, 2019), Araújo's ecological dynamics (Abrahamson and Sánchez–García, 2016), and Vérillon and Rabardel's instrumented activity theory (Shvarts et al., 2021). Quantitative analyses of students' motor and sensory activity have enabled the research collaboration to pioneer the demonstration of conceptual phenomenology as perceptual assembly of sensorimotor behavior (Abdu et al., 2023; Tancredi et al., 2021).

Embodied-design activity architectures are "pan-media," in the sense that they can be implemented in a range of human — computer interaction platforms. As such, the Trainer can cater to students of diverse sensory capacities and needs. For example, Trainers have been built for sighted students' remote-action (Howison et al., 2011) or hands-on tablet manipulation (Abrahamson et al., 2011) yet also for enhanced accessibility (PhET, 2021), including haptic devices for students who are blind or visually impaired (Lambert et al., 2022).

As we enter the systemic era in theorizing mathematics education (Abrahamson, 2015), we foresee increasing adoption of constructs and methods from dynamic systems theory. The Mathematics Imagery Trainer, while supporting student development of deep conceptual understanding, could furnish the empirical context for investigating the pivotal epistemic role of learning to move in new ways.

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