IJMEST

International Journal of Mathematical Education in Science and Technology

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/tmes20

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To cite this article: Ratih Ayu Apsari & Dor Abrahamson (24 Nov 2024): Dancing geometry: imagining auxiliary lines by reflecting on physical movement, International Journal of Mathematical Education in Science and Technology, DOI: 10.1080/0020739X.2024.2427099

To link to this article: https://doi.org/10.1080/0020739X.2024.2427099

Published online: 24 Nov 2024.

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY
https://doi.org/10.1080/0020739X.2024.2427099

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Dancing geometry: imagining auxiliary lines by reflecting on physical movement

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ABSTRACT

Constructing auxiliary lines is an important component skill in solving geometry problems, and yet it is difficult to teach, precisely because these lines are 'invisible' until they are actually drafted. Is there any intuitive resource that geometry students could possibly draw on to develop this skill? Is there any domain of human activity where we all naturally entertain imaginary lines, even if we are not aware of doing this? And yet, if so, how would these tacit imaginary lines come forth to be geometrical auxiliary lines? It turns out that dancers spontaneously imagine linear structures, known as attentional anchors, to help them enact their movements. These attentional anchors are drawn out in the dancer's subjective perception and are, therefore, invisible to others. Notwithstanding, we have used an embodied design-based research framework to create a gridded floor mat where students can render their covert dance-oriented attentional anchors as overt geometry-oriented auxiliary constructions. Situated in the cultural context of traditional Balinese dance, this practitioner-oriented paper demonstrates several activities for global classroom use by way of sharing some empirical results from implementing this pedagogical approach with young learners. An appendix lists a set of additional activities for dance-based geometry exploration.

ARTICLE HISTORY
Received 19 July 2024

KEYWORDS
Attentional anchor; auxiliary lines; embodied cognition; dance; design-based research; geometry

MATHEMATICS SUBJECT CLASSIFICATIONS
97G20; 97C70; 97E50; 97C30

1. Introduction

Auxiliary lines are diagrammatic elements that geometricians heuristically supplement into their working space as a means of perceiving latent figural properties that may lead to critical inferences towards solving problems (Palatnik & Dreyfus, 2019; Polya, 1957). For instance, consider the following case of the Two-Rectangle Problem. The task is to determine the area of rectangle *BDEF* given that the area of rectangle *ABCD* is 40 units (Figure 1(a)). Granted, one could use a computational method to calculate the length and width of *BDEF*. However, we could afford a more elegant solution by drawing an auxiliary line *CX* (see the red dashed line in Figure 1(b)). This strategy allows us to uncover a 'hidden' spatial relationship between the two rectangles, an insight that demonstrates how auxiliary lines can reveal important spatial connections that, otherwise, are not immediately apparent.

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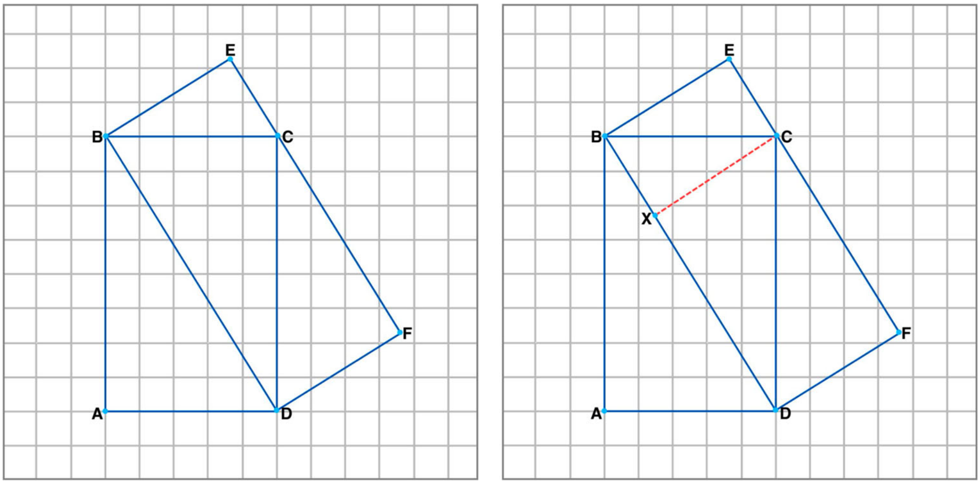


(a) Original problem: What is the area of *BEDF*, if the area of *ABCD* is 40 units?

(b) Supplementing an auxiliary line, CX, suggests new avenues toward a solution.

Figure 1. The two-rectangle problem. (a) Original problem: What is the area of *BEDF*, if the area of *ABCD* is 40 units? (b) Supplementing an auxiliary line, CX, suggests new avenues towards a solution.

Figure 1 demonstrates the utility of auxiliary lines for solving geometry problems by dividing and uniting geometric entities (Palatnik & Sigler, 2018). These lines enable us to apply logical reasoning, eventually establishing a compelling argument, such as justifying the alleged equivalency of the two rectangles. As such, auxiliary lines serve as valuable tools for re-construing diagrams – seeing them anew – towards successfully resolving a given problem.

Despite the important role that auxiliary lines play in geometrical problem-solving, teachers encounter difficulties in guiding students to construct these lines independently (Fan et al., 2017; Herbst & Brach, 2006). As educational design-based researchers inspired by embodiment and radical-constructivist theorisations of human learning (Abrahamson, Tancredi et al., 2023), we approached this pedagogical challenge – namely, '*How should we foster students' skill of constructing auxiliary lines?*'–by asking, in turn, '*Are there any everyday-life situations where students intuitively generate analog structures of these geometry elements?*' Our rationalisation was that if we identified appropriate every-day situations, we may be able to leverage them as our instructional resource for grounding auxiliary lines in students' intuitive skills. Namely, we would create classroom conditions that elicit those intuitive skills and then devise a method for formalising these behaviours as fitting the formal norms of geometry. As such, we required some general framework for grounding mathematical concepts in intuitive skill. As we now explain, we chose to operate within the framework of embodied design.

Embodied design (Abrahamson, 2014), the pedagogical approach employed in our project, is derived from the embodied cognition paradigm from the cognitive sciences, which argues that conceptualising the world begins from experiences of physically acting on the world (Johnson, 2015). Embodied designs create opportunities for students to solve motor-control problems through engaging in sensorimotor exploration that leads to new ways of perceiving the environment; these perceptions are then steered towards formalisation by way of prompting students to incorporate traditional mathematical instruments into their movement strategies. As such, students can first experience a concept through bodily engagement with the environment and only later appropriate mathematical resources as means of enhancing the enactment, evaluation, or explanation of their embodied strategies (Abrahamson, Nathan et al., 2020).

Our quest for an embodied design that would support students in developing the skill of constructing auxiliary lines led us quite serendipitously[1] to investigate the literature on dance pedagogy, which affirms that dancers frequently create imaginary figurative elements that they 'see' around them to navigate space and facilitate the precise execution of their dance movementss (Clark & Ando, 2014; Movement Research, 2012). Further, Hutto and Sánchez-García (2015) coined the notion of *attentional anchors* – perceptual resources, such as imaginative linear constructions, that assist individuals in enacting movements, for example, imagining a tall rectangle guides the juggling of balls. A specific example of using attentional anchors in dance occurs in Gambuh, Balinese traditional dance-drama. Gambuh is usually performed on a rectangular stage area surrounded by bamboo fences called *kalangan*, which is decorated with typical natural and cultural elements, like greenery, lances, and two umbrellas, each on the right and left of the entrance. In turn, these decorative elements serve also as points of reference for the dancers (Bandem, 1978). Instructors scaffold dancers' visual and physical orientation to kalangan elements, for instance, by prompting the students to 'find the (right/left) umbrella' (Panji, 2020; Suteja, 2024). Here, the umbrella functions as an attentional anchor, a sensory information structure whose apprehension facilitates movement. This function of attentional anchors in dance, namely restructuring the dancer's orientation to the sensory information in their environment so as to promote the effective enactment of their practice, parallels the role of auxiliary elements in geometry, namely bringing forth latent linear configurations to enable the appreciation of relations that promote a solution to a given problem.

Unsurprisingly, therefore, the construct of attentional anchors was imported from the philosophy of movement science to research on mathematics education, when Abrahamson and Sánchez-García (2016) employed the construct to theorise the perceptual behaviours of students who, like dancers, solve motor-control problems by suddenly 'seeing' linear structures that will later lead to conceptual insight. Some previous examples of mathematical tasks that implicitly invite students to create attentional anchors as a means of reducing complex coordination problems into more manageable tasks can be seen in: (a) the Mathematics Imagery Trainer activities, which challenge students to enact a target bimanual movement that maintains invariant some figural, relational, or quantitative property they thus learn (Abrahamson, Trninic et al., 2011); and (b) the collaborative construction of body-scale polyhedra, such as an icosahedron, where students experience attentional shifts towards simple embedded forms, such as triangles (Palatnik, 2022).

This paper introduces *dancing geometry*, a module of mathematics activities that implement the embodied design framework as situated within culturally authentic practice and using only non-digital material resources. Specifically, we designed a novel learning environment that integrates Balinese traditional dance methodology with formal mathematical discourse. As in all action-based embodied-design activities, dancing geometry first solicits students' attentional anchors that enable their coordinated performance of complex movements and only then guides students to signify these attentional anchors mathematically (Abrahamson & Trninic, 2015). We are not the first scholars to explore Balinese dance's underlying ethnomathematical structures, such as precise geometrical forms inherent to the dancers' postures and movements (Dewi et al., 2022; Radiusman et al., 2021). However, we may be the first educational researchers to leverage and evaluate the pedagogical potential of these choreographic routines.

Whereas Balinese dance traditionally comprises numerous gestures, stances, and movements performed simultaneously by different body parts, thus requiring expert skill, our current design employs only very basic dance elements from this tradition and, as such, could be used globally and by individuals with varying levels of dance experience. Notwithstanding, inviting students to perform these basic postures rigorously is expected to foster the construction of attentional anchors. In summary, this paper presents a novel activity designed to introduce students to geometry content through dance experiences. Specifically, the activity elicits students' perceptions for movement while offering resources to objectify these implicit attentional anchors as explicit auxiliary lines as they solve a series of Balinese dance tasks.

The mathematical concepts we aim to develop through the activity are related to the concept of angle. Through the designed activities, students will work on constructing auxiliary lines to measure and justify an angle and later use them to classify two-dimensional objects. We chose angle as the exploration content due to the various challenges of teaching and learning angles in elementary school. For instance, young students struggling with angle conservation thought that applying a transformation to an angle (e.g. rotation) changes its aperture (Bütüner & Filiz, 2017; Devichi & Munier, 2013). Referring to NCTM geometry standards for Grades 3–5, our designed activities target students' analysis and reasoning skills in geometry, inviting them to investigate geometrical objects' characteristics, properties, and their relationships (NCTM, 2000, 3.G.1, 4.G.1–3, 5.G.3–4).

In the remainder of this paper, we explain the resources we created for students to ground the formal geometrical practice of building auxiliary lines in the naturalistic dance methodology of imagining attentional anchors. For each activity, we elaborate on the theoretical background of the design and how it is connected to the teaching and learning of specific targeted mathematical concepts. Using photographs and transcriptions extracted from video data – which we collected while implementing the activity with Grade 5 students in Bali, Indonesia, and California, United States – we will demonstrate how children have collaboratively used our resources to transition from dance to geometry and, in turn, think geometrically about new dance moves. An appendix will spell out suggestions for geometry teachers to use the resources in their classrooms. Whereas we worked with Balinese children who are familiar with the basic vocabulary of their heritage choreographic tradition, teachers operating in other indigenous and diaspora cultural contexts, whether historical (e.g. ballet, square dance) or contemporary (e.g. hiphop), could plausibly modify our suggested activities to fit their movement traditions and geometrical explorations.

2. Design solution

Figure 2 features *GRiD* (Geometry Resources in Dance), a diagrammatic floor mat designed to serve mathematics students as a material platform for bridging together from dance to geometry and back again. In our study, we conjectured that students' intuitive ability to produce attentional anchors as movement solutions in Balinese dance practice could be tapped to train them to construct auxiliary lines as geometry solutions.

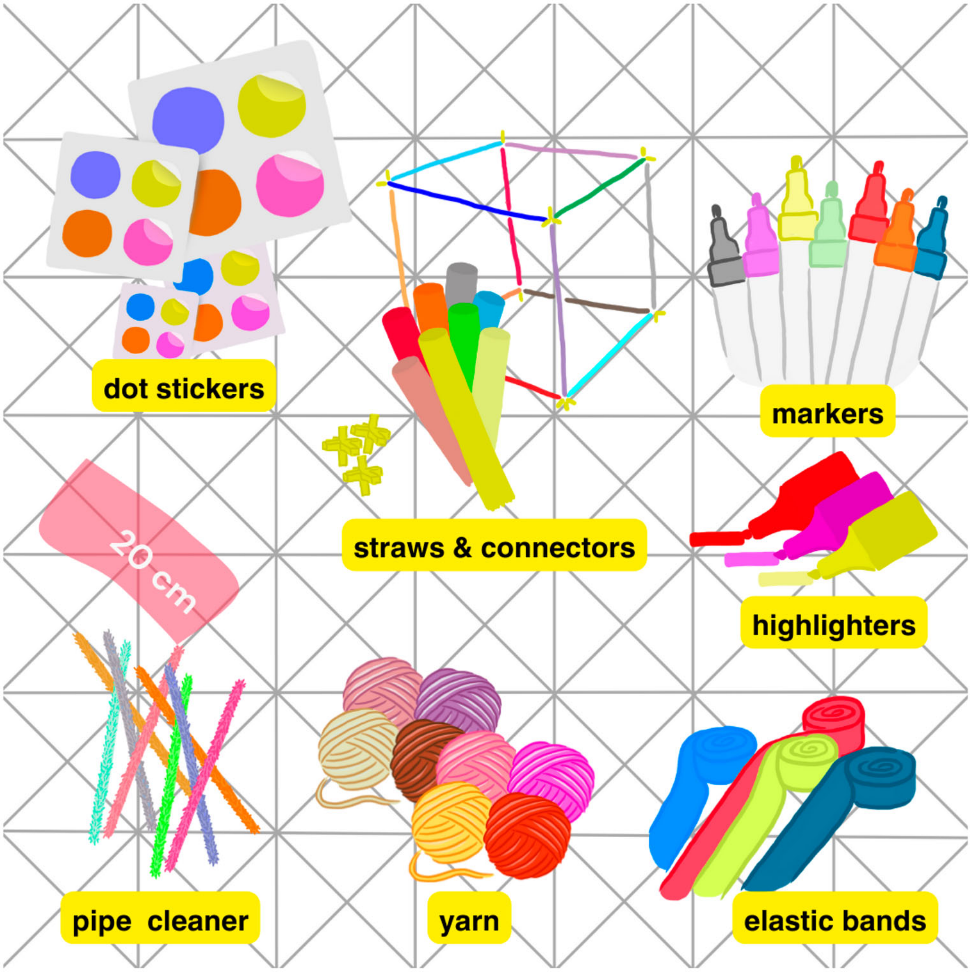


Figure 2. GRiD's material components (graphic artwork: Made Ariawan). GRiD comprises a gridded mat, markers, stickers, highlighters, straws, connectors, yarn, pipe cleaners, and elastic bands. Students are to use these resources as their means of objectifying attentional anchors, which they implicitly use in dance, as auxiliary lines that they explicitly use in geometry.

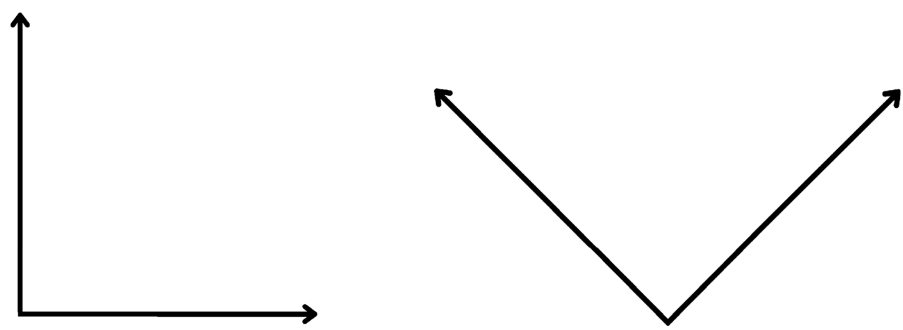
GRiD's mat dimensions can range from 1 m × 1 m to 1.5 m × 1.5 m to accommodate either individuals or groups of up to four individuals at a time. To prolong its use, the mat may be covered with transparent plastic. Various accessories, such as colourful markers, dot stickers, colourful yarns, and construction straws with connectors, pipe cleaners, and elastic bands can be utilised to facilitate exploration, discovery, and discussions.

The next section of this paper presents in detail three sample dancing-geometry activities from the Balinese context to explain their global mechanism as educational opportunities. We propose a design that provokes and mediates perspectival conflicts as a medium for students to learn the targeted mathematical concepts in which auxiliary lines are utilised. We also demonstrate how these activities might be used in the classroom by presenting some of our participant fifth graders' spontaneous solutions to individual and collaborative movement problems. The appendix of this paper elaborates on the sequences of activity and the suggested prompts and questions for interested teachers to implement in their classrooms.

2.1. Activity 1: justifying a ninety-degree angle

The activity is rooted in the most basic posture of Balinese traditional dance, the feet configuration Tapak Sirang Pada (TSP): heels meet, feet rotated out, creating a form that would be described in Euclidean geometry as a ninety-degree angle or an orthogonal linear relation. The task is for students to enact the position, evaluate whether their feet are configured at 90°, and explain their reasoning. The follow-up for the activity is for students to step to different locations (forward, backward, to the side) and create the TSP again. Students are asked to explain how they recreate the form in other places and directions.

A key challenge of this activity stems from a perspectival conflict between traditional textbook presentations of 90°, namely, as perceived from a third-person perspective ('L-shaped', Figure 3(a)), and the embodied dance presentation, as created by the feet, as perceived from a first-person perspective ('V-shaped', Figure 3(b)). Cognitive analysis of this perspectival conflict suggests an epistemic asymmetry between the embodied vs. text-book presentations, where the V-shape is grounded in naturalistic yet unarticulated sensorimotor schemes, while the L-shape is familiar as a cultural icon of ninety-degree concept image (Vinner & Hershkowitz, 1980) yet does not evoke any such scheme. Gerofsky (2011) has juxtaposed these two types of perspectives, respectively, as 'being' (internal, egocentric) vs. 'seeing' (external, allocentric) vs., with the former statistically correlated with deeper mathematical understanding. At the same time, previous studies have found that students struggle to recognise a ninety-degree angle when it is not presented in the canonical L-shaped form (Bütüner & Filiz, 2017; Devichi & Munier, 2013). The instructional challenge thus becomes, Can we create conditions for students to recognise the V-shape formation as 90°? That is, could we possibly guide students to perceive the V-shape as a figural variant on the L-shape that maintains its geometric property of angle magnitude?

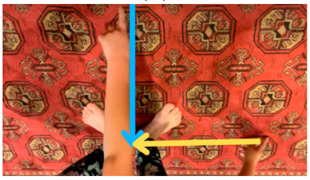


(a) Text-book presentation **(b)** Dance-based embodiment

Figure 3. Two perceptual perspectives on a right angle. Geometry textbooks typically present a right angle as L-shaped, with orthogonal horizontal and vertical rays (3a), whereas, dancers perceive it as V-shaped, with the feet-rays extending diagonally. (a) Text-book presentation. (b) Dance-based embodiment.

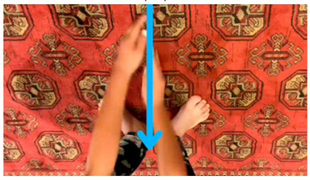
Transcript 1. The traces of invisible lines.

(a)



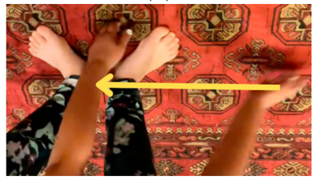
A: I think of it almost like, for example... [*blue and yellow arrows trace her simultaneous left hand (LH) and right hand (RH), index fingers' converging gestures*],

(b)



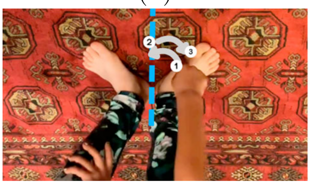
A: ...if... this section [*blue arrow traces her bimanual gesture*]

(c)



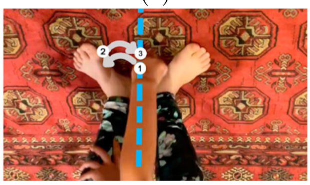
A: and this section [*yellow arrow traces her gesture*] would be a right angle,

(d)



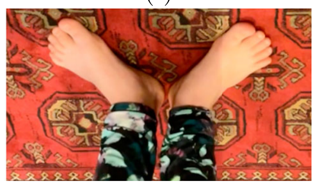
A: this would be like the 45 degrees [*gray arrow traces her back and forth RH gestures, the dashed blue line traces her earlier vertical gesture as in Image 1b*],

(e)



A: and then that would be the 45 degrees [*now RH traces to the left of the imaginary vertical partition*].

(f)



A: So, I kind of think of it like in angles.

Note: The photographs show Anna's feet as seen from above, with her hand occasionally highlighting features of her posture.

As educational designers, we approach documented learning challenges by analysing their cognitive underpinnings and, in turn, ideating novel instructional responses that include dedicated activities with new material resources (Collins, 1992). In particular, *we wondered whether the didactical dance context of geometrically choreographed physical movement on the GRiD mat might create unique pedagogical opportunities for students to appreciate the V-shaped linear configuration as 90° by somehow reconciling its apparent perspectival conflict with the more familiar L-shaped configuration*. The cognitive work of this V-vs.-L reconciliation, we further conjectured, could involve students in reflecting over their own perceptual experience of perspectival conflict as well as through logical–deductive operations that draw on their geometrical knowledge.

The execution of TSP, as is the case with all Balinese dance postures, must be meticulous. Dancers studiously appraise their postures for accuracy, and they are accountable to their teachers for their method of ascertaining the quality of executing these postures. As such, asking students to justify the adequacy of their TSP posture elicits their pre-articulated personal references, whereby other students could follow their reasoning. Consider the following Transcript 1, in which a student, pseudonym Anna (A), spontaneously invokes horizontal and vertical auxiliary dance lines to argue that her feet constitute angle bisectors of two abutting right angles. Thus, a 'not-much-here' foot posture quickly becomes a learned geometric exposition…

When Anna was presented with the GRiD tarp (see Figure 4), she immediately referred to its indicated GRiD lines as constituting the imaginary dance lines she had invoked in explaining her posture (compared to Transcript 1). She then placed three dot stickers on

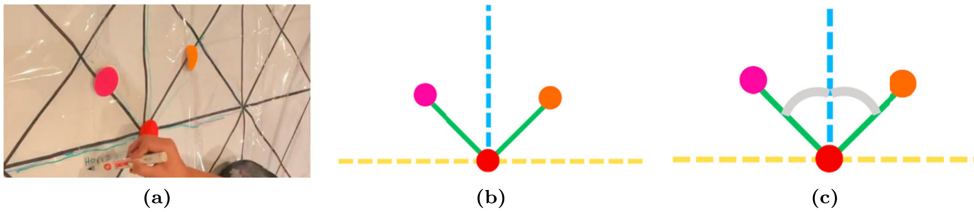


Figure 4. Anna's GRiD construction of auxiliary lines to justify her TSP posture. Anna constructed a system of auxiliary lines (yellow and blue) and points to overtly show the covert lines she uses to evaluate her dancing posture. The green lines mark the direction of her feet as diagonal to the auxiliary lines. The dot stickers respectively mark the positions of her heels (red) and toes (pink and orange). The gray arcs indicate Anna's incarnation of a 90° angle formed by two 45° angles, as in her explanation in Transcript 1(d)–(e).

the tarp (see Figure 4(a)) to mark the respective locations of the junction of her heels (one sticker) and the tips of her toes (two other stickers). Anna then drew lines connecting the single heel sticker to the two toe- stickers (see Figure 4(b)); the lines, as Anna explained, thus came to indicate the angle rays (see Figure 4(c)).

Per our embodied design objective, Anna spontaneously recognised GRiD embedded utilities for enhancing the enactment, evaluation, and explanation of her TSP performance. *In adopting GRiD resources as frames of action and reference, Anna implicitly appropriated the formal mathematical practice of geometrical argumentation, where overt auxiliary lines materialise her covert attentional anchors*. Once Anna's attentional anchors became auxiliary lines, Anna and the researcher were on the same page, literally, where they could engage together in geometric reasoning about the dance posture.

2.2. Activity 2: understanding square as a special case of rhombus

The following activity involves physical actions and geometrical reasoning that elaborate on the TSP activity, above, and so it would be advisable to begin with the TSP activity before advancing to this one.

Once students are confident in performing TSP and justifying their posture, assign them to pairs. Ask them to stand face-to-face, so that the ends of their toes meet, creating a quadrilateral (see Figure 5). Students' task will be to reason about the shape of this four-feet configuration. Whereas students' foot size naturally varies across the classroom, teachers can ask students to ignore these differences and, instead, assume the four lengths are identical. Under circumstantial contexts where students engage in this activity individually, such as a student who is participating remotely through video-conferencing, the teacher could ask the student to create a new TSP that faces their previous TSP from Activity 1.

Similar to Activity 1, here again we highlight a conceptual difficulty caused by perceptual capacities and traditions. Just as in the case of L-shaped vs. V-shaped right angles, here students will likely be more familiar with the image of squares in their cardinal standing-on-an-edge orientation (vertical and horizontal edges) than the rotated standing-on-a-vertex orientation. We conjectured that students would debate over the identity of the shape – whether it was a rhombus or a square – depending on their angle of sight. As such, the quadrilateral is a perspectively ambiguous form (cf. Abrahamson, 2019). We hoped to

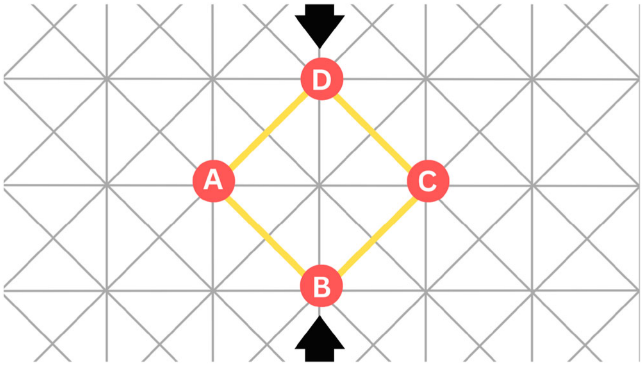


Figure 5. Configuration of two TSPs. Student 1's heels meet at Point B, and Student 2's heels meet at Point D. The two students' toes meet at Points A and C. Their foot lengths (yellow lines) are taken as equivalent. From Activity 1, it is known that $\angle ABC$ and $\angle ADC$ are right angles.

facilitate a logical resolution of this perspectival ambiguity, whereby a square is a rhombus with right angles. As such, a pedagogical heuristic underlying this activity is for students to appreciate the epistemic power of geometric argumentation to resolve apparent conflict. Attention to the constituent properties of an object, such as its angles, can challenge a classification based solely on appearances and perceptual habits.

Moreover, we hoped that the social context of these activities would enhance individual students' reflections and geometrical proving through what Schwarz and Baker (2017) call *collaborative argumentation* – respectful and supportive collective logical inquiry. The social context of peer inquiry, we surmised, could offer unique affordances for reconciling perspectival construals of the linear configuration as square vs. rhombus, because these vying perspectival orientations could be distributed over two or more students, thus constituting the two perspectives as simultaneous (you and me at the same time) rather than sequential (me and then me again). That is, the linear configuration would become a socially distributed ambiguous figure (I see duck *while* you see rabbit; (Abrahamson, Bryant et al., 2009; Benally et al., 2022). Consequentially, we further conjectured, the conflict could be heightened and honed as a logical paradox that, in turn, might be resolved when each participant endorses the veracity of their peer's contemporaneous view (Gopnik & Rosati, 2001); the conflicted views could thus be experienced as necessarily complementary rather than mutually exclusive. In summary, our design approach is to identify, analyse, and then deliberately elicit students' perspectival conflict, which has impeded their learning, as a productive opportunity enabling learning, by creating structured instructional conditions, including material, purposeful, and social resources, that foster students' reconciliation of these tensions (Abrahamson, 2014; Abrahamson & Wilensky, 2007).

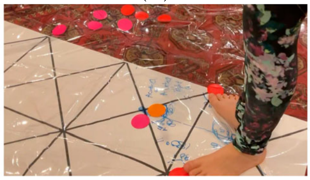
When working on this problem, Anna initially saw a rhombus, and then later, she saw a square. As Transcript 2 details, Anna proposed a geometric condition of squareness: Given that all four edges of the quadrangle are equal, then the shape is square *iff* all its four angles are also equal. However, as you will see below, it took Anna some time to finally convince herself that, indeed, all the angles are equal. After a while, Anna performed the embodied reasoning of enacting each of the four angles with her feet in TSP to argue that they were all right angles. We now join Anna, as she begins to reconsider her assertion that the shape is a rhombus but not a square.

Transcript 2. Embodied reasoning: combines enacting and inscribing geometry.

(a)

A: Actually [*assume the TSP posture*], mmm do we have our feet like right here,
R: ehem
A: we can also try

(b)



A: ...moving our feet in the direction of this side [*moves to create the adjacent angle*],
R: ehem

(c)

A: ...not just this side [*returns to the position as in 2(a)*]
R: ehem ehem

(d)

A: [*now returns to the position as in 2(b)*] and if our two feet right here too, then that [*points to the point where her heels met*]

(e)

A: ...and we do the exact, we do the same positioning as this side [*moves to 2(a)*]
R: ehem
A: and this side is a ninety degrees angle, that would mean

(f)

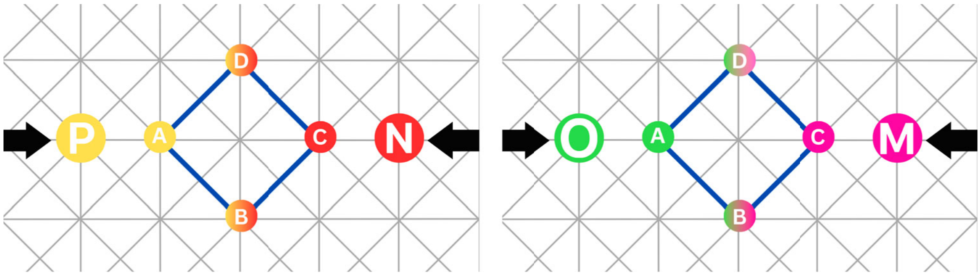
A: ...this side [*moves to 2(b)*] is a ninety-degree angle, too.

Following her strategy in Transcript 1, Anna has established a new, embodied measurement tool serving her geometric argumentation: her feet's figural construction in performing TSP. Since she had previously constructed a valid argument that her TSP opens at ninety degrees (Figure 4), she evaluated the other angles by showing that she could perform TSP there as well.

Whereas students' analysis of the TSP geometry in Activity 1 then served them in Activity 2 to argue that the double-TSP is a square, we found that collaborating students might self-initiate this problem of the square while still in Activity 1. Here, Merry (M), Nath (N), Owen (O), and Philip (P), and Ratih (R, the researcher/teacher) spontaneously investigated the case of two TSPs. It started with Nath's proposal that to justify his TSP, he checks on a rhombus configured by connecting his TSP with a point opposite to his heels. To help other students follow Nath's idea, Ratih asked Philip to stand in the position Nath was referring to, followed by Merry and Owen in their respective places (see Figure 6). As such, they worked on the problem still before they had established that TSP forms a right angle, thus unwittingly leading them to face the square-vs.-rhombus dilemma..., only to raise the question of whether each TSP is a right angle.

Finding themselves in a bootstrapping situation, where they needed to justify a single TSP's angle to determine the double-TSP shape, Merry realised that different points of view lead to different perceptions of the same object, so that conflicted inferences might be resolved (compare with situation of ambiguity when solving spatial geometry problem, Palatnik & Abrahamson, 2022). She conveyed her reasoning to her group members, as Transcript 3 recounts.

In Transcript 3, Merry proposed a strategy to perceive the angle differently by changing her point of view (3a). Nath and Philip initially rejected her idea (3b), but later, Owen confirmed it (3c–d). The perspectival conflict, as is shown in Transcripts 2 and 3,

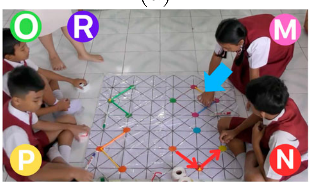


(a) (b)

Figure 6. The group configured the 2 TSPs next to each other. Nath paired with Philip (6a), and Merry paired with Owen (6b). Points A marks the heels" meeting point, respectively of Philip (6a) and Owen (6b), and Points C – for Nath (6a) and Merry (6b). Nath's and Philip's toes met at Points B and D (6a), as did Merry's and Owen's (6b). All of their feet's lengths (blue lines) are taken as equivalent.

Transcript 3. Different perspectives lead to different perceptions.

(a)

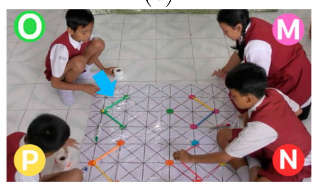


M: [*reorients her body to face the corner see blue arrow*] From this side, it is a right angle.
P: Acute
N: It's acute [*traces along his angle, see red arrows*]
R: if Merry looks from that side, she said it is a right angle.

(b)

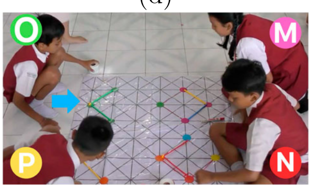
N:From here [*reorients his body, see blue arrow*], it's an acute.,
P:[*tilts his head, see blue arrow*] Mine's the most acute.

(c)



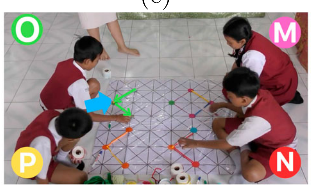
O:[*reorients to L-shape*] Yes, from here, it's a right angle [*see blue arrow*].
P:Hah? [*tilts his head*]

(d)



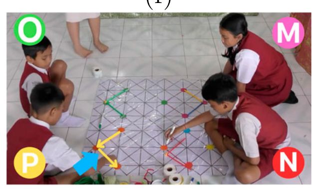
O: From here, it's acute [*back to the V-shaped orientation, see blue arrow*].
P:[*orients his body to the corner*]

(e)



R:Oh, I see. How does it change depending on where you see it?
O: [*reorients to the other L-shape view, traces along angle, see green arrows*]
P: [*tilts his head back to the L-shape orientation*]

(f)



P: [*adjusts his body to that L-shaped orientation, see blue arrow*] Oh, that's right. From here, it's a right angle [*traces the angle, see yellow arrows*]

is a crucial learning moment, because it creates for the students an intellectual need (Harel, 2013) for more rigorous reasoning to justify their claims regarding an angle's measure. Apparently, students may believe that an angle is either acute or right depending on one's point of view. In Piagetian terms, one might say that they have not yet conserved the angle (cf. Piaget, 1941) with respect to the parameter of orientation. As we shall see, this epistemic quandary may offer an auspicious learning opportunity for students engaged in mutually respectful collaborative argumentation. Figure 7 offers one

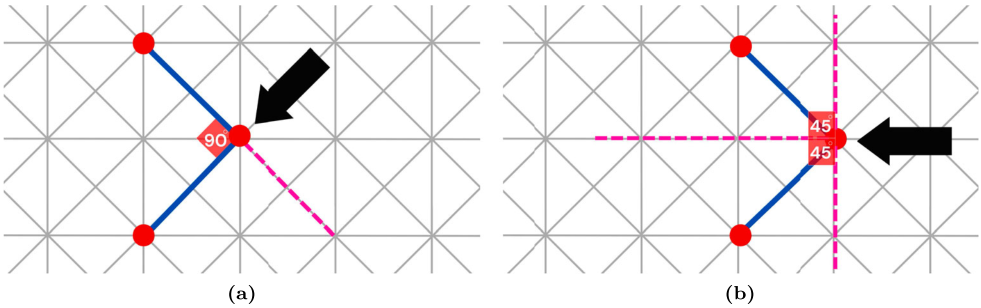


Figure 7. Possible further arguments with GRiD to justify that TSP forms a right angle: (a) adjusting one's visual perspective (black arrow) to maintain an orthogonal construal of the *L*-shaped right angle and further arguing that its measure derives from being half a straight angle; and (b) maintaining the centre perspective to construct two right angles (to the left and right of the vertical axis), where each foot (blue diagonal lines) serves as the bisector of its respective right angle, thus entrapping a 45° angle with the vertical axis; together, these two 45° angles form a right angle.

possible resolution of this quandary using GRiD's affordances, as viewed from a student's perspective.[2]

The auxiliary lines (pink dashed line) in Figure 7 performed a function to allow students to use a known result (Palatnik & Sigler, 2018), i.e. half of 180° (7(a)) or two times the half of 90° (7b). Having successfully justified that their TSP's aperture measures as a 90° angle, the next step was for students to likewise investigate other angles on the constructed shape. Whereas each student had felt confident that the angle at their heels is 90° (recall Figure 6, $\angle BAD$ and $\angle BCD$, they had yet to evaluate the angle where their toes meet (Figure 6, $\angle ABC$ and $\angle ADC$).

Tackling the problem, Nath (N) employed his straw construction[3] to prove that all four angles of the quadrangle are equal. Nath's peers followed his proof construction (Transcript 4(a)–(d)), correcting his momentary confusion over the angle orientation (Transcript 4(e)–(f)). In the interest of clarity for the purposes of this exposition, we have superimposed red lines and a yellow circle on the transcript images of Nath's original straw construction.

In Transcript 4, Nath began his argumentation by invoking the geometric structure of Philip's TSP at the location where it had been performed. Nath then attends to the unknown toe-to-toe angles. Using his plastic construction straws, Nath argued that the toe-to-toe angle is congruent with the heel-to-heel angle. Noting that Nath had thus accounted for three of the four-footed quadrangle angles – Philip's heel-to-heel angle and the two Nath–Philip toe-to-toe angles – but not yet for his own heel-to-heel angle, the researcher challenged Nath to further examine whether those three angles were congruent with his own TSP. The prompt was intended to probe the students' budding perspectival fluency in perceiving the same angle from different orientations. Yet, whereas Nath correctly pointed to the original location of his feet (see Transcript 4d), he placed his straw construction elsewhere (cf. Transcript 4e). Soon, Nath's peers Philip and Owen, who had been following his argumentation, called out Nath's error and helped him amend it by offering monosyllabic verbal evaluations '*no*' and '*yes*' (see Transcript 4(e) and (f)), and then Mary manually moved the straws to their correct place and in the correct orientation (Transcript 4(f)).

Transcript 4. Straw construction as a measurement tool.

(a)

N: Here, here [*grabs his straw construction*]

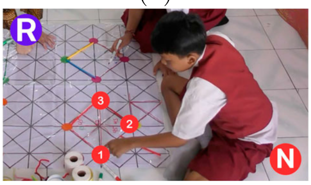
(b)

N:[*rotates the straw construction to the adjacent side*] To here.
R:Oh, I see. You can check from there [*addressing the group*]. Is that the same as the one at your feet, Nath?

(c)

N:[*lifts his straw construction and ''translates'' it to the opposite side*]
R: Hmmm... What about on your feet's TSP?
N: Same.

(d)



R: Where are your TSP's feet, Nath?.
N: This [*points to his leg*]. Eh, this one [*points at (2), and then (1), (2), (3)*]
R: Yes, so where should you put it [*refers to the straw*]?

(e)

N: Like this? [*moves the straws to the other side*]
P: No, no, no!
N: N: Here here [*adjust the straw construction*]

(f)

M: No! Like this, at your feet! [*moves the straws to Nath's feet line*]
O & P: Yes!

Further, we can see how Nath uses his straw construction as a measurement tool to argue that all four angles in the quadrangle are equal. Nath's situated geometrical reasoning is similar to Anna's strategy (see earlier in Transcript 2) – using one's feet as a measurement tool, an embodied yardstick of sorts, to compare angles across locations, specifically to negotiate the perspectival conflict of perceiving the right angle as L-shaped vs. V-shaped. Moreover, both students used supplementary semiotic devices – markers (Anna) and straws (Nath) – to extract an imprint of the feet onto the environment. Still, the straw construction with the right-angle joiner offers a technical ergonomic advantage over marking the angle, by virtue of the constructions's material affordances for rapidly and reliably translating the angle across locations without changing one's posture or perspective.

2.3. Activity 3: an angle's aperture is independent of the length of its rays

As students now understood that the orientation of an angle does not change the *magnitude of its aperture*, we moved forward to emphasise that the angle's aperture does not depend on the *length of its rays*, either. Taking once again the Piagetian perspective, we now turn to support students' conservation of angle measure with respect to the parameter of ray length. Both of these conservation parameters, while appearing trivial to mathematically informed individuals, are far from trivial to pre-conservational students. Many elementary school students believe the angle rays' lengths impact the angle's measure (Ozen Unal & Urun, 2021; Sari et al., 2021).

To explore the relation between an angle's ray-length and measure, we invited students to work on a hypothetical situation in which someone with longer feet than them wishes

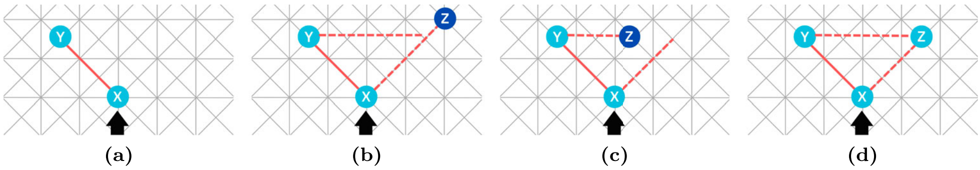
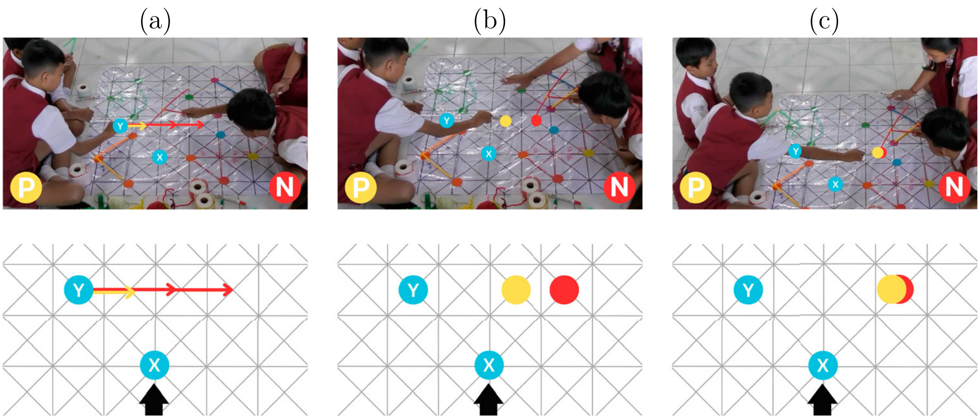


Figure 8. Determine the position of an unknown point with the help of auxiliary lines. Given points X and Y (a), find Point Z such that $\angle YXZ$ is 90°, where Lengths $XY = XZ$. Solving this problem requires the construction of auxiliary lines triangulating both X and Y, otherwise Z could be in incorrectly placed (as in 8(b) or (c)). The correct position of Z is achieved when two auxiliary lines, XY and XZ, are intersected (c). Moreover, the lengths of angle rays XY and XZ are greater than in the earlier Activity 1 and 2 (Figure 5 and 6), thus creating opportunities for students to appreciate that an angle's measured aperture (here, 90°) is conserved over variation in ray length.

Transcript 5. The construction of auxiliary lines with two reference points.



(a)
N: [*gesturing an auxiliary line as traced by the red arrow, taking a short pause, and continuing the line*]
P: Their right foot is here [*gesturing an auxiliary line as traced by the yellow arrow*]

(b)
P: Here [*points at the position indicated by the yellow dot*].
N:Here [*discontinues tracing his imaginary line and points at the position indicated by the red dot*].

(c)
N:Here [*still pointing at the position indicated by the red dot*]
P: Oh yeah, right, here [*points to the new position, see yellow dot*]

to perform the TSP position. The first part of the problem asks where the right toe should end (Z), given the location of the heels (X) and left toe (Y) on the mat (Figure 8(a)). To determine the position of Z, students need to use both X and Y as reference points (Figure 8(b)–(d)) and then determine a means of copying Length XY onto an appropriately angled ray trajectory from X to find Z.

Transcript 5, narrates how Nath and Philip independently constructed an auxiliary line, even as they used different reference points, which led to different proposed locations for Z. The photo images maintain the same notation systems for Points X and Y as in Figure 8. Immediately below each photo image is a 'cleaned up' schematic illustration of the geometric construction, to simplify the reader's interpretation of the rich GRiD information that includes residual markings from the previous inquiry.

Working on this problem, students need to be able to construct their auxiliary line from Y to Z because it is not available on the GRiD (compared to, for instance, the line that connects points *A* and *C* in Figure 6). As such, we challenged students to go beyond what is already made available by the GRiD's lines. In addition, recalling Activity 1, students realised that *Z* has to be equidistant from *X* as *Y* is from *X*, because the two feet of our hypothetical long-footed person are equal in length. The line *XZ*, which represents that other foot, serves as an auxiliary line that helps determine the position of *Z*.

As evident in Transcript 5, Philip and Nath arrived at different proposals for locating *Z* (5b). Whereas they both traced the horizontal line from *Y* (5(a)) with respect to the orientation of TSP as shown by the black arrow (simplified illustration, 5(a)–(c)), Nath referenced back to *X*, as evident in the video data, coinciding with a short pause he took before continuing his hand motion towards the red dot position (5(b) and (c)). Therefore, rather than letting the auxiliary line stop at any non-specific location (e.g. as Philip proposed, see Transcript 5(b)), Nath was convinced that *Z*'s position should be unique (Transcript 5(b) and (c)). Examining Nath's answer, Philip apparently realised that he, too, should use *X* as another reference point to triangulate the position of *Z* (Transcript 5(c)).

Having determined *Z*'s correct position, students were then asked to inquire whether the feet in Figure 8 also configure a 90° angle, a question that some students were still considering. Recall that in Activity 1 students had devised a measurement tool to investigate TSP angles. This tool, whose rays may fall short in the new context of the hypothetical long-footed dancer, might be reintroduced now back onto GRiD as a handy means of reflecting together on relations between ray lengths and angle measures, ultimately supporting the conservation of angle amidst the ray-length parameter.

3. Summary

Auxiliary lines are useful diagrammatic means of solving geometry problems, because they invoke 'hidden' structural properties of the problem space conducive to the solution of construction or proof problems (Palatnik & Dreyfus, 2019; Polya, 1957). Responding to students' documented difficulty with building contextually useful auxiliary lines (Gridos et al., 2022; Wang et al., 2018), we created GRiD, a novel, low-cost diagrammatic floor mat designed to situate geometry studies in traditional ethnic movement practices.

Our evaluation studies of GRiD are leading us to believe that it can serve as a useful support for students to experience the purposeful construction of auxiliary lines. The first activity encouraged students to generate auxiliary lines to measure a right angle. The second activity prompted students to build auxiliary lines for recognising objects and arguing angle conservation. The third activity led students to employ auxiliary lines as a construction tool. In general, as we demonstrated through three exemplary activities, GRiD creates structured opportunities for students to experience, reflect on, represent, and discuss their imaginary perceptual construction of geometric structures, thus learning through invoking and socially reconciling perspectival tensions. As they enact dance movements and solve choreographic problems on GRiD, students materialise their tacit movement-oriented perceptual structures (attentional anchors) in the form of auxiliary lines that they gesture, name, trace, mark, and construct, thus meaningfully and collaboratively engaging in situated geometric argumentation. Throughout the activity, the negotiation of perspectival conflict is mediated by embodied reasoning, where students engage spontaneously in multimodal collaborative argumentation, using their body, hand gestures, verbal language, drawings, and tinkering with available resources.

Formal geometrical reasoning can be rooted in traditional dancing practice through solving movement activities that gradually shift students towards mathematical discursive registers and, vice versa, use mathematical reasoning to improve dance practice. To optimise classroom results, we recommend that teachers encourage and endorse students' multimodal informal communications, and not only formal mathematical utterances, by immersing students at once in the equally experiential worlds of both dance and mathematics.[4]

Notes

1. This adverb is apt: an etymology of the word will lead us to a Renaissance romance about Serendip, originally Sri Lanka, and yet our own tale will soon arrive at another island in South-East Asia.
2. A future publication coming out of the first author's dissertation research project will detail how a group of students developed this proof, assisted by their teacher. Here we mention this line of reasoning anecdotally only to illustrate GRiD's versatile mediation of auxiliary lines for geometric argumentation. As they facilitate this activity, teachers may wish to walk around the mat so as to view it from the unique perspective of each student, so that they can scaffold each student's unique reasoning.
3. During their attempt to solve the dilemma of 'squaring the rhombus' (see Transcript 3), each student had placed two straws on the mat to mark the position of their TSP. Nath took the action further by creatively connecting his two straws at their vertex using a connector, thus creating a portable straw construction. Later, all the other students imitated Nath's straw construction by using a provided connector.
4. Please refer to Appendices for additional recommended activities and facilitation ideas.

Acknowledgements

The authors wish to thank Alik Palatnik for his constructive comments on an earlier draft. The first author also expresses her deep gratitude to the Indonesia Endowment Fund for Education (LPDP) as the scholarship donor of her Ph.D.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendices

Appendix 1. Activity 1: Ninety-degree angle

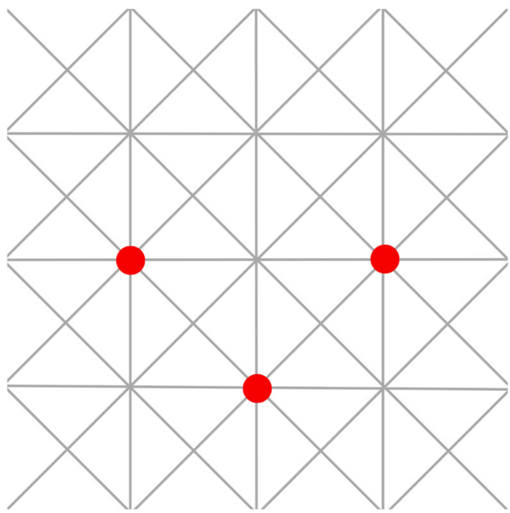
A.1 Description

The activity is rooted in the primary feet's configurations in the Balinese traditional dance called *Tapak Sirang Pada* (TSP): heels meet, toes open to the corner, creating a ninety-degree angle. The task is for students to enact the position, evaluate if their feet are opened at a ninety-degree angle, and explain their reasoning.

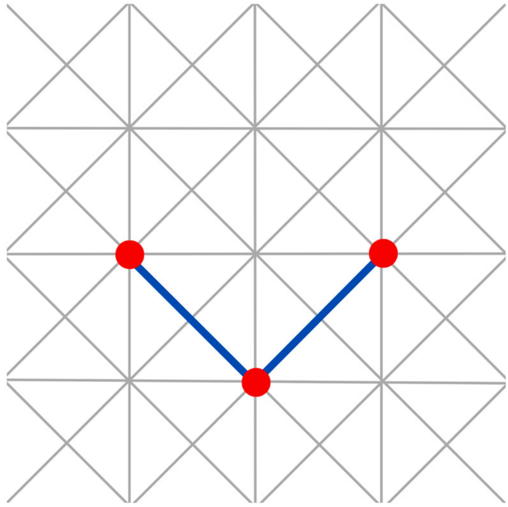
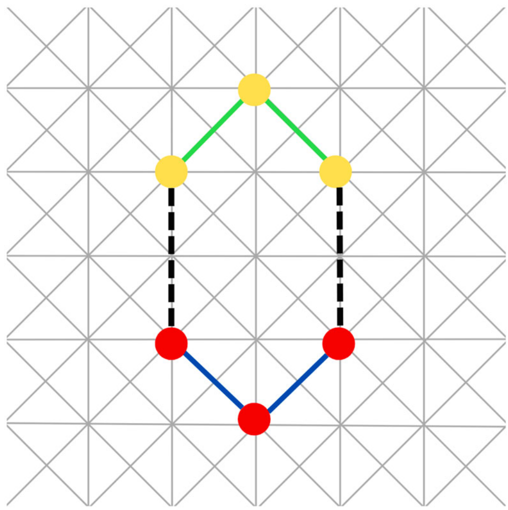
A.2 Students' organisation

- Individual
- Individuals in pairs or groups of four

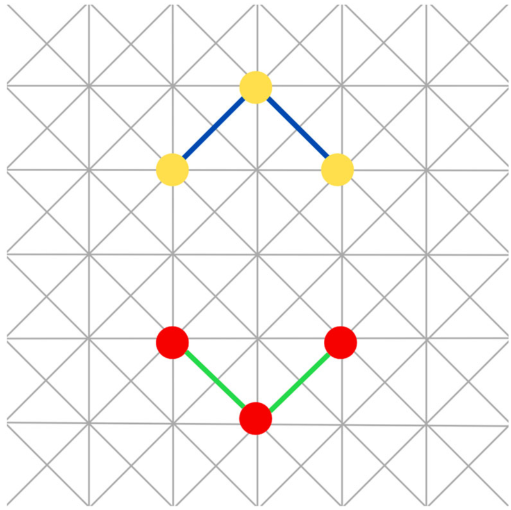
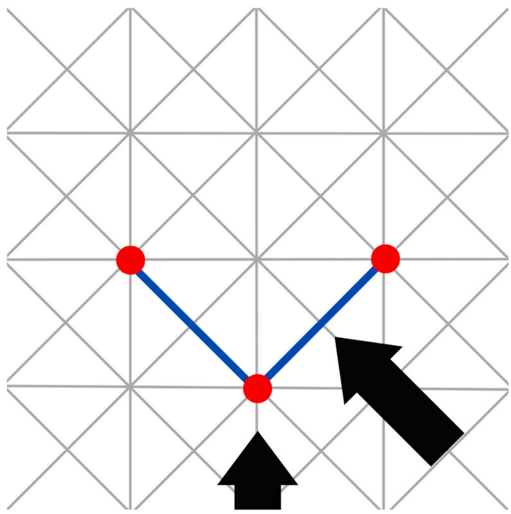
A.3 Learning trajectories

No.	Instruction	Possible solution
1.	On GRID, ask students to perform a 90-degree angle feet position. Ask them if they are confident in their foot configuration and why or why not.	Heels together, and toes open to the corner.
2.	Ask students to mark their feet position, preferably with dot stickers. If the sticker placement is not exactly on the GRiD points, ask students to check they follow the TSP rules correctly: heels meet, toes open to the corner.	Students place three stickers: one for the point where their heels meet and one for each of the ends of their toes. 

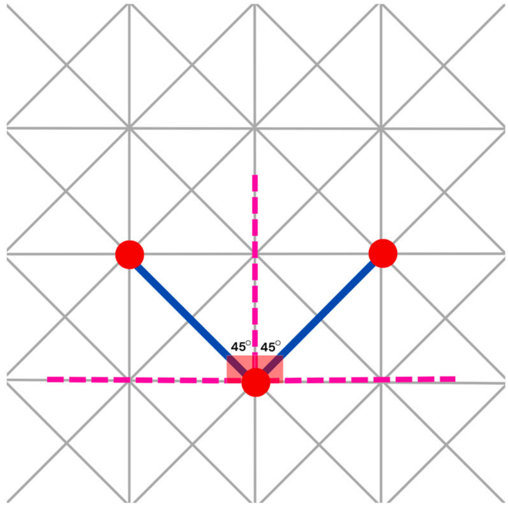
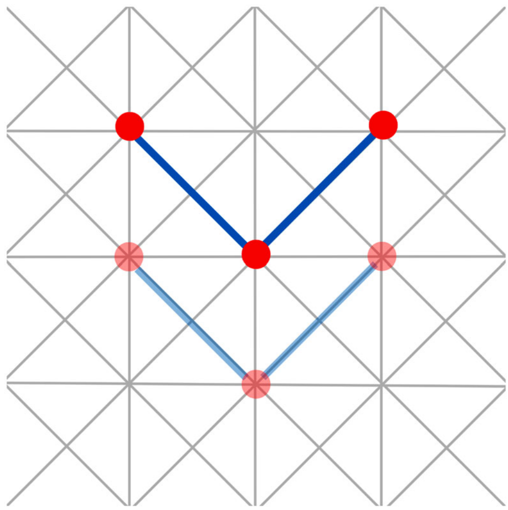
(*continued*).

No.	Instruction	Possible solution
3.	Ask students to draw their feet's lines with markers.	Students draw lines from the sticker that represents their heels to each of the other stickers that represent the end of their toes. 
4.	If they are in pairs or groups, students evaluate each other: Do they open their feet similarly? Why or why not? How do we ensure everyone opens their feet similarly?	They observe each other from their position. 

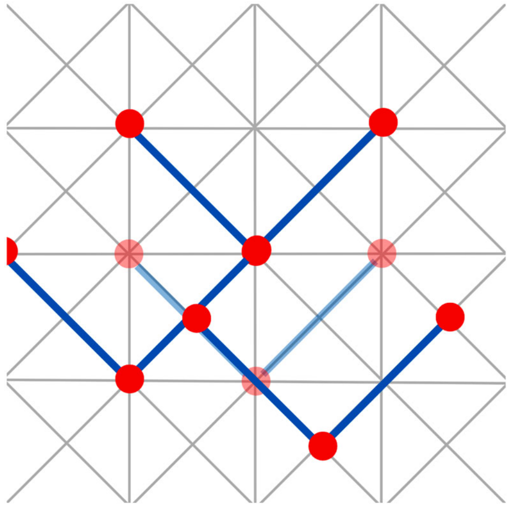
(*continued*).

No.	Instruction	Possible solution
		They observe each other by moving around and using their feet to check the stickers' placement of their friends. 
5.	Ask students why they think they do the foot configurations correctly at a 90-degree angle.	Students will adjust their heads or bodies to perceive the L-shaped ninety-degree. 

(*continued*).

No.	Instruction	Possible solution
		Students will create auxiliary lines to help them reason about a ninety-degree angle. 
6.	Ask students to take one step forward and perform TSP in a new location. How would they justify if their posture is correct?	

(*continued*).

No.	Instruction	Possible solution
7.	Repeat step 6 in various directions: backward, to the right, and the left side. The number of steps can be adjusted based on the availability of space.	

Appendix 2. Activity 2: square is a special case of Rhombus

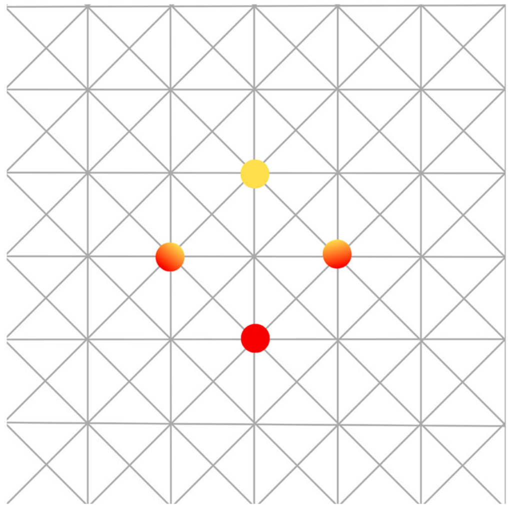
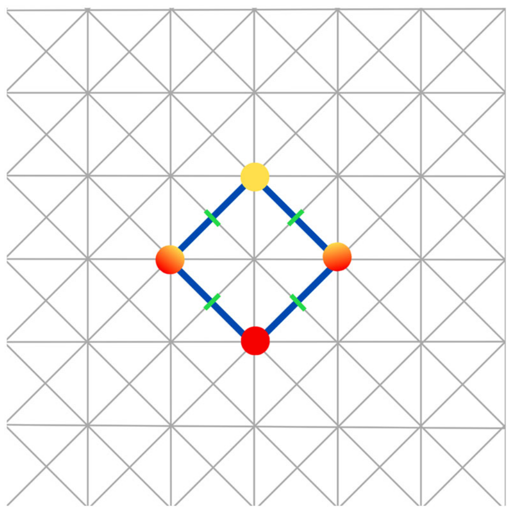
A.4 Description

The activity elaborates on TSP feet's configurations. The task is for students to create TSP in pairs on GRiD, face-to-face, and let the ends of their toes meet. They will need to reason about the shape of their feet's configuration. The main challenge of this activity also relies on the perspectival conflict of seeing 'rhombus' or 'square', which leads them to reason mathematically about characteristics of geometrical objects, i.e. a square is a rhombus with equal angles.

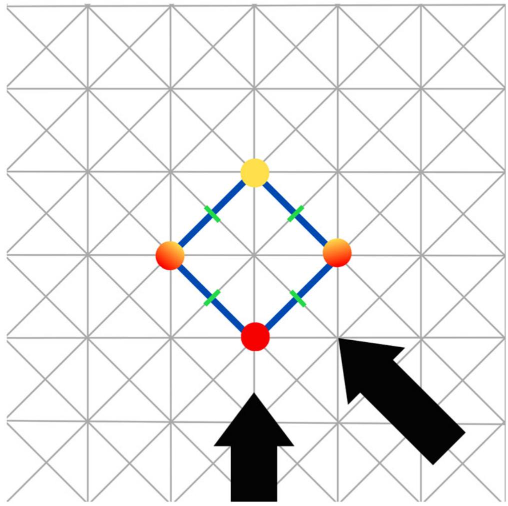
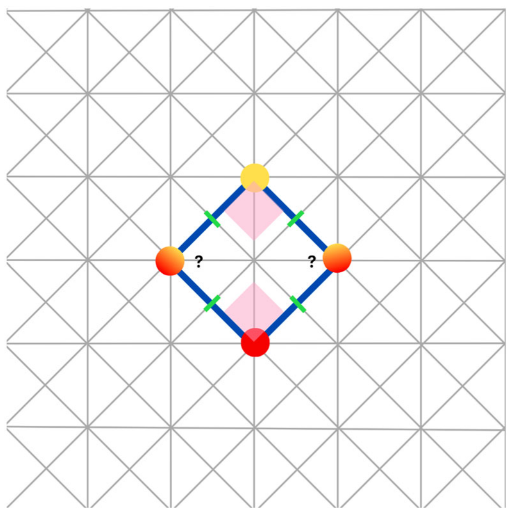
A.5 Students' organisation

- Individual
- Pairs
- Pairs in groups of four

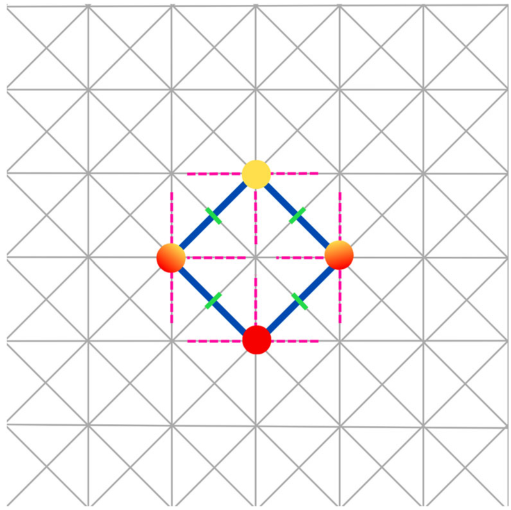
A.6 Learning trajectories

No.	Instruction	Possible solution
1.	On GRID, ask two students whose feet sizes are relatively equal to perform a 90-degree angle foot position face-to-face, with the ends of their toes met. Ask students to put stickers. Ask students to say the name of the shape they constructed together.	Students use dot stickers to indicate their positions. 
2.	Ask students to draw the feet' lines. If their soles are not equal in length, we can ask them to assume the lengths are the same and place the stickers accordingly. The equal sides are clear from the assumption given in the problem. What about the angles?	Students draw diagonal lines using markers or make them with construction straws or yarns or elastic bands to represent their feet. 

(*continued*).

No.	Instruction	Possible solution
		Since it has all equal sides, students would probably call the shape a rhombus or diamond if they look at it from the centre or align with their orientation when they are standing. Some probably say it is a square if they look at it from the corner. 
3.	Ask them again to convince you about the shape that they think they created together. Can the same shape change just because you see it from a different angle? If not, what is the *real* shape they are constructing together?	Some students might try to justify the other angles of the constructed shape. 

(*continued*).

No.	Instruction	Possible solution
		One possible method is to use similar reasoning to what they did to work with the problem in Activity 1.  After realising that the shape has equal length and equal angles, students can conclude that it is a square.

Appendix 3. Activity 3: angle and its arrays

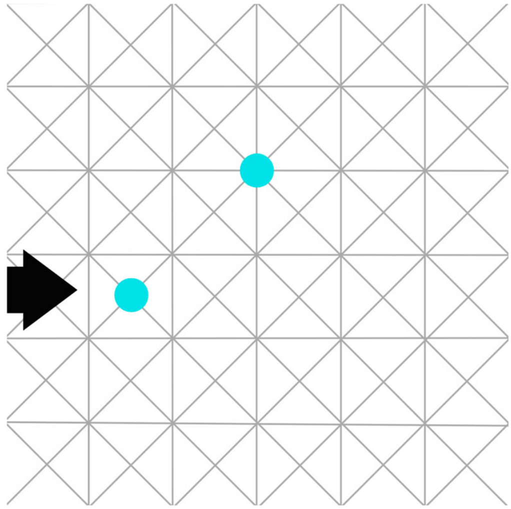
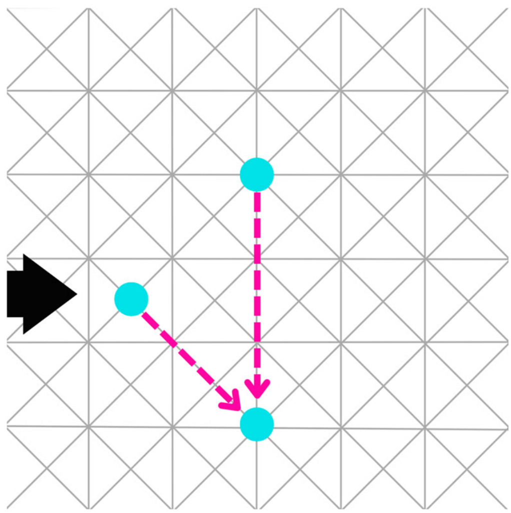
A.7 Description

This activity can take place before or after Activity 2. Still elaborating on TSP feet' configurations, the task is for students to predict the location of the end of one of the toes, given the position of the meeting point of the heels and the end of another toe. Students will work in a hypothetical situation in which someone whose feet are longer than theirs also wants to learn the TSP position.

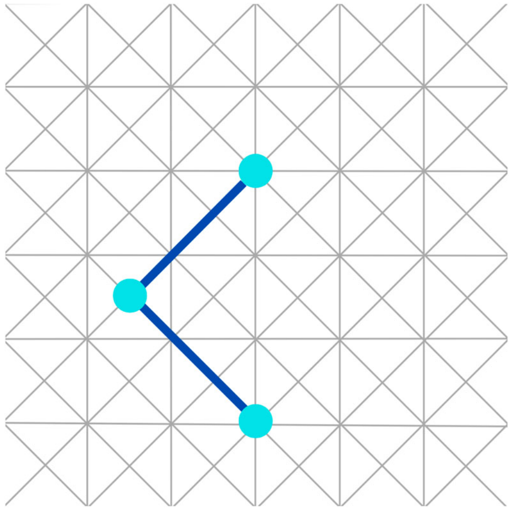
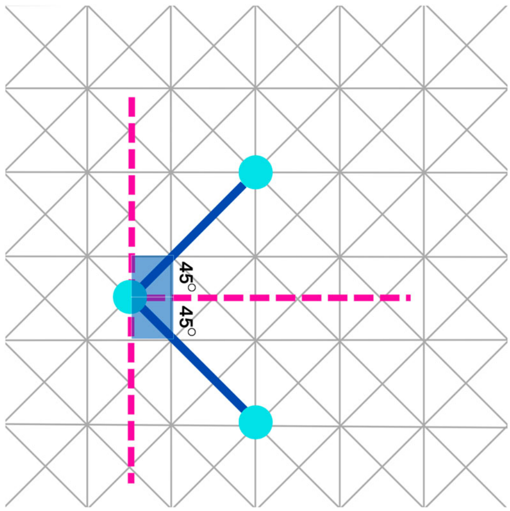
A.8 Students' organisation

- Individual
- Pairs
- Pairs in groups of four

A.9 Learning trajectories

No.	Instruction	Possible solution
1.	Place a new sticker on the GRID. Tell a story about someone with longer feet who wants to perform the TSP position. Place another sticker and tell them that this is where the other person's left toe ends. Let them know in which direction this hypothetical person is performing TSP.	
2.	Ask students to place a sticker to indicate where that person's right toe ends.	

(*continued*).

No.	Instruction	Possible solution
3.	Ask students to draw the lines where that person's feet would be.	
4.	Ask students to explain if they think this hypothetical person also opens their feet like them, or in other words, at ninety degrees.	
5.	Ask students to find the position of this person's partner if they have the same length of feet and want to create a square with their TSP.