I see what you’re thinking: Embodied collaborative argumentation

Alik Palatnik[1], Dor Abrahamson[2, 3] and Catalina Lomos[4]

[1]The Hebrew University of Jerusalem, Israel; alik.palatnik@mail.huji.ac.il

[2]University of California Berkeley, USA, [3]Korea University, Korea

[4]Luxembourg Institute of Socio-Economic Research, Luxembourg

A group of 5 pre-service teachers engaged in constructing and exploring a body-scale icosahedron model. Triangulating audio–video and mobile eye-tracking data, we analyzed the roles of motor actions and sensory perceptions in the emergence and propagation of geometrical insight. We present an exemplary episode where Gail multimodally expresses a valid argument for finding the icosahedron’s half-height. Another participant’s eye-tracking data revealed that his perception of Gail’s manual actions helped him interpret and then replicate non-verbal features of her argument. The study contributes to our understanding of embodied cognitive processes in authentic educational contexts, particularly the role of perception–action engagement with concrete media in collaborative mathematics tasks. Teacher awareness of these perceptuomotor micro-processes might enhance the pedagogical efficacy of integrating embodied activities into classroom geometry instruction.

Keywords: Embodied argumentation, collaborative learning, preservice teachers, spatial geometry

Introduction

There has been growing interest in 4E (embodied, enactive, embedded, and extended) approaches to mathematical learning, particularly in geometry (e.g., Abrahamson et al., 2020). These approaches, unlike traditional ones, emphasize physical interaction and sensory experiences in developing mathematical understanding. Geometry—a study of shape and space, where spatial visualization and reasoning are vital components of learning—is particularly amenable to the pedagogical implementation of these approaches. Through manipulating physical models, moving around and through the models, measuring them, and using various gestures, learners engage in embodied argumentation, a collaborative process where action-based experiences are integral for reasoning and proving (e.g., Palatnik & Abrahamson, 2022; Walkington et al., 2018).

This paper introduces the concept of *embodied argumentation*—a form of mathematical argumentation in which learners use physical actions, such as interactions with objects, as well as gestures, to construct, support, and communicate mathematical proofs and explanations. The empirical context of the investigation is a geometric task involving the collaborative construction of a human-sized Platonic solid followed by exploring its global features, such as determining its total number of vertices. Drawing on insights from gestural reasoning, collaborative problem-solving, and Proofs Without Words (PWW, Marco et al., 2022), this study examines how pre-service teachers use their bodies, physical interactions with the model, and gestures to develop and communicate geometric arguments. The goal of this research is to contribute to the field’s understanding of how embodied cognition manifests in collaborative mathematical argumentation and to investigate the educational potential of integrating embodied activities into geometry instruction.

Theoretical Background

The late David Tall suggested that mathematical reasoning evolves through three worlds: the embodied, the symbolic, and the formal–axiomatic world. The embodied world, where learners use physical interactions and sensory experiences to grasp mathematical concepts, provides the foundation for math before transitioning to symbolic and formal reasoning (Tall, 2003). One of the examples presented by Tall (2003) shows a Proof Without Words (PWW) on the sum of the *n* first integers as the manipulation of pebbles (embodied), then proof by arranging a sum of two lines of numbers, one of them in reverse order (symbolic), and finally proof by induction (axiomatic). Tall noted that the first two proofs have "clear human meaning, the first translating naturally into the second. The induction proof, on the other hand, often proves opaque to students, underlining the gap that occurs between the first two worlds and the formal world" (p. 8).

Embodied cognition posits that human thinking is deeply rooted in physical interactions with the environment (e.g., Barsalou, 2008). In mathematics education, this framework has supported claims that learners engage with concepts through gestures, movement, and manipulating physical objects: moving in a new way by perceiving in a new way and then naming in a new way (Palatnik et al., 2023). This embodied view is especially valid in geometry, where physical interaction with geometric figures, such as objects and diagrams, is inseparable from understanding (Herbst et al., 2017).

An embodied approach to geometry instruction aligns with research suggesting that embodied reasoning can support the development of geometric understanding. Nathan et al. (2021) emphasize that intuition and insight in geometry often arise through embodied experiences, such as gestures, that allow learners to transition from informal exploration to more formal proof. Their study provides evidence that dynamic depictive gestures, which involve motion-based transformations of mathematical objects (e.g., dilation or rotation), significantly support the production of valid proofs, whereas static gestures are less effective in fostering mathematical generalizations necessary for proofs. Other studies investigating learners interacting with physical models—such as 3D geometric shapes—show they gain insights through bodily movements (Benally et al., 2022). These actions form the basis of geometric argumentation and reasoning, as learners use their bodies to represent, explore, and communicate geometric properties (Palatnik & Marco, 2024).

Embodiment and dynamism are constitutive to geometric reasoning, whether observable by others or only covertly experienced by individuals through multimodal processes, like mental transformations, or in the form of imaginary auxiliary constructions. We speculate an experiential spectrum between covert and overt multimodal mental activity. We further assume that covert and overt aspects of mathematical reasoning afford complementary contributions: at times, it is helpful to imaginatively manipulate a geometric structure, unbound by available media, whereas, at other times, it could be vital to materially realize these mental figments in the form of publicly inspectable models. A case in point is PWW-based activities, where learners engage in *gap-filling*, which is the process of identifying missing components in visual representation and constructing formal proof (Marco et al., 2022). In the absence of formal symbolic displays, PWW artifacts provoke mathematical thinking and support legitimate forms of argumentation (Nelsen, 1993); they help learners bridge embodied experiences and formal argumentation. Indeed, Marco and Shvarts (2024) implicated several key sensory-motor processes involved in gap-filing. As such, engaging students in interpreting PWW diagrams fosters opportunities to experience authentic mathematical reasoning, which, as Lakatos (1976) insisted, is the actual lived phenomenological underbelly of the eventual written proofs.

Still, nurturing students' capacity to understand PWW—what they are as epistemic forms and what epistemic games we are to play with them (Collins & Ferguson, 1993)—may take holistic entrainment. From a didactical perspective, one might wish to scaffold mathematics students' epistemic practices, cognitive routines, and interpersonal argumentation by way of occasioning for them opportunities first to engage in the *overtly* manifest enactment of geometric reasoning prior to introducing problems more so conducive to *covert* reasoning. One powerful cultural tradition incorporating overt geometric actions is construction activities, for example, the historical pedagogical methodology established by Friedrich Fröbel (Brosterman, 1997). Whereas Fröbel designed his geometry construction tasks for kindergarten students, our philosophical and theoretical commitment to embodiment perspectives, as well as our empirical research, have led us to believe that high-school students, too, should build knowledge through building objects (Palatnik & Abrahamson, 2022). Namely, implicit to the hands-on pedagogical regimens characteristic of our studies into geometry education is an enactivist argument that humans are life-long sensorimotor learners—we never actually graduate the sensorimotor stage Piaget theorized as ending in early childhood (Abrahamson, 2022). Yet these polemic epistemological debates would avail from empirical data. As such, as we now argue, introducing material stuff into geometry pedagogy may shift the focus of *research* on geometry pedagogy, drawing attention to students' authentic praxis and multimodal phenomenology of manipulating concrete objects, possibly revealing the critical cognitive roles of perception and action that embodied mathematics-education researchers allege.

Recent research on embodied learning in proving and argumentation has largely focused on gestures' role in mathematical reasoning. Walkington et al. (2019) show how collaborative gestures support group reasoning, while Nathan et al. (2021) highlight that dynamic gestures may enhance proof production by simulating geometric transformations. While these studies provide valuable insights into the body's role in mathematical reasoning, they focus almost exclusively on gestures as the manifestation of embodied learning. A notable gap remains in understanding other forms of embodied cognition, such as physical interaction with manipulatives, sensorimotor experiences, or spatial reasoning through body movement, and how these contribute to mathematical proving and argumentation. Furthermore, research in authentic classroom settings, where students and teachers collaboratively develop embodied argumentation, is still missing. Further research is needed to investigate these broader manifestations of embodied learning in various educational contexts.

The research questions guiding our study were: What are the roles of motor action and sensory perception in mathematical argumentation? What is the educational potential of fostering opportunities for embodied argumentation in geometry instruction?

Illustrative case study of embodied argumentation

Our empirical context is a spatial geometry activity in which five pre-service mathematics teachers participated voluntarily: Aya, Gail, Ruth, Tim, and Yoni (pseudonyms). They were tasked to construct and analyze a body-scale Platonic solid (Figure 1). Whereas we had used this activity extensively in researching geometry education (Benally et al., 2022; Palatnik & Abrahamson, 2022, under review; Rosenbaum et al., 2024), we had yet to examine experiences related to argumentation.

The session was audio-video recorded. During the activity, three participants wore mobile eye-tracking (MET) devices (Pupil Lab Neon), generating Multimodal Learning Analytics (MMLA) data.

Our data further comprised a stimulated-recall group interview, in which participants discussed their experience of collaboratively constructing the model. We prepared the activity transcript, juxtaposing the three MET sources and adding photos to capture movements, actions, and gestures. We transcribed the SRI interview. We coded data by marking key moments in the construction and exploration processes to capture elements of cooperative action and argumentation. Utterances were transcribed and coded, marking aspects of perception and action. We later juxtaposed key video moments in the construction process with relevant testimonials from the stimulated-recall interview.

We focus our analysis on the episode that begins at the 16th minute of the activity when the participants had already successfully constructed the icosahedron model and answered questions about its number of edges, vertices, and faces. The next question, however, presented a different challenge: “Assume the polyhedron was placed on one of its faces and filled with water. What would the shape of the water’s surface be?”

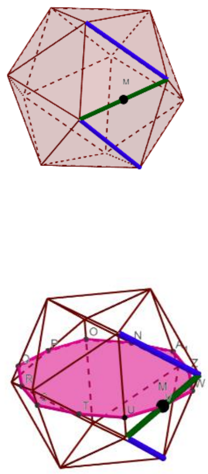
 a	 b	 c	 d	 e
Because *this* is here, [slides her right hand down along the edge, starting from the top base (top blue segment on 1e)]	*that* is there. Symmetry. [slides her left hand up along the edge from the bottom base (bottom blue segment on 1e), establishing symmetry]	And then we have… [puts both hands on the ends of the center edge, where her previous movements had ended; slides her hands simultaneously to the middle of the edge]	*this* [The meeting point (M on 1e) is the middle of the edge and precisely at half the model's height when it stands on the face.	
Note: (a)-(d) blue arrows indicate the movement of Gail’s hands; a red circle is an eye-tracking marker of Tim’s gaze; (e) a scheme of Gail’s argumentation and the form of the cross-section—dodecagon				

Figure 1: Gail’s embodied argumentation

The participants perceived this problem as non-trivial. Aya remarked, “Oh, interesting, because it’s not... no, it’s not symmetric. Part of (the edges) will be cut in the middle, and part will be cut…” Realizing they needed to establish the height of the cross-section, the participants quickly began approximating the position of the model’s half-height above the ground. Yoni intuitively placed his hand on an edge close to the correct height but expressed uncertainty. Other participants offered suggestions on where the half-height might be located and, based on their approximations, attempted to count the vertices of the imagined cross-section. They debated between a pentagon and a hexagon (both suggestions are incorrect) but were unsure of the correctness of the answer.

The group then split into two sub-groups, which, due to the large physical size of the model, became separated: Aya and Yoni on one side and Ruth, Gail, and Tim on the other. While partially aware of each other's actions, they worked somewhat independently. Aya first suggested that the cross-section at half-height would include one of the vertices. Aya and Yoni then agreed that two vertices on the same edge were not the same height from the ground. Their argumentation was supported by the observation of the model while touching the vertices, sliding the hands along the edges, simulating and comparing each other actions

Meanwhile, Gail engaged in searching for the half-height in a different way. First, she approximated the half-height of the icosahedron by comparing it to the distance between the palms of her hands put vertically. She took several such measurements with her hands. Then, she stepped back to change her perspective to be able to capture the whole model from a distance. She observed the model from different angles. At 16:08, when the instructor asked, "Where is it, exactly, half of a height?" Tim initially supported the idea that it was located at one of the vertices. However, Gail confidently stepped toward the model and indicated (correctly) horizontally with the palm of her hand: "No. This is half of a height." When the instructor asked, "Why is this half of a height?" Gail produced a concise argument through a series of three coordinated actions (Figure 1).

Tim's eye-tracking data revealed that he followed Gail's gestures closely. His gaze tracked each movement (marked by a red circle in Figure 1). Tim fixed his gaze and pointed at the midpoint of the edge before Gail completed her third movement. Based on this MMLA data, we conclude that he understood the argument before she finished her demonstration. He exclaimed, "Wow, cool!" recognizing the solution.

<table>
<tr><td>
a</td><td>
b</td><td>
c</td><td>
d</td></tr>
<tr><td>"Yes. Because this [slides a right hand down]</td><td>and this are the same height.[slides a right hand up]</td><td>And here, [slides both hands simultaneously toward the edge's middle]</td><td>this is its middle.</td></tr>
<tr><td colspan="4">Note:(a)-(d) blue arrows indicate the movement of Tim's hands; a red circle is an eye-tracking marker of Aya's gaze; Gail continues to hold her hands, indicating the half-height point</td></tr>
</table>

Figure 2: Tim's embodied argumentation

Note that the eye-tracking and audio data of Aya and Yoni revealed that they were not yet aware of Gail's embodied proof. Having endorsed Gail's argument, the instructor directed the other subgroup's attention: "Guys, did you hear what Gail said?" This prompted Aya, Yoni, and Ruth to look toward the first group, and Tim quickly took over the explanation (see Figure 2). While Tim's actions enact the same chain of reasoning as Gail's, his actions are not precisely the same—he captured the essence of argumentation while varying the movements' motorics.

Further, the participants established that the cross-section had 12 vertices: six midpoints of the edges, as Gail demonstrated, and six additional points (see Figure 1e), which they found using a spare wooden edge as a line of intersection of the cross-section plane with the icosahedron's edges.

Discussion

The illustrative case study shows that embodied argumentation is not merely an aid but an integral part of how pre-service teachers construct and communicate geometric proofs (cf., Alač & Hutchins, 2004, on scientific reasoning). The analysis of the icosahedron task reveals how physical actions and perceptual processes constitute the mathematical argumentation of pre-service teachers.

Firstly, Gail's embodied argumentation exemplifies how a sequence of physical actions can serve as a form of mathematical proof. By sliding her hands along the edges and establishing symmetry, she transformed an abstract geometric property into a concrete perceptual experience for her and the group. This aligns with and expands Schenck et al. (2020), who argue that gestures externally represent geometric relationships, making abstract concepts accessible.

The activity in which a group of peers first constructed and then explored the model prepared Gail and other participants to establish a more intimate connection to the mathematical object to externalize their reasoning, supporting both her insight and other arguments of the group (see Nathan et al., 2021 on intuition, insight, and proof). When she prepared her argumentation, Gail used her hands to approximate distances and moved toward and from the model, thus changing perspective and angles of view (literally zooming in and out of the model). Gail's embodied argumentation emerged gradually through simulation, changes of gaze, and repositioning of the body in relation to the model (see also Palatnik & Marco, 2024).

Secondly, Tim's eye-tracking data reveals how constructing mathematical arguments involves a succession of perception-action loops (Shvarts et al., 2021). Tim's ability to follow Gail's gestures and anticipate the identification of the midpoint of the edge demonstrates how perception is tightly coupled with physical action in embodied argumentation. As Walkington et al. (2018) suggest, gestures can act as collaborative tools, allowing learners to share their reasoning. In this case, Tim's tracking of Gail's physical movements enabled him to quickly understand and replicate her argument for the rest of the group.

Moreover, the internalization, reproduction, and endorsement of Gail's embodied proof by her peer—a future teacher, suggests he finds it valid. Given that the participants are prospective teachers, this acceptance provides a valuable opportunity to influence classroom culture by broadening the scope of argumentation to include movement, manipulation, and gestures involving physical models. In the milieu of the embodied collaborative construction activity, the group accepted the argument as valid.

In contemporary classroom culture, mathematical proof is typically limited to formal, symbolic logic (Herbst & Brach, 2006). However, embodied argumentation—which includes physical actions, manipulation of artifacts, and gestures—offers a compelling expansion of what constitutes valid reasoning. Rooted in embodied cognition, which posits that human thought is deeply interconnected with physical interactions (Barsalou, 2008), this form of reasoning is particularly relevant in classroom discourse, where participants can perceive each other actions on shared artifacts. Manipulating objects or engaging in physical interactions, such as rotating models, constructing auxiliary elements, or positioning geometric shapes, often reveals mathematical relationships that might be difficult to express verbally (Palatnik et al., 2023). These actions, along with gestures such as pointing to a diagram or highlighting spatial connections, are not supplementary but integral to mathematical argumentation. Expanding the definition of proof to include embodied argumentation aligns with the interactive nature of classroom learning, where norms are shaped by teachers and students collaboratively constructing knowledge (Stylianides et al., 2022).

Expanding the definition of proof to include embodied argumentation reflects the multimodal ways students interact with mathematical notions, promoting a more inclusive understanding of proof. By fostering a classroom culture that embraces embodied argumentation, educators can better support students' reasoning processes, particularly in spatial and geometric contexts, where physical interaction with objects significantly enhances comprehension. This shift also echoes practices in professional communities, such as in research settings, where manipulating physical models and performing gestures play a role in the collaborative negotiation of meaning and problem-solving (e.g., Roth, 2001). Embodied argumentation may be similarly integrated into sedentary classroom practices, including PWW-based activities (Marco et al., 2022), where reasoning based on visual information is a stepping stone for the gap-filling process, leading to more formal proving methods.

In conclusion, recognizing embodied argumentation as a legitimate form of proof in classroom discourse expands the cultural norms of mathematics education, making it more reflective and more inclusive of how humans do mathematics, naturally integrating its embodied and formal aspects.

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