Intellectica, 2024/1, **80**, pp. 149-174

Enactivist Learning: Grounding Mathematics Concepts in Emergent Perception for Action

Dor ABRAHAMSON*

ABSTRACT. Harnad's *symbol-grounding problem* launches a quest to found a new Learning Sciences paradigm innovating and evaluating pedagogical resources that enable students to develop meaning for mathematical concepts. A proposed paradigm, *embodied design*, integrates design-based research efforts to innovate interactive resources, facilitation methodologies, multimodal learning analytics, and empirical findings all resonant with Varela's enactivist tenets of cognitive epigenesis. These tenets, we argue, readily inform a heuristic design framework, which we then exemplify with the Mathematics Imagery Trainer, an interactive technological architecture that occasions opportunities for students to develop new conceptual choreographies grounding curricular content. Solving motor-control problems, students first discover *attentional anchors*, dynamic perceptual structures mediating task-effective coordinated action; they then use available mathematical instruments as frames of action and reference to enhance the enactment, evaluation, and explanation of these solution strategies, thus shifting into disciplinary discourse, where the attentional anchor serves as a percept-to-concept pivot—from doing to thinking-about-doing. With its emphasis on learning through sensorimotor exploration, embodied design caters to mathematical practice inclusive of students with sensory, motor, and cultural differences.

Keywords: Constructivism, Embodied design, Embodiment, Enactivism, Field of promoted action, Mathematics education, Mathematics Imagery Trainer, Perception, Sensorimotor, Symbol-grounding.

RÉSUMÉ. Apprentissage énactif : ancrer les concepts mathématiques à travers l'émergence d'une perception qui guide l'action. Le problème de l'ancrage des symboles posé par Harnad initie la quête d'un renouvellement paradigmatique dans le domaine des *Learning Sciences* afin de permettre la conception et l'évaluation de ressources et dispositifs pédagogiques qui permettent de donner du sens aux concepts mathématiques. Une de ces alternatives paradigmatiques, la conception « incarnée », se fonde sur des recherches orientées par la conception pour innover dans le développement de ressources interactives, de méthodes de facilitation et d'accompagnement, d'analyses multimodales des apprentissages, et de résultats empiriques, tous cohérents avec les principes de l'enaction développés par Varela. Nous soutenons que ces principes peuvent informer de manière heuristique et féconde la conception. Nous illustrerons notre propos à l'aide d'un dispositif technologique et interactif pour l'enseignement des mathématiques, the *Mathematics Imagery Trainer*, conçu dans la perspective d'offrir aux étudiants des opportunités pour développer de nouvelles formes de chorégraphies conceptuelles permettant d'ancrer les contenus enseignés. En résolvant des défis basés sur des tâches motrices, les élèves découvrent d'abord des ancres attentionnelles, des structures perceptuelles dynamiques médiatrices

* Embodied Design Research Laboratory - Berkeley School of Education - University of California Berkeley, USA — Korea University. dor<at>berkeley.edu

d’actions coordonnées en vue d’une réalisation efficace de la tâche ; ils utilisent ensuite les instruments mathématiques disponibles comme cadres d'action et comme référence pour améliorer la mise en œuvre, l'évaluation et la justification de ces stratégies, passant ainsi dans l’univers disciplinaire, l’ancre attentionnelle ayant un rôle de pivot dans le passage de la perception au concept – du faire à la réflexion sur le faire. En mettant l’accent sur l’apprentissage par des explorations sensori-motrices, la conception incarnée favorise une pratique des mathématiques inclusive permettant d’accueillir des élèves présentant des différences sensorielles, motrices, ou culturelles.

Mots-clés : Constructivisme, conception « incarnée », dimension incorporée, énactivisme, espace d’actions encouragées, didactique des mathématiques, dispositif de formation à l’imagerie mathématique, perception, sensori-moteur, ancrage des symboles.

OVERVIEW

Situating a Discussion of Mathematical Meaning in the Embodied Design Research Paradigm

Varela *et al.* (1991) state the following:

> “In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action; and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided.” (pp. 172–173)

If, indeed, cognitive structures emerge from recurrent sensorimotor patterns that enable action to be perceptually guided, then that should necessarily include also the neural substrates of mathematical reasoning. At first blush, one might gape at a theoretical proposal that mathematical concepts, which folk psychology labels as “abstract”, could possibly emanate from the biologically rudimentary activity of sensorimotor exploration. As such, one would rightfully wonder what mathematics education could possibly look like that adhered to these enactivist principles. How might these principles guide the creation and facilitation of educational activities that lead to mathematical understanding? And what would a research program look like that evaluated a proposed design for enactivist mathematics teaching and learning? How could we monitor and measure whether the would-be cognitive structures have emerged? And how would these presymbolic neural substrates give rise to conscious reflection, professional discourse, and the production of formal semiotic artifacts, such as numbers, tables, and graphs?

The Embodied Design Research Laboratory’s long-term design-based research program evaluates whether and how enactivist epistemology can serve as a heuristic framework guiding the creation of pedagogical resources for grounded enculturation into K–16 curricular concepts. Accordingly, we create learning environments where novices figure out how to move—ergo, how to perceive and think—in new ways that occasion opportunities for guided appropriation of disciplinary discourse.

EDRL’s activities engage participants in solving motor-control problems, such as performing complex bimanual manipulations of virtual artifacts to generate designated feedback. Specific movement forms that solve the problems have been pre-designed to instantiate the new concepts as dynamical conservations. For example, moving both hands simultaneously, traversing proportionate accumulations, along orthogonal trajectories, is conjectured to foster proportionate reasoning.

EDRL's mixed methods triangulate multimodal learning analytics, that apply dynamic-systems-theory quantitative gauges of manual–visual coordination, with micro-genetic ethnographic and ethnomethodological analysis of verbal–gestural conversational utterance and artifact generation.

EDRL's findings evidence that study participants spontaneously develop new attentional anchors—perceptual orientations toward the problem situations—as their means of coordinating sensorimotor enactments of solution movements. In turn, these emergent perceptual orientations, often comprising imaginary auxiliary structure, are reified via available semiotic forms, including mathematical instruments and language, into stable indexable entities—consciously accessible ontologies that students describe, represent, measure, symbolize, and calculate.

There are now embodied designs for a variety of mathematical concepts, including mechatronic devices for sensomotorically diverse students. Case-study analyses of enactivist tutorials offer guidelines for educating mathematical perception by cueing attentional anchors or otherwise steering their discovery. As such, we empirically corroborate the enactivist hypothesis, elaborate its micro-process, and explicate its instructional application.

This paper explains the philosophical and theoretical motivation of our design-based research into the enactive roots of mathematical concepts. We offer the reader a case-study walk-through of how we implement enactivist tenets as heuristic design guidelines in building and evaluating interactive platforms where students learn to move, perceive, and think in new ways.

1 – Introduction

Welcome to Pirézia: The Problem of Meaningless Mathematics

You land in Pirézia national airport.[1] For argument's sake, let's assume here that you don't speak or read Pirézian. You want to collect your luggage, so you look for overhead signs telling you where the carousels are. There are many brightly lit signs, and they are all written only in Pirézian. Confused, you wander over to the information desk. The receptionists are most polite, but they only speak Pirézian. Instead, they hand you a dictionary. Relieved, you open the dictionary. You look up at one of the signs and stare at what appears to be the key word. This word is a string of unfamiliar symbols. You flip through the dictionary, back and forth, and eventually you locate that word as an entry. Progress! Ok, you try to read the translation of the word. Alas, it is not a translation. It is a definition of the word in Pirézian. Hmm, sure, …right. No problem. Undaunted, you look up *those* Pirézian words, from that definition, in the same dictionary, only to find that their entries, too, …are in Pirézian. And so forth. I think you get it. The dictionary appears to be an intact yet perfectly closed system. It is impervious to you. You are familiar with the dictionary as a

[1] Readers may be challenged to locate this country on Google Maps. May we, instead, recommend Wikipedia?

common artifact, you recognize its structure and logic, and you can follow its inter-referring web of signs. And yet the actual meanings of these signs are never revealed to you. Meaning is elsewhere. Bon voyage.

My Pirézian exordium paraphrases on Harnad (1990), an experimental psychologist who offered this hypothetical situation as a parable for what he calls the "symbol-grounding problem." The meanings of symbols, argues Harnad, are not inherent to the symbols themselves nor to the system of symbols. Rather, when we experience symbols as meaningful, it is because these symbols evoke particular *non*-symbolic cognitive resources, primarily sensory impressions.

Well, we've covered the case of luggage and language, but what does all this imply for something else, perhaps something we need to learn? How about mathematics? Where or what on earth are the meanings of "2+3"? Or, just, what does the "+" mean? As you behold "+", how do you make sense of it? How do you grasp it? How do you be-hold it?... Per Harnad, what non-symbolic cognitive resources do you tacitly bring to bear in making sense of "+"? Perhaps, upon introspection, you realize that "+" conjures for you a subtle perception of cumulation, that is, an inner gesture of "adding stuff." If so, can you demonstrate to me what "adding stuff" means, but without using any words? Would you, perhaps, draw your hands toward each other in an overt, covert, or even imaginary motor performance? Because, if so, then that nuanced "adding stuff" schema is your symbol-grounding for "+". It is your dynamic concept image (cf. Tall & Vinner, 1981; see Abrahamson *et al.*, 2014, pp. 79–80).

Knowledge, then, is not in language itself. Piaget (1970), predating Harnad, concurs:

> " [L]e point de départ de ces constructions [logico-mathématiques], au plan du comportement, n'est pas le langage, mais qu'aux niveaux sensori-moteurs on en trouve les racines dans les coordinations générales des actions (ordre, emboîtements, correspondances, etc.)." (p. 73–74)[2]

And yet, the show-me pragmatists among our readers may judiciously inquire, what could all this philosophy and theory mean for teaching and learning mathematics? Indeed, one might remonstrate that, but of course, scholars of education have forever known that meanings are not in symbols—we've known this at least going back to Jean-Jacques Rousseau, Friedrich Fröbel, Maria Montessori, and their likes. As Émile's tutor emphatically adjured us three centuries ago:

> "Pourquoi toutes ces représentations ? que ne commencez-vous par lui montrer l'objet même, afin qu'il sache au moins de quoi vous lui parlez ! …. En général, ne substituez jamais le signe à la chose que quand il vous est impossible de la montrer ; car le

[2] For non-native speakers of Pirézian: "[T]he formation of logical and mathematical structures in human thinking cannot be explained by language alone, but has its roots in the general coordination of actions." (Piaget, 1971, p. 19)

> signe absorbe l'attention de l'enfant et lui fait oublier la chose représentée."[3]

Alors, it turns out that the idea of grounding a symbol in *l'objet même* has featured in educational discourse for quite a while now. *Dommage*, though, this historical pedagogical heuristic is not at all obvious to today's scholars of mathematics education. Consider Thompson's apologia.

Thompson (2013) tells us a poignant story about visiting a US classroom, where youth were studying the geometrical concept of an angle. That day, the students were learning to use a protractor to measure an angle, which they would then write down on their worksheets. His conversations with the students led Thompson to realize that they did not have any idea what an angle actually is. To be sure, the diligent students were most adept at determining angle measures using the protractor instrument, and yet they could not begin to explain to Thompson the meaning of these measures. Not a single student (nor the teacher, alas, for that matter) understood the angle measure as quantifying the ratio between the angle aperture and the full circle, for example, that the meaning of "40°" is derived from 40/360, where 360 is an historical convention, and therefore "40°" means that the angle arms (rays) are 1/9 -of-a-circle open. The students only had what Skemp (1976) called "instrumental knowledge" about angles, no "relational knowledge." Thinking back to our earlier example of "+" and the cumulation gesture, the students didn't have any analog gesture to express what the protractor was measuring (cf. Hardison, 2019). Thompson then candidly reflects back on decades of research in the late 20th century, when he and his colleagues believed that children learn a mathematical concept by translating across its inter-signifying semiotic displays, for example, among an equation of a linear function, its graphical display, and its tabular embodiment. Yet now, Thompson realizes that,

> "Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us…. I am now saying that I was mistaken. …. [I]t may be wrongheaded to focus on graphs, expressions, or tables as representations of function, but instead focus on them as representations of *something* that, from the students' perspective, is representable, such as some aspect of a specific situation. The key issue then becomes twofold: (1) To find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) Orient students to draw connections among their representational activities in regard to the situation that engendered them." (pp 39–40, my emphasis)

Eh bien, centuries after Rousseau's *l'objet même*, we're still looking for *something*—we're still living in Pirézia! How can this be? Where has the wisdom

[3] "What is the use of all these symbols; why not begin by showing him the real thing so that he may at least know what you are talking about? .... As a general rule—never substitute the symbol for the thing signified, unless it is impossible to show the thing itself; for the child's attention is so taken up with the symbol that he will forget what it signifies." (Rousseau, [1762], p. 170)

of Enlightenment gone? Perhaps the chastising ghost of Nicolas Bourbaki[4] still quietly stirs doubt among those who would hope for students to ground the meaning of symbols in sensorimotor experience. Perhaps some of us believe, like Piaget, that once children have mastered basic sensorimotor skills and reached the proverbial formal operational stage of cognitive development, they no longer avail of physical interaction for learning new logico-mathematical concepts. Then again, perhaps we do believe that new mathematical concepts call for new sensorimotor experiences, only that we haven't figured out what the sensorimotor experience should be for certain mathematical concepts, for example an angle, proportionality, or an asymptote. Perhaps, furthermore, we do appreciate that mathematical ideas should be grounded in sensorimotor experience *and* we even have figured out what these sensorimotor should be, and yet we just don't know how to foster such sensorimotor experience, and so we give up. *Que faire ?*

An objective of this paper is to evaluate whether technology—writ large, to include mechanical, electronic, and mechatronic interactive devices—can help us escape Pirézia to a land where we might ground mathematical symbols in sensorimotor experiences. One premise of our escape from Pirézia is to heed the wry comment quipped by Glenberg (2006) in his Commentary on the future of educational technology:

> "[O]ne can view most of my reasons for skepticism as challenges for the future development of technology that is sensitive to the principles of biological cognitive systems." (p. 271)

More broadly, by looking to understand and harness "biological cognitive systems" we join Dreyfus and Dreyfus (1999), who, in their essay on French Phenomenology philosopher Maurice Merleau–Ponty, express their frustration with prevalent Cartesian paradigms in the cognitive sciences:

> "Until cognitive scientists recognize [the] essential role of the body, their work will remain a mixed bag of ad hoc successes and, to them, incomprehensible failures." (p. 118)

In the next section, we will be examining just what these "biological cognitive systems" might be that technology should accommodate if we are to avoid "incomprehensible failures" in creating learning environments where students ground mathematical symbols. But we want to go beyond articulating theory and deriving design principles by actually practicing design and conducting empirical studies. That is, *we are looking to evaluate whether theories of biological cognitive systems that incorporate the body in explaining human learning, teaching, and thinking can be applied by way of creating novel activities for the betterment of mathematics pedagogy*. In choosing to elaborate on philosophical and psychological scholarship through designing and evaluating instructional resources, we are inspired by Clements and Sarama

[4] Bourbaki is the collective name of a team of mostly-French mathematicians. Founded in the early 1930's, Bourbaki aimed to purge the mathematical discipline of any ontological grounding in human sensory experience, including spatial–temporal media, modes, and modalities. This movement was notoriously influential in the United States, where Bourbaki's tenets were instantiated in the form of a nation-wide failed pedagogical approach known as "New Math" that required young children to disavow their early number sense and, instead, meaninglessly manipulate meaningless formal symbols.

(2015), whose Commentary on a special issue centered on early mathematics education chastised the contributing authors for a lack of application:

> "Mathematics education should not be an "implication" tagged on to the end of studies from developmental and cognitive psychology. Mathematics education research and cognitive research should be interwoven enterprises." (p. 251)

Interweaving education research and cognitive research is the bread and butter, or perhaps even the meat and potatoes, of an investigative approach associated with the Learning Sciences, design-based research (Bakker, 2018; Collins, 1992; Easterday *et al.*, 2016). In *design-based research* (sometimes named *design research*), investigations of educational phenomena are vested in cycles of assembling and appraising pedagogical methodology. Design-based research projects typically make contributions to the field by way of: (a) corroborating and refining theories of learning, teaching, and cognition in the sociocultural context; (b) generating and validating educational prototypes through evaluating them with end-users; and (c) developing heuristic design frameworks applicable in broader contexts (Abrahamson & Wilensky, 2007).

This section has dialogued with Harnad's stipulation that for symbols to make sense to us—or, rather, for us to make sense of symbols—the symbols must be grounded. And they must be grounded, we took from Rousseau, Dreyfus, Glenberg, Thompson, and others, through engaging our biological cognitive systems in embodied worldly interaction. At the same time, as Piaget and other cognitive developmental psychologists insist, the learning trajectory is not from the symbol to its grounding but, *au contraire*, from sensorimotor experience to the symbol, by way of semiotic mediation (Bartolini Bussi & Mariotti, 2008). And so, another premise of our attempted escape from Pirézia will be to examine whether philosophy of cognitive science offers guiding tenets for creating educational activities where students experience first, signify later (cf. Nathan, 2012). At the same time, we acknowledge and document the critical role of teachers—cultural agents—in facilitating students' dialectic symbol-grounding process (Flood, 2018).[5]

The next section appreciates the philosophy and science of enactivism as a coherent intellectual source for guiding educational design-based research on mathematical cognition, teaching, and learning. A subsequent section will then demonstrate what such pragmatic application of enactivism to education could look like. We highlight that the proposed pedagogical methodology is an equitable educational practice that caters to intersectionally diverse students by including educational offerings for neurally and sensorially atypical children; as well as post-colonial restorative-justice participatory designs for Indigenous people of hetero-European linguistic epistemology. The paper ends with a summative symbol-grounding exercise.

---

[5] Questioning the psychologically possible and pedagogically optimal directionality of symbol-grounding—whether from a symbol to a sensorimotor experience or vice versa—becomes complexified from the perspective of dynamic systems theory. Some educational researchers view material and semiotic resources as functionally equivalent in simultaneously transforming students' environment, where sensorimotor interactions lead to the development and stabilizations of new perceptions (Shvarts & Abrahamson, 2023).

2 – A Solution: Enactivist Philosophy of Mind

We're still talking about meaning, the meaning of things, including symbol-things. We began this article with a visit to Pirézia, where we worried about the *absence* of meaning. We diagnosed this absence of meaning as the problem of symbols that are not grounded in sensorimotor experience. We submitted that biological cognitive systems require sensorimotor experiences to develop meaning for signs, and we suggested that educational interventions might include technological artifacts that would occasion opportunities for students to develop sensorimotor grounds for mathematical symbols. We left open the question of how we might approach this task of creating educational activities for students to develop these sensorimotor capabilities. As design-based educational researchers, we have a sense of what we are looking for—what it should deliver—but we're not yet quite sure what intellectual edifice could guide our educational development. Put simply, how do we get kids to move in new ways that would enable them understand some particular mathematical concept? Enter philosophy.

Now we turn to a philosophical paradigm in the cognitive sciences concerned with the nature and origin of meaning. This epistemological theory, called *enactivism*, looks to explain how goal-oriented interactions between biological systems and their environment give rise to neurally sustained capacities—cognitive structures—which mediate and regulate adaptive behaviors. For higher-order organisms, these cognitive structures may, in turn, become available for reflection, objectification, and semiotic expression—they may become things, new ontologies to think and talk about, that is, things by which both to articulate individual reasoning and, dialectically, regulate social mediation of shared perception to enable coordinated joint action, collaborative planning, mythology, etc.[6] Things that lend meaning to symbols.

The rationale and motivation of this section is to lay out the enactivist paradigm from philosophy in a way that I will then operationalize as an *applied heuristic framework* for building educational environments. These enactivist environments—what Reed and Bril (1996) might call *fields of promoted action* (see also Abrahamson & Trninic, 2015)—are designed to occasion opportunities for students initially to develop new cognitive structures facilitating their adaptive sensorimotor engagement with the environment (*l'objet même*) and only later—through the mediated appropriation of available semiotic artifacts—come to realize that these cognitive structures ground what turns out to be a new mathematical concept. So, we're in the business of designing activities for students to develop a grip on mathematics (Abrahamson, 2021; Hutto, 2019). As I put it elsewhere (Abrahamson, 2009), *we construct means for constructing meaning*.

Talking about the meaning of mathematical symbols can be difficult. Let's make our life easier by first talking about the meaning of material phenomena—

[6] See Donald (1991, 2001) for an archeo-anthropological account of *mimesis*, the evolved neurocognitive capacity for thinking about action. See Goodwin (1993), Schegloff (1997), and Mondada (2014) on the spontaneous emergence of ontologies as a social solution to maintaining conversational intelligibility supporting the coordination of action. See Radford (2014) on a semiotic–cultural theory of objectification as it plays out in mathematics education.

regular stuff we encounter in our everyday life. What are the meanings of things? It seems useful to surmise that meanings may not be absolute, that they are ad hoc. The meaning of something is what some person might think to do with it to get something done under some given circumstances. For example, the meaning of a banana could be a paperweight for holding down napkins onto your picnic table on a windy summer's day in the verdant mountainous glades of South-East Pirézia. So, meaning is subjective, functional, tacit, and contextual, where features of the thing in question come forth as affording apparent utilities to get something done. And yet the meaning of things can become objective, explicit, universal, and generalized in cultural practice, once normative routines and discourse pertaining to the thing are collectively established. Perhaps the paperweight-ness of bananas will become a thing—enterprising Pirézians will create ornate little brass bananas as artisanal paperweights, to use when organic bananas are out of season, and they might compose verses extolling the artifact, depict it on their flag, sell it to Hungarian tourists in rustic gift stores, and so on.

In this vein, the enactivist cognitive scientists Varela *et al.* (1991) are interested in how the stuff of new tacit meanings first coalesces, when an organism attempts to engage adaptively with its ecological environment to satisfy an existential need. To begin to understand the enactivist way of thinking, however, we must shift away from our everyday mode of thinking about things—a mode of thinking that is ineluctably vested in Western languages' ontological parsing, syntactic positioning, and referential tokenizing (Barton, 2008; Urton, 1997; Verran, 2001)—toward a pre-linguistic phenomenological immersion in the environment (Abram, 1996; Sheets-Johnstone, 1999; Petitmengin, 2007). In a sense, I'm asking you to stop thinking about thinking and just think, or, better, just *be*—be a human being—even as I recognize and apologize for the paradox of my request.

For enactivists, such as Varela, and similar to ecological psychologists (Gibson, 1977; Heft, 1989), the fundamental ontological unit for understanding biological organisms' ethological behaviors is neither the organism per se (the "subject") nor the environment per se (the "object") but always inherently an organism–environment atomistic relation—not a binary dualism but a monist *duality*, "the quality or state of having two commensurate (mutual and reciprocal) aspects" (Turvey, 2019, p. 327). This organism–environment originary duality is necessarily dynamic, action-prone, and perceptually governed—it is *enactive*, in the sense that organisms' neurally potentiated capacity to behave functionally in their ecology is constituted only by virtue of it being carried out, whether actually, imaginatively, or some blend thereof. The being of knowing is in doing. As Piaget (1968) wrote, "il n'existe pas de structure sans une construction" (p. 120).[7] As such, knowledge, or, better, the

[7] "There is no structure apart from construction" (Piaget, 1970, p. 140). Turner (1973) elaborates on this idea. "'Construction,' or the process through which structures are formed, is thus the most important concept in Piaget's theory of structure. Construction consists of an adaptive interaction between a system or entity already organized at some level, which plays the functional role of 'subject,' and its objective environment. The adaptive orientation of the 'subject' is toward the achievement of a more stable equilibrium within the total system constituted by itself and its environment. To achieve this goal, it must make a series of accommodations to the objective conditions imposed by its environment, and incorporate these accommodations into its own structure as the basis of its future behavior. The subject attempts to

phenomenology of *knowing*, is in the doing that instantiates this knowing. Whereas the capacity for knowledge is neurally potentiated, the experience of knowledge is necessarily in its situated application that may involve both natural phenomena and cultural artifacts, themselves fashioned so as to support our actions.

Meaning, per enactivism, first rises through the biological organism's ecologically adaptive perception–action loops that marshal new forms of goal-oriented sensorimotor activity into functional systems. The enactivist manifesto states the following profound tenets:

> "In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided." (Varela *et al.*, 1991, p. 173)

The quotation, above, is a dense encapsulation of a rather complex theoretical model, which may be unintuitive to scholars trained in traditional epistemology. To unpack the enactivist theoretical model and begin to apply it to educational practice, let us attempt, in the next section, to explicate each of its key constructs as capturing the process of learning a mathematical concept and, as such, implicating how mathematics should be taught. That is, as a design-based researcher, I find it particularly useful to interpret an epistemological model by hypothesizing a process of learning in terms of *how we would design an environment that implemented the model*. In doing so, the design-based researcher asks: (1) What would an enactivist mathematics pedagogy look like?; and (2) What would a research program look like that evaluated whether the proposed pedagogy is indeed implementing the framing theory? The following section will explicate the enactivist motto by way of a design exemplification—an enactivist design for fostering symbol-grounding.[8]

3 – Towards an Enactivist Pedagogy

I am a design-based educational researcher bent on creating interactive digital resources for students to learn particular targeted concepts. For example, I may wish to create an interactive technological platform for young students to learn the mathematical concept of proportionality. As they engage in activities centered on these designed educational resources, students are to develop new *cognitive structures*. I think of these cognitive structures as constituting proto-mathematical capacities conducive to grounding the particular mathematical notion in question.[9] So, as I consider a design for proportion, I ask myself what

encompass each new set of accommodations on the basis of its capacity to 'assimilate' objective reality at its existing level of structural development" (p. 355). How to theorize this would-be pre-existence of some "objective reality" becomes a nettled point of debate that will not be expanded here.

[8] Focusing on the enactivist proposal as elaborated by Francisco Varela and his colleagues, this paper will not discuss a parallel branch of enactivism propounding Umberto Maturana's emphases on the problematics of observation. See Drury and Tudor (2023) for an elegant historical exposition on the drifting apart of enactivist schools, which they characterize as Maturana's "autopoietic enactivism" and Varela's "radical enactivism.".

[9] I owe this sense of "proto" to Resnick (1992), who speaks of "protoquantities" as developmental antecedents to more advanced mathematical reasoning.

these new cognitive structures might be that would ground an enactive understanding of proportion.

To recall, we are looking to ground mathematical ideas in sensorimotor experiences. As such, we think of mathematical *learning as moving in new ways*. In practice, we want kids first to figure out how to move in some new way, before they come to realize that this way of moving will become the meaning of some mathematical idea. That comes later. My practical methodology, which I call the *action-based genre of embodied design* (Abrahamson, 2014), consists of the following components. Note that the listing of these components, below, should not be taken as specifying steps in a rigorous sequence of resource development. Rather, we find educational design to be more like solving a puzzle—it is a cyclic, emergent, negotiated optimization process, where key insights occur sporadically, often through coming to understand the experience of our critical partners, the study participants. Or, as Georges Perec put it,

> « On en déduira quelque chose qui est sans doute l'ultime vérité du puzzle : en dépit des apparences, ce n'est pas un jeu solitaire. »[10]

3.1 - From Notion to Motion— The Designer Animates the Mathematical Concept as a Dynamic Enactment

What *perceptually guided action* should students perform, so that they come to enact the notion of proportion? Answering this question begins by pondering what the notion of proportion means to *me*, what it *is* for me. *Donc, que sais-je ? Comment le sais-je ?* How do I think of proportion? How do I think proportionately? How do I… become proportion?! (cf. Gerofsky, 2011) What is my tacit enactment of proportion? For example, how would I gesture "proportion?" If proportion be an image, what, then, are my "subtle inner micro-gestures that are performed to elicit, stabilize, recognize, evaluate, rule out or enrich this image, as well as the bodily sensations and feelings that accompany this process" (Petitmengin, 2017, p. 104)?[11] As such, I am looking to instantiate the targeted notion of proportion—to *phenomenalize* it (Pratt *et al.*, 2006)—as some motor action or, perhaps, a coordination of two or more motor actions.

Earlier, we considered the cumulative action of two hands coming together to enact the arithmetic operation of addition (the meaning of "+"). A motor action that instantiates a mathematical *relation*—such as proportion—might, moreover, enact a *dynamic conservation* of a spatially realized quantitative property across a span of exemplars (cf. Leung *et al.*, 2013, on discerning invariants in dynamic geometry environments, such as GeoGebra). For example, we might enact the notion of proportional progression by first stacking both hands flat on the table, our makeshift "zero," and then raising both hands while continuously increasing the vertical spatial interval between them. The hands are rising at different speeds, yet each is rising at its respective constant speed—perhaps the top hand is rising double as fast as the lower hand—and yet a

---

[10] "One will deduce from it something which is undoubtedly the ultimate truth of the puzzle: in spite of appearances, it is not a solitary play" (Perec, 1978, from La Vie mode d'emploi. Translated by D. Bellos, 1987).

[11] Claire Petitmengin studied and co-authored with Francisco Varela.

comparison of their respective heights above the desk would evidence that they are maintaining a constant ratio, for example 1:2. Or you might note that the top hand is always double as high as the bottom hand; the bottom hand is always half as high as the top hand, and so on. I think of any such enactment as a *conceptual choreography*, a *mathematical kata* (Abrahamson, 2014; Abrahamson & Shulman, 2019).

The various ways of perceptually maintaining a dynamic conservation—in the case of proportion these are: (a) continuously increasing the interval; (b) moving at different yet respectively constant speeds; (c) keeping a multiplicative relation fixed at *x/y*; or (d) at *y/x*; (e) keeping the hands aligned with an imaginary anchor point on the desk, down to one side[12]; and so on—each require a different way of orienting perceptually toward the moving hands, each foregrounds and attends to a different element or property of the situation, each is enabled by a different cognitive structure, each potentiates different mathematical meaning. Such plurality of methods for enacting one and the same mathematical notion (*i.e.*, proportion) makes concretely manifest the concept's *polysemy*, its cognitive cluster of conceptually complementary referents. Still, it is one thing to be able to enact each of these proto-mathematical polysemous solutions to a single motor-control problem. It is another thing to understand *why* or *how* this polysemy could be possible. We have found educational benefits in having students investigate enactive polysemy, for example, by asking them *why* the top hand moving double *as fast* as the bottom necessarily implies that the top hand is always double *as high* as the bottom hand. Investigations into enactive polysemy promote coherent conceptual understanding by recruiting and integrating diverse curricular content (Abrahamson *et al.*, 2014) that, otherwise, is liable to remain inert and compartmentalized (Bereiter & Scardamalia, 1985).

3.2 - Building an Activity—The Designer Situates a Proto-Mathematical Enactment as a Means of Performing a Task

Up to this point, I have selected a mathematical notion I would like students to learn, and I have devised a generic movement form as a proposed phenomenalization of the notion. As explained earlier, I conceptualize the movement form as proto-mathematical, in the sense that learning to perform the form fosters prospective grounding for the mathematical notion; the experience of grounding is immanent to the new cognitive structures that students tacitly develop as their means of enabling their perception to guide the enactment of the new movement form.

Ok, clear enough, I hope. Yes? No? But now, the enactivist designer asks, how might students learn to enact this movement form? I cannot give them performance instructions by way of mathematical description, because, well, they do not yet know the mathematical concept they are to learn by performing the movement form! We appear to be facing an *epistemological* anomaly, because we expect students to build a whole that is greater than the sum of

[12] I owe this strategy to Rob Goldstone, personal communication, at Comal, Berkeley, circa 2012, over flutes innumerable of fair-trade organic mezcal.

available parts (Bereiter, 1985).[13] At the same time, we face a potential *motivational* problem: How might we engage students in an activity by which they are to learn to perform some action (Ba & Abrahamson, 2021)? Finally, we face a *pragmatic* and *axiological* problem: How can we support numerous students who may not have access to school-based enactivist pedagogy?

One way of fostering, motivating, and ethically administering the enactment of a new movement form is to install it as an operation on an interactive environment and, moreover, as a condition for performing a task. For example, if you want to motivate the performance of a palm rotation, you could situate it as a means of opening a jar. And so, the designer needs to envision some activity, where the target action would be the means of achieving a task objective. In so doing, I must ensure that students receive immediate feedback on the quality of their performance. If you will, I need to build some "jar," so that you can readily monitor whether or not you are opening it.[14] Thus, the target conceptual enactment is designed to constitute a continuous movement solution to an ongoing motor-control problem.[15] How could digital technology implement this design principle?

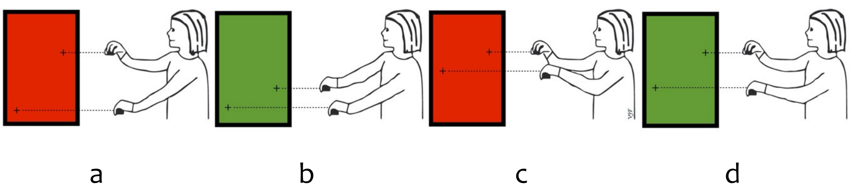

Figure 1
A Mathematics Imagery Trainer for Proportion (art: Virginia J. Flood)

One can think of the enacted form as a *proximal action*, where the child's hand movements effect a technologically mediated *distal action* on the environment (Abrahamson & Bakker, 2016). For example, Figure 1 features an exemplar of a technological design architecture we call a *Mathematics Imagery Trainer* (Abrahamson, 2012b; Abrahamson & Howison, 2008; Howison *et al.*, 2011), where enacting a particular movement form—here, raising the hands in a

[13] For further readings on the "learning paradox," see Prawat (1999), Neuman (2001), Hoffmann (2003), Norton (2009), and Abrahamson (2012a).

[14] Note that opening a jar is an ongoing action. At the human–jar interface is the lid, the jar's contact surface. This lid is shaped in a basic geometrical form, a circle, which enables its smooth frictive threading onto the vessel. Amidst the circle's figural simplicity, the requisite movement form of mobilizing the lid is of considerable motor complexity. Over and over again, at each micro-moment, we must kinesthetically adjust our anatomy, contorting our arm, forearm, hand, and fingers and adjusting the power of our grip so as to accommodate the inherent logic of an inert artifact, which, in turn, minutely elevates us as it opens. The movement form of opening a jar is not abrupt (ballistic)—it is a continuous engagement with an object, where we rapidly and iteratively correct our actions according to the jar's mute "rule." One can imagine a toddler first figuring out how to open and close a jar, perhaps a toy wood jar, and, through that, perhaps learning about circles.

[15] Unlike standard tasks in DGE (dynamic geometry environments), where the digital objects may prevent students from performing task-irrelevant actions, our designs let students freely manipulate the object and discover the constraints for themselves (Abrahamson & Abdu, 2020).

coordinated bi-manual scheme that instantiates a continuous proportional progression—causes a red screen to become green. The student's task is to figure out which proximal actions cause the desired distal effects and to become sufficiently proficient at performing this feat. In so doing, the student must figure out how to wield the mediating artifact as an instrument for reaching and engaging with the environment. That figuring out is where new sensorimotor patterns emerge.[16]

Figure 1 sketches four key stages in a paradigmatic learning experience, as the child figures out how to remote-manipulate two screen cursors so as to make the screen green and keep it green while moving, where the task has been set at a 1:2 ratio: (a) Initially, the screen is red, and the student places the cursors at some location that does not accord with the yet-unknown rule of the black box; (b) Through exploration, she finds a location pair that fits the rule, where her right hand is twice as high up along the screen than her left hand; (c) She believes that the rule is to keep a constant interval between the hands, and so she has raised her hands keeping the same distance (compare to *b*)—consequently the screen goes red, because the system's embedded ratio rule has been violated; finally, (d) She discovers that the interval between her hands should increase as her hands rise (Abrahamson *et al.*, 2011).

3.3 - The Emergence of Proto-Mathematical Cognitive Structures— The Designer Analyzes What Structure the Student Should Construct

Let's focus on this interval between the hands—this nothing that is. I find it an utter elusive wonder that our interactions with the environment bring forth to our attention something we had not noticed before. We invent an instrument, in this case an invisible instrument—a spatial gap—that enables us to get our work done by perceptually guiding our actions. This emergent mental instrument—that is, a stabilized ecologically embedded sensorimotor pattern that enables perception to guide action—is the enactivist cognitive structure, and our

[16] Per Vérillon et Rabardel (1995), the technology plays a role as a digital artifact (*une boîte noir*) that the student must "instrumentalize" as their means of effecting the desired distal action. But for that, and through that, the student must "instrument" their proximal actions, that is, the artifact comes to extend their intentionality onto the environment. It is there, in this process of instrumenting, that lies the potential for learning to enact a new movement form. As Rabardel (1993) states, « [D]ans les situations où la résolution du problème passe par la mise en oeuvre d'artefacts, de tels schèmes familiers constituent la composante schème des instruments dont les artefacts forment l'autre composante. Or, non seulement ces schèmes ont une genèse, mais comme les artefacts, ils peuvent se voir attribuer de nouvelles significations. La genèse des schèmes, l'assimilation de nouveaux artefacts aux schèmes (donnant ainsi une nouvelle signification aux artefacts), l'accommodation des schèmes (contribuant à leurs changements de signification), sont constitutifs de cette seconde dimension de la genèse instrumentale : les processus d'instrumentation » (pp. 116–117). For further embodied-design elaboration of Vérillon and Rabardel's theory of Instrumented Activity Situations, see Shvarts *et al.* (2021). "[I]n situations where the resolution of the problem passes through the implementation of artefacts, such familiar schemes constitute the scheme component of the instruments of which the artefacts form the other component. However, not only do these schemes have a genesis, but like artefacts, they can be attributed new meanings. The genesis of schemes, the assimilation of new artefacts to schemes (thus giving a new meaning to artefacts), the accommodation of schemes (contributing to their changes in meaning), are constitutive of this second dimension of instrumental genesis: the processes of instrumentation." (edited Google translation)

biological capacity to create these cognitive structures is the enactivist answer to the would-be epistemological paradox that has plagued philosophers searching for meaning. Our neurocognitive capacity of stabilizing emergent sensorimotor patterns is our biological epistemic means of constructing meaning.

Ok, let's take stock of where we are. We have a working context—the Mathematics Imagery Trainer for Proportion—to take on the pedagogical challenge posed for educational designers by the enactivist dictum quoted above: "(2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided." And we have characterized the interval as the emergent cognitive structure that enables the student to perceptually guide the performance of task-appropriate actions to solve the motor-control problem. Perhaps I should elaborate on this process just a tad more.

It is difficult to coordinate raising your hands at different speeds. It is easier to manipulate a single object. The student tacitly crystalizes this interval-form through their sensorimotor exploration; the form becomes interpolated into the student's perception–action loop; this form enables the student to sustain a new task-effective *sensorimotor pattern*. The interval between the hands may first emerge as a candidate cognitive structure when one still erroneously believes that the interval should be constant (in Figure 1, compare images b & c). This tool turns out to be inadequate, and so one explores further, until an adjusted sensorimotor pattern emerges, because it turns out to better generate the desired feedback, that is, to enable one to perform the task. *Enfin, on peut le faire !*

The interval pops up from negative space to become a thing one can manipulate, think about, talk about, measure, and copy to paper (Abrahamson *et al.*, 2011; Bongers, 2020; Flood *et al.* 2016). We call this cognitive structure an *attentional anchor*—it is an aspect of the environment whose perception enables task-effective action (Abrahamson & Sánchez-García, 2016; Hutto & Sánchez-García, 2015). Whereas attentional anchors may emerge spontaneously in the course of solving a motor-control problem, they might also be highlighted, indexed, or otherwise implied by instructors (Flood *et al.*, 2020).

3.4 - From Action to Symbol[17]*—Designing for Mathematical Modeling of Attentional Anchors*

How do attentional anchors come to ground mathematical practice? We now examine what happens when the "rubber" of enacting proto-mathematical movement forms meets the "road" of formal mathematical practice

The cognitive birth of mathematical thinking is momentous, for it decomposes and recomposes our flowing know-how. The moment we become conscious of the tacit attentional anchor—the moment we come to think *about* it—monistic subject–object duality is rattled, parsed, severed, and cleft asunder into explicit articulated particles that we experience as being things out there. As thinking-for-speaking kicks in, a discursive wedge (Sfard, 2007, p. 603) breaks-down our naïve experience of a tool as being *ready-to-hand*, rendering it *present-at-hand*—"The environment announces itself afresh" (Heidegger, [1927],

[17] I owe this phrase to Bamberger (1999).

pp. 105-106). Now we're not just doing, we're thinking about doing, as couched in syntactic taxonomy. Once referred to, an attentional anchor can serve as an enactive-to-semiotic pivot (cf. Bartolini Bussi & Mariotti, 2008, p. 757). A sign is born.

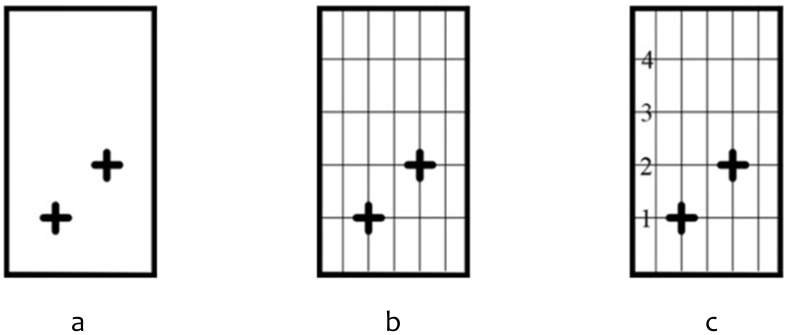

Figure 2
The Introduction of Enactive-cum-Semiotic Resources Into the Sensorimotor Learning Environment Through Three Schematic Interface Modes

Figure 2a presents the vacant screen of a Mathematics Imagery trainer. In this continuous space, students figure out how to move their hands simultaneously to make the screen green, and they use qualitative language to describe their strategy, for example, "The higher I raise my hands, the bigger I make the distance between them." At that point, the tutor—whether a human or an artificially intelligent pedagogical avatar—turns on a grid of lines, as sketched in Figure 2b. Students recognize in this new digital artifact embedded utilities for enhancing their strategy. They utilize the grid to enhance their: (a) enactment, by adopting the lines as perceptual markers of action destination (a pragmatic enhancement); (b) evaluation, by using the lines to measure the interval with greater precision (an epistemic enhancement); or (c) explanation, by pointing to the lines to improve the intelligibility of their strategy (a discursive enhancement). And yet, in the course of assimilating the horizontal lines into their perception–action loop, the strategy itself changes, as does students' thinking about the strategy. Now the space is discrete, not continuous, and students find themselves moving their hands sequentially, not simultaneously: they raise one hand until the cursor reaches a line, then they raise the other hand farther up until they find green. Soon the student exclaims their quantitative, not qualitative, strategy: "For every one line I go up on the left, I go up two lines on the right." Thus, the grid shifts surreptitiously from constituting an enactive instrument to a mathematical frame of reference. When, finally, we activate numerals as well (see Figure 2c), students draw on their arithmetic skills to recognize the multiplicative constant inherent to the proportional progression, for example, they note that the right-hand cursor is always double as high as the left-hand cursor (Abrahamson *et al.*, 2011; Abrahamson *et al.*, 2014).

Radical constructivists have recognized the importance of facilitating an epistemic elaboration from an emergent cognitive structure to reflection and formalization. Students begin by "having" a dynamic image (Pirie & Kieren, 1994), yet, given auspicious social conditions, will then advance to "'languaging' and giving it form in the domain of consensual coordination of action in which we exist as human beings" (Steffe & Kieren, 1994, p. 723, citing Maturana, 1978). The grid and numerals used in the Mathematics Imagery Trainer for Proportion demonstrate how, as a designer, I need to determine what supplementary resources might be introduced into the activity space to augment the student's pragmatic, epistemic, or discursive experience and, in so doing, shift the student's socio-cognitive engagement from immersed action into normative mathematical discourse *about* their action.

But the designer's work does not end in creating artifacts. It matters *how* the new resources are introduced and positioned relative to the task. For example, if the resources are framed as irrelevant or as potential distractors, the student may be less inclined to appropriate them as enactive or semiotic enhancers (Abrahamson *et al.*, 2012, p. 82). Elsewhere, we have elaborated on the tutor's nuanced role in facilitating enactive learning: see Flood *et al.* (2020, 2022) for analysis that draws on Co-Operative Action (Goodwin, 2013) and the ethnomethodological approach to conversation analysis (Mondada, 2014); and see Shvarts and Abrahamson (2019, 2023) for analyses that draw on dynamic systems theory (Kelso, 1995) and cultural–historical psychology Vygotsky (1926/1997).

3.5 - What's Going On?— The Design-Based Researcher Monitors the Learning Process

Finally, wearing the researcher hat, I need to embed into the learning environment instruments that will enable me to gather the empirical data I require to perform *multimodal learning analytics* (MMLA) of students' actions, products, and utterances (Abrahamson, Worsley *et al.*, 2022; Worsley & Blikstein, 2014). As a Little Prince once told us,

> "Les grandes personnes ne comprennent jamais rien par elles-mêmes, et c'est ennuyeux pour les enfants d'être toujours et toujours en train de leur expliquer des choses."[18] (A. de Saint-Exupéry)

For example, I may wish to track students' manual and optical movements as they work on a task, and triangulate these with students' verbal–gestural expression, because doing so could help me determine what sensorimotor patterns are emerging into cognitive structures enabling the perceptual guidance of task-effective motor action. I could then run quantitative analyses of these sensorimotor behaviors to identify and characterize features and trends of this emergence, such as through Recurrence Quantification Analysis (RQA, Marwan *et al.*, 2007).

MMLA of our empirical implementations has revealed an array of attentional anchors that study participants develop spontaneously as their perceptual means of enacting the goal movements in the Mathematics Imagery Trainer tasks (Abrahamson *et al.*, 2015; Duijzer *et al.*, 2017). Furthermore, using RQA, we

---

[18] "Grown-ups never understand anything by themselves, and it is tiresome for children to be always and forever explaining things to them." (A. de Saint-Exupéry).

were able to demonstrate that students' manual actions (Tancredi *et al.*, 2021), eye movements (Abdu *et al.*, 2023), and hand–eye coordination (Tancredi, Abdu, *et al.*, 2022) manifest characteristic properties of complex dynamic systems in flux, such as emergence, reduction of entropy, and phase transitions.

With that, we conclude our proposed answers to the two driving questions of this essay: (1) What would an enactivist mathematics pedagogy look like? And (2) What would a research program look like that evaluated whether the proposed pedagogy is indeed implementing these enactivist ideas? We call this entire research paradigm *embodied design* (Abrahamson, 2009, 2012c, 2014, 2019; Abrahamson *et al.*, 2020; Abrahamson, Dutton, & Bakker, 2022; Abrahamson *et al.*, 2023). The scope of our embodied design research program includes the development of mechanical, digital, and mechatronic educational resources for a range of K–16 concepts (Alberto *et al.*, 2021) and serving students of intersectional diversity, such as students with sensory and cognitive difference (Lambert *et al.*, 2022; Tancredi, Chen, *et al.*, 2022), minoritized multilingual students (Liu & Takeuchi, 2023), students at remote locations (Shvarts & van Helden, 2021), and Indigenous students with unique epistemological–linguistic heritage (Benally *et al.*, 2022). As embodied design becomes adopted by schools, we look to understand how best to integrate the activities into existing classroom practices, perhaps transforming these practices (Kosmas & Zaphiris, 2023).

4 – Farewell to Pirézia

You land in Mathematics national airport. For argument's sake, let's assume you don't speak or read Math. You want to find your relations. You see a sign showing "$tan\theta$". What does $tan\theta$ mean to you? Let's try the following. I'm going to ask you to move your hands, perhaps as though you were conducting an orchestra, or, better, painting on a canvas with two brushes. So, take a seat, and here's what I invite you to do.

Imagine a canvas in front of you, within comfortable reach. Your left hand will move up and down on the *left* side of the canvas, and your right hand will move right and left on the *bottom* of the canvas. In a sense, your left hand is moving along a *y*-axis, and your right hand is moving along an *x*-axis. Your hands can meet at the bottom-left corner of the canvas, the origin point of the two axes. Is this making sense? Ok, there's more. Your hands will be moving *orthogonally*, yes? Because they're tracing along the vertical and horizontal axes. And they will move *simultaneously*… Can you do that, all the while keeping the hands on orthogonal trajectories? But there's one more thing. Sorry. Your hands need to be moving *proportionately* as well. I'm choosing the ratio 1:2. How's that for you? It means your right hand must always be twice as far away from the imaginary origin point as the left hand. Work on this. Enjoy it. As we say in Mathematics, knock yourself out!

When we asked young Dutch children to perform this challenging task on a tablet, they bewildered us with the originality and variety of their creative perceptual solutions to this coordination problem. They spontaneously developed effective attentional anchors to facilitate their enactment of this simultaneous, orthogonal, proportional bimanual movement form (Shayan *et al.*, 2017). We were tracking their eye movements during this experiment, however these gaze data would become available to us only *post facto*, when the experiment was over and the data were analyzed and overlaid onto the video footage. As such, we could not tell in real time *how* the students were performing the task, and so the tutor–researcher asked them. Several study participants alluded to some diagonal line running down the screen from the top-left corner down to the bottom-right corner. One child patiently explained to the confused interviewer, "Het blijft, zeg maar, op één lijn"—"It stays, let's say, on one line."

"What line?" you might wonder. "Where is it? What is this little prince talking about?!" Patience, please. Back to our canvas. Hands up, every body move![19] Place your hands at some pair of locations that are at 1:2 distances from the imaginary origin. Now, "see" a diagonal line between your left and right hands.[20] Once this line has stabilized in your perception, move *the line* to the right, keeping it angled just the same way. That is, do what it takes—simultaneously, orthogonally—so that the diagonal moves to the right without changing its orientation. Can you do that? If you attend exclusively to the line, you'll find that you have raised your left hand a bit and have moved your right hand double as much to the right. But if you attend to one of your hands…, you might "lose" the diagonal line. So, stay focused, attending to the diagonal line. Now try moving this line to the left. Stopping to think about it, what did your hands do?

The line is your attentional anchor. It's not really there, but, then again, what is *ever* really there? What is reality anyway, but a figment of our perception? You now have a new cognitive structure enabling your perception to guide your actions of enacting the challenging movement (Mechsner *et al.*, 2001). In turn, we can *discuss* this perceptual activity mathematically (Abrahamson & Mechsner, 2022). Still, we must now transition from hand-waving to numbers; from legerdemain to prestidigitation. So, let's do that now.

If you think of your left hand as running up the vertical side of a right triangle, and the right hand running along its horizontal side, then the diagonal is the triangle's hypothenuse. As you move the diagonal to the left and right, the sides of this right triangle change, but a constant ratio is maintained between their lengths. All these dilating and contracting triangles are geometrically similar—they are precisely the same shape only bigger or smaller. But the line always leans at the same angle.

We could stop there. But let's push this just a bit more into trigonometry. The line's angle of leaning is at about 27° to the horizon ($arctan\ .5 = 26.565°$). When we divide the length of our 1:2 triangle's vertical side by its horizontal side, we get .5. In trigonometric terms we talk about this this quotient of .5 as a *function* of the angle of ~27°, which we mark as θ. We call the function *tan*, so $tan\theta = .5$. No matter how big or small a right triangle is, if the quotient of its measured sides is .5, then the angle opposite the shorter side will always measure at ~27°.[21]

[19] I owe this pun to Petrick (2012).

[20] The imaginary line is a *projection* that runs between the hands (cf. Kirsh, 2009).

[21] Trigonometric functions, such as $tan\alpha$, are often introduced through the unit circle. For examples of action-based embodied design for trigonometry, the reader is referred to the work of Anna Shvarts (Alberto *et al.*, 2021).

So now you speak Math. The symbol has been grounded. Giant step for humankind. Welcome to Mathematics. Enjoy your stay!

CONCLUSION

Mathematical thinking, teaching, and learning are enactive cognitive activities distributed over routinized manipulations of socially established semiotic resources that emerged through cultural–historical practice. To participate meaningfully in mathematical work, individuals need to develop appropriate cognitive structures. By cognitive structures, we refer to neural substrates orchestrating consistent forms of purposeful interaction with the environment. These cognitive structures guide the identification and engagement of environmental circumstances affording adaptive perception–action loops. Cognitive structures lend us a sense of a figured world populated by ontologies, some of which may be idiosyncratic and others culturally shared, thus facilitating our social traffic. The embodied design paradigm organizes an ongoing research program to innovate, develop, and evaluate learning activities that implement enactivist philosophy in the form of interactive educational resources. The paradigm encompasses theoretical constructs, design principles, and methodological techniques centered on the pedagogical objective that students develop cognitive structures enabling them to ground the symbols and discourse of disciplinary practice.

REFERENCES

Abdu, R., Tancredi, S., Abrahamson, D., & Balasubramaniam, R. (2023). A complex-systems view on mathematical learning as hand–eye coordination. In M. Schindler, A. Shvarts, & A. Lilienthal (Eds.), Eye-tracking research in mathematics education [Special issue]. *Educational Studies in Mathematics.* https://doi.org/10.1007/s10649-023-10279-0

Abrahamson, D. (2009). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics, 70*(1), 27–47.

Abrahamson, D. (2012a). Rethinking intensive quantities via guided mediated abduction. *Journal of the Learning Sciences, 21*(4), 626–649. https://doi.org/10.1080/10508406.2011.633838

Abrahamson, D. (2012b). *Mathematical Imagery Trainer—Proportion (MIT-P) IPhone/iPad application (Terasoft).* In iTunes. https://itunes.apple.com/au/app/mathematical-imagery-trainer/id563185943

Abrahamson, D. (2012c). Discovery reconceived: Product before process. *For the Learning of Mathematics, 32*(1), 8–15.

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction, 2*(1), 1–16. https://doi.org/10.1016/j.ijcci.2014.07.002

Abrahamson, D. (2021). Grasp actually: An evolutionist argument for enactivist mathematics education. *Human Development, 65*(2), 1–17. https://doi.org/10.1159/000515680

Abrahamson, D., & Abdu, R. (2020). Towards an ecological-dynamics design framework for embodied-interaction conceptual learning: The case of dynamic mathematics environments. In T. J. Kopcha, K. D. Valentine, & C. Ocak (Eds.), Embodied cognition and technology for learning [Special issue]. *Educational Technology Research and Development, 69*, 1889–1923. https://doi.org/10.1007/s11423-020-09805-1

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. In N. Newcombe & S. Weisberg (Eds.), Embodied cognition and STEM learning [Special issue] [journal article]. *Cognitive Research: Principles and Implications, 1*(1), 1–13. https://doi.org/10.1186/s41235-016-0034-3

Abrahamson, D., Dutton, E., & Bakker, A. (2022). Towards an enactivist mathematics pedagogy. In S.A. Stolz (Ed.), *The Body, Embodiment, and Education: An Interdisciplinary Approach* (pp. 156–182). New York, NY, Routledge.

Abrahamson, D., Gutiérrez, J.F., Charoenying, T., Negrete, A.G., & Bumbacher, E. (2012). Fostering hooks and shifts: Tutorial tactics for guided mathematical discovery. *Technology, Knowledge, and Learning, 17*(1–2), 61–86. https://doi.org/10.1007/s10758-012-9192-7

Abrahamson, D., & Howison, M. (2008). *Kinemathics: Kinetically induced mathematical learning* UC Berkeley Gesture Study Group (E. Sweetser, Organizer), December 5, 2008 . https://edrl.berkeley.edu/wp-content/uploads/2019/06/Abrahamson-Howison-2008_kinemathics.pdf

Abrahamson, D., Lee, R.G., Negrete, A.G., & Gutiérrez, J.F. (2014). Coordinating visualizations of polysemous action: Values added for grounding proportion. *ZDM Mathematics Education, 46*(1), 79–93. https://doi.org/10.1007/s11858-013-0521-7

Abrahamson, D., & Mechsner, F. (2022). Toward synergizing educational research and movement sciences: A dialogue on learning as developing perception for action. *Educational Psychology Review, 34*(3), 1813–1842. https://doi.org/10.1007/s10648-022-09668-3

Abrahamson, D., Nathan, M.J., Williams-Pierce, C., Walkington, C., Ottmar, E.R., Soto, H., & Alibali, M.W. (2020). The future of embodied design for mathematics teaching and learning [Original Research]. *Frontiers in Education, 5*(147). https://doi.org/10.3389/feduc.2020.00147

Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Journal of the Learning Sciences, 25*(2), 203–239. https://doi.org/10.1080/10508406.2016.1143370

Abrahamson, D., Shayan, S., Bakker, A., & van der Schaaf, M.F. (2015). Eye-tracking Piaget: Capturing the emergence of attentional anchors in the coordination of proportional motor action. *Human Development, 58*(4–5), 218–244. https://doi.org/10.1159/000443153

Abrahamson, D., & Shulman, A. (2019). Co-constructing movement in mathematics and dance: An interdisciplinary pedagogical dialogue on subjectivity and awareness. *Feldenkrais Research Journal, 6*, 1–24. https://feldenkraisresearchjournal.org/index.php/journal/article/view/13/8

Abrahamson, D., Tancredi, S., Chen, R. S. Y., Flood, V. J., & Dutton, E. (2023). Embodied design of digital resources for mathematics education: Theory, methodology, and framework of a pedagogical research program. In B. Pepin, G. Gueude, & J. Choppin (Eds.), *Handbook of Digital Resources in Mathematics Education*. Berlin, Springer.

Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: Signifying sensorimotor enactment in fields of promoted action. *ZDM Mathematics Education, 47*(2), 295–306. https://doi.org/10.1007/s11858-014-0620-0

Abrahamson, D., Trninic, D., Gutiérrez, J.F., Huth, J., & Lee, R.G. (2011). Hooks and shifts: A dialectical study of mediated discovery. *Technology, Knowledge, and Learning, 16*(1), 55–85. https://doi.org/10.1007/s10758-011-9177-y

Abrahamson, D., & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning, 12*(1), 23–55.

Abrahamson, D., Worsley, M., Pardos, Z.A., & Ou, L. (2022). Learning analytics of embodied design: Enhancing synergy. *International Journal of Child–Computer Interaction, 32*, 100409. https://doi.org/10.1016/j.ijcci.2021.100409

Abram, D. (1996). *The Spell of the Sensuous: Perception and Language in a More-than-Human World*. New York, NY, Random House.

Alberto, R., Shvarts, A., Drijvers, P., & Bakker, A. (2021). Action-based embodied design for mathematics learning: A decade of variations on a theme. *International Journal of Child–Computer Interaction*, 100419. https://doi.org/10.1016/j.ijcci.2021.100419

Ba, H., & Abrahamson, D. (2021). Taking design to task: A dialogue on task-initiation in STEM activities. *Educational Designer, 4*(14), 1–21. http://www.educationaldesigner.org/ed/volume4/issue14/article54/

Bakker, A. (Ed.). (2018). *Design Research in Education: A Practical Guide for Early Career Researchers*. New York, NY, Routledge.

Bamberger, J. (1999). Action knowledge and symbolic knowledge: The computer as mediator. In D. Schön, B. Sanyal, & W. Mitchell (Eds.), *High Technology and Low Income Communities* (pp. 235–262). Cambrige, MA, MIT Press

Barton, B. (2008). *The Language of Mathematics: Telling Mathematical Tales*. Berlin, Springer.

Bartolini Bussi, M.G., & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L.D. English, M.G. Bartolini Bussi, G.A. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education*, 2nd revised edition (pp. 720–749). Mahwah, NJ, Lawrence Erlbaum Associates.

Benally, J., Palatnik, A., Ryokai, K., & Abrahamson, D. (2022). Learning through negotiating conceptually generative perspectival complementarities: The case of geometry. *For the Learning of Mathematics, 42*(3), 34–41.

Bereiter, C. (1985). Towards the solution of the learning paradox. *Review of Educational Research, 55*(2), 210–226.

Bereiter, C., & Scardamalia, M. (1985). Cognitive coping strategies and the problem of "inert knowledge". In S.F. Chipman, J.W. Segal, & R. Glaser (Eds.), *Thinking and Learning Skills: Current Research and Open Questions* (Vol. 2 [Research and open questions], pp. 65–80). Mahwah, NJ, Lawrence Erlbaum Associates.

Bongers, T.J.D. (2020). *Transfer of embodied experiences in a tablet environment towards a pen and paper task*. Unpublished Master's thesis (Applied cognitive psychology). Utrecht University.

Clements, D.H., & Sarama, J. (2015). Discussion from a mathematics education perspective. In J. Torbeyns, C. Gilmore, & L. Verschaffel (Eds.), The acquisition of preschool mathematical abilities: Theoretical, methodological and educational considerations [Special issue]. *Mathematical Thinking and Learning, 17*(2–3), 244–252. https://doi.org/10.1080/10986065.2015.1016826

Collins, A. (1992). Towards a design science of education. In E. Scanlon & T. O'shea (Eds.), *New Directions in Educational Technology* (pp. 15–22). New York, NY, Springer.

Donald, M. (1991). *Origins of the Modern Mind: Three Stages in the Evolution of Culture and Cognition.* Cambridge, MA, Harvard University Press.

Donald, M. (2001). *A Mind so Rare: The Evolution of Human Consciousness*. New York, NY, W.W. Norton & Company.

Dreyfus, H.L., & Dreyfus, S.E. (1999). The challenge of Merleau-Ponty's phenomenology of embodiment for cognitive science. In G. Weiss & H.F. Haber (Eds.), *Perspectives on Embodiment: The Intersections of Nature and Culture* (pp. 103–120). New York, NY, Routledge.

Drury, N., & Tudor, K. (2023). Radical enactivism: A guide for the perplexed. *Journal of Theoretical and Philosophical Psychology*. https://doi.org/10.1037/teo0000225

Duijzer, A.C.G., Shayan, S., Bakker, A., van der Schaaf, M.F., & Abrahamson, D. (2017). Touchscreen tablets: Coordinating action and perception for mathematical cognition. In J. Tarasuik, G. Strouse, & J. Kaufman (Eds.), Touchscreen tablets touching children's lives [Special issue] [Original Research]. *Frontiers in Psychology, 8*(144). https://doi.org/10.3389/fpsyg.2017.00144

Easterday, M.W., Rees Lewis, D.G., & Gerber, E.M. (2016). The logic of the theoretical and practical products of design research. *Australasian Journal of Educational Technology, 32*(4), 125–144.

Flood, V.J. (2018). Multimodal revoicing as an interactional mechanism for connecting scientific and everyday concepts. *Human Development, 61*(3), 145–173. https://doi.org/10.1159/000488693

Flood, V.J., Harrer, B.W., & Abrahamson, D. (2016). The interactional work of configuring a mathematical object in a technology-enabled embodied learning environment. In C.-K. Looi, J.L. Polman, U. Cress, & P. Reimann (Eds.), *"Transforming Learning, Empowering earners"*. Proceedings of the International Conference of the Learning Sciences *(ICLS 2016)* (Vol. 1, "Full Papers," pp. 122–129). International Society of the Learning Sciences.

Flood, V.J., Shvarts, A., & Abrahamson, D. (2020). Teaching with embodied learning technologies for mathematics: Responsive teaching for embodied learning. *ZDM Mathematics Educaion, 52*(7), 1307–1331. https://doi.org/10.1007/s11858-020-01165-7

Flood, V.J., Shvarts, A., & Abrahamson, D. (2022). Responsive teaching for embodied learning with technology. In S. Macrine & J. Fugate (Eds.), *Movement Matters: How Embodied Cognition Informs Teaching and Learning* (pp. 179–195). Cambridge, MA, MIT Press.

Gerofsky, S. (2011). Seeing the graph *vs*. being the graph: Gesture, engagement and awareness in school mathematics. In G. Stam & M. Ishino (Eds.), *Integrating Gestures* (pp. 245–256). Amsterdam, John Benjamins.

Gibson, J.J. (1977). The theory of affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, Acting and Knowing: Toward an Ecological Psychology* (pp. 67–82). Mahwah, NJ, Lawrence Erlbaum Associates.

Glenberg, A.M. (2006). Radical changes in cognitive process due to technology: A jaundiced view. In S. Harnad & I.E. Dror (Guest Eds.), Distributed cognition [Special issue]. *Pragmatics & Cognition, 14*(2), 263–274. https://doi.org/10.1075/pc.14.2.07gle

Goodwin, C. (2013). The co-operative, transformative organization of human action and knowledge. *Journal of Pragmatics, 46*(1), 8–23. https://doi.org/10.1016/j.pragma.2012.09.003

Hardison, H.L. (2019). Four attentional motions involved in the construction of angularity. In S. Otten, A.G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 360–369). University of Missouri.

Harnad, S. (1990). The symbol grounding problem. *Physica D, 42*, 335–346.

Heidegger, M. [1927]. *Being and Time* (J. Macquarrie & E. Robinson, Trans.). Harper & Row, 1962.

Heft, H. (1989). Affordances and the body: An intentional analysis of Gibson's ecological approach to visual perception. *Journal for the Theory of Social Behaviour, 19*(1), 1–30.

Hoffmann, M.H.G. (2003). Peirce's 'diagrammatic reasoning' as a solution of the learning paradox. In G. Debrock (Ed.), *Process Pragmatism: Essays on a Quiet Philosophical Revolution* (pp. 121–143). Amsterdam, Rodopi.

Howison, M., Trninic, D., Reinholz, D., & Abrahamson, D. (2011). The Mathematical Imagery Trainer: From embodied interaction to conceptual learning. In

G. Fitzpatrick, C. Gutwin, B. Begole, W. A. Kellogg, & D. Tan (Eds.), *Proceedings of the annual meeting of The Association for Computer Machinery Special Interest Group on Computer Human Interaction: "Human Factors in Computing Systems" (CHI 2011)* (Vol. "Full Papers", pp. 1989–1998). ACM Press.

Hutto, D.D. (2019). Re-doing the math: Making enactivism add up. *Philosophical Studies, 176*(3), 827–837. https://doi.org/10.1007/s11098-018-01233-5

Hutto, D.D., & Sánchez-García, R. (2015). Choking RECtified: Embodied expertise beyond Dreyfus. *Phenomenology and the Cognitive Sciences, 14*(2), 309–331. https://doi.org/10.1007/s11097-014-9380-0

Kelso, J.A.S. (1995). *Dynamic Patterns: The Self-organization of Brain and Behavior*. Cambridge, MA, MIT Press.

Kirsh, D. (2009). Projection, problem space and anchors. In N. Taatgen, H. van Rijn, & L. Schomaker (Eds.), *Proceedings of the 31st Annual Conference of the Cognitive Science Society* (pp. 2310–2315). Mahwah, NJ, Lawrence Erlbaum Associates.

Kosmas, P., & Zaphiris, P. (2023). Improving students' learning performance through Technology-Enhanced Embodied Learning: A four-year investigation in classrooms. *Education and Information Technologies*. https://doi.org/10.1007/s10639-022-11466-x

Lambert, S.G., Fiedler, B.L., Hershenow, C.S., Abrahamson, D., & Gorlewicz, J.L. (2022). A tangible manipulative for inclusive quadrilateral learning. *The Journal on Technology and Persons with Disabilities, 10*, 66–81.

Leung, A., Baccaglini-Frank, A., & Mariotti, M. A. (2013). Discernment of invariants in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.

Liu, S., & Takeuchi, M.A. (2023). Embodied mathematical pedagogy to liberate racialized and multilingual bodies. *Educational Studies in Mathematics, 112*(2), 267–287. https://doi.org/10.1007/s10649-022-10185-x

Marwan, N., Romano, M. C., Thiel, M., & Kurths, J. (2007). Recurrence plots for the analysis of complex systems. *Physics Reports, 438*, 237–239.

Maturana, H. (1978). Biology of language: The epistemology of reality. In G. Miller & E. Lenneberg (Eds.), *Psychology and Biology of Language and thought* (pp. 1–27). New York: Academic Press.

Mechsner, F., Kerzel, D., Knoblich, G., & Prinz, W. (2001). Perceptual basis of bimanual coordination. *Nature, 41*(6859), 69–73. https://doi.org/10.1038/35102060

Mondada, L. (2014). The local constitution of multimodal resources for social interaction. *Journal of Pragmatics, 65*, 137–156. https://doi.org/10.1016/j.pragma.2014.04.004

Nathan, M.J. (2012). Rethinking formalisms in formal education. *Educational Psychologist, 47*(2), 125–148. https://doi.org/10.1080/00461520.2012.667063

Neuman, Y. (2001). Can the Baron von Münchausen phenomenon be solved? An activity-oriented solution to the learning paradox. *Mind, Culture & Activity, 8*(1), 78–89.

Norton, A. (2009). Re-solving the learning paradox: Epistemological and ontological questions for radical constructivism. *For the Learning of Mathematics, 29*(2), 2–7.

Petitmengin, C. (2007). Towards the source of thoughts: The gestural and transmodal dimension of lived experience. *Journal of Consciousness Studies, 14*(3), 54–82.

Petitmengin, C. (2017). Enaction as a lived experience: Towards a radical neurophenomenology. *Constructivist Foundations, 12*(2), 139–147. http://constructivist.info/12/2/139.petitmengin

Perec, G.H. (1978). *La Vie mode d'emploi*. Paris, Hachette.

Petrick, C.J. (2012). *Every body move: Learning mathematics through embodied actions*. University of Texas at Austin. (Unpublished doctoral dissertation).

Piaget, J. (1968). *Le structuralisme*. Paris, Presses Universitaires de France.

Piaget, J. (1970). *Structuralism* (C. Maschler, Trans.). New York, NY, Basic Books.

Piaget, J. (1971). *Genetic Epistemology* (E. Duckworth, Trans.). New York, NY, Columbia University Press.

Piaget, J. (1970). *L'épistemémologie génétique*. Paris, Presses Universitaires de France.

Pirie, S.E B., & Kieren, T.E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*(2–3), 165–190.

Pratt, D., Jones, I., & Prodromou, T. (2006, July). *An elaboration of the design construct of phenomenalisation*. Paper presented at the Seventh International Conference on Teaching Statistics, Salvador, Bahia, Brazil, July 2–7, 2006.

Prawat, R.S. (1999). Dewey, Peirce, and the learning paradox. *American Educational Research Journal, 36*, 47–76.

Rabardel, P. (1993). *Les hommes et les technologies ; une approche cognitive des instruments contemporains*. Armand Colin. ffhal-01017462f

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM Mathematics Education, 46*, 349–361. https://doi.org/10.1007/s11858-014-0591-1

Reed, E. S., & Bril, B. (1996). The primacy of action in development. In M.L.Latash & M T. Turvey (Eds.), *Dexterity and its Development* (pp 431–451). Mahwah, NJ, Lawrence Erlbaum Associates.

Resnick, L.B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R.A. Hattrup (Eds.), *Analysis of Arithmetic for Mmathematics Tteaching* (pp. 373–429). New York, NY, Lawrence Erlbaum Associates.

Rousseau, J.-J. [1762]. *Emile or on Education* (A. Bloom, Trans.). New York, NY, Perseus, Basic Books, 1979.

Schegloff, E.A. (1997). Practices and actions: Boundary cases of other-initiated repair. *Discourse Processes, 23*(3), 499–547.

Sfard, A. (2007). When the rules of discourse change, but nobody tells you - making sense of mathematics learning from a commognitive standpoint. *Journal of Learning Sciences, 16*(4), 567–615.

Skemp, R.R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.

Shayan, S., Abrahamson, D., Bakker, A., Duijzer, A.C.G., & Van der Schaaf, M.F. (2017). Eye-tracking the emergence of attentional anchors in a mathematics learning tablet activity. In C.A. Was, F.J. Sansosti, & B.J. Morris (Eds.), *Eye-tracking Technology Applications in Educational Rresearch* (pp. 166–194). IGI Global. https://doi.org/10.4018/978-1-5225-1005-5.ch009

Sheets-Johnstone, M. (1999). *The Primacy of Movement*. Amsterdam, John Benjamins.

Shvarts, A., & Abrahamson, D. (2019). Dual-eye-tracking Vygotsky: A microgenetic account of a teaching/learning collaboration in an embodied-interaction technological tutorial for mathematics. *Learning, Culture and Social Interaction, 22*, 100316. https://doi.org/10.1016/j.lcsi.2019.05.003

Shvarts, A., & Abrahamson, D. (2023). Coordination dynamics of semiotic mediation: A functional dynamic systems perspective on mathematics teaching/learning. In T. Veloz, R. Videla, & A. Riegler (Eds.), Education in the 21st century [Special issue]. *Constructivist Foundations*, *18*(2), 220–234. https://constructivist.info/18/2

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics, 107*(3), 447–469. https://doi.org/10.1007/s10649-021-10053-0

Shvarts, A., & van Helden, G. (2021). Embodied learning at a distance: From sensory-motor experience to constructing and understanding a sine graph. *Mathematical Thinking and Learning*, 1–28. https://doi.org/10.1080/10986065.2021.1983691

Steffe, L. P., & Kieren, T. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education, 25*(6), 711–733.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/doi:10.1007/BF00305619
Tancredi, S., Abdu, R., Abrahamson, D., & Balasubramaniam, R. (2021). Modeling nonlinear dynamics of fluency development in an embodied-design mathematics learning environment with Recurrence Quantification Analysis. *International Journal of Child–Computer Interaction*, 100297 . https://doi.org/10.1016/j.ijcci.2021.100297
Tancredi, S., Abdu, R., Balasubramaniam, R., & Abrahamson, D. (2022). Intermodality in multimodal learning analytics for cognitive theory development: A case from embodied design for mathematics learning. In M. Giannakos, D. Spikol, D. Di Mitri, K. Sharma, X. Ochoa, & R. Hammad (Eds.), *The Multimodal Learning Analytics Handbook* (pp. 133–158). New York, NY, Springer International Publishing. https://doi.org/10.1007/978-3-031-08076-0_6
Tancredi, S., Chen, R.S.Y., Krause, C. M., & Siu, Y.T. (2022). The need for SpEED: Reimagining accessibility through Special Education Embodied Design. In S.L. Macrine & J. M. Fugate (Eds.), *Movement Matters: How Embodied Cognition Informs Teaching and Learning* (pp. 197–216). Cambridge, MA, MIT Press.
Thompson, P.W. (2013). In the absence of meaning…. In K. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 57–94). New York, NY, Springer.
Turner, T. (1973). Piaget's structuralism (review article). *American Anthropologist, 75*(2), 351–373.
Turvey, M. T. (2019). *Lectures on Perception: An Ecological Perspective*. New York, NY, Routledge / Taylor & Francis.
Urton, G. (1997). *The Social Life of Numbers: A Quechua Ontology of Numbers and Philosophy of Arithmetic*. Austin, TX, University of Texas Press.
Varela, F. J., Thompson, E., & Rosch, E. (1991). *The Embodied Mind: Cognitive Science and Human Experience*. Cambridge, MA, MIT Press.
Verran, H. (2001). *Science and an African Logic*. Chicago, IL, The University of Chicago Press.
Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101.
Vygotsky, L.S. [1926]. *Educational Psychology* (R.H. Silverman, Trans.). Boca Raton, FL, CRC Press LLC, 1997.
Worsley, M., & Blikstein, P. (2014). Using multimodal learning analytics to study learning mechanisms. In J. Stamper, Z. Pardos, M. Mavrikis, & B.M. McLaren (Eds.), *Proceedings of the 7th International Conference on Educational Data Mining* (pp. 431–432). Institute of Education.