EMBODIED LEARNING OF ALGEBRA FOR BRAILLE READERS

A. Van Leendert[1, 2, 3], M. Patel[4], S. G. Lambert[5], D. Abrahamson[4], J. L. Gorlewicz[5], A. Shvarts[2]
[1]Royal Visio, [2]Utrecht University, [3]Erasmus MC, the Netherlands;
[4]University of California, Berkeley, [5]Saint Louis University, US

The rise of digital educational technologies and the emergence of embodied theories have transformed the teaching and learning of mathematics. However, this often neglects the sense of touch, putting Braille readers at a disadvantage. This paper introduces the initial iteration of designing TacTiles—a tactile interaction system for learning algebra. In the design, we combine two models used for teaching algebra, namely algebra tiles, and balance and adapt the affordances of our interaction system based on tryouts with Braille readers. The system allows for modeling expressions, factoring, and solving equations. Students found grasping equations while balancing tiles intuitive and insightful. However, factoring was challenging, as users tended to arrange the tiles randomly. In addition, the system requires an interface with Braille displays and sound to store the steps in solving equations. This study contributes to the advancement of innovative technologies and our understanding of Braille readers' algebra comprehension and beyond.

INTRODUCTION

Over the past decade, the extensive development of digital educational technologies and theories of embodied learning have given rise to new ways of teaching and learning: designers are building interactive spaces that support new forms of action by providing specific affordances and continuous feedback—*instrumented fields of promoted actions* [1]. Those technological environments make discovering mathematics to be a natural continuation of human sensorimotor skills. Despite these advances, digital learning often focuses on sight and sound and neglects the use of touch, thus excluding Braille readers. To address this gap, our research focuses on developing interactive technologies that allow individuals to incorporate both tactile and auditory sensations into their learning experience, while receiving digital feedback on their interactions. Research on a tangible device for geometry learning, called the Tangible Manipulatives for Quadrilaterals (TMQ), illustrates the advantages of the TMQ over traditional, static media and its ability to serve as an inclusive geometry-learning tool [2]. In the current study, we turn to algebra learning.

Traditional mathematical notation provides sighted learners structural cues, with symbols and numbers arranged at different heights above, on, and below the baseline [3]. These symbols, numbers, and operators have distinct visual features that are easily perceived through spatial combinations of straight lines and curves [4, 5]. The special features of this representation, in combination with the simultaneous character recognition by sight [6, 7], enable students with typical vision to grasp an equation overview in no time [8]. To illustrate this, consider the following equation:

$$\frac{(x+2)^2}{\sqrt{3}} = \frac{3(x+2)}{\sqrt{3}}$$

Students with typical vision quickly recognize that this is an equation involving fractions. They immediately notice that both fractions share the same denominator and recognize the left-hand expression as quadratic and the right-hand expression as linear. This information guides them directly in solving the problem.

Traditionally, Braille readers use Braille when processing algebraic expressions and equations- a practice that differs from sighted learners. Braille is a linear notation, in which all characters are arranged at the same height (Table 1). The characters differ by the presence or absence of a raised dot in different places. This lack of redundancy can lead to decoding errors that are difficult for Braille readers to detect, given the condensed and low-context nature of algebraic expressions. To gather information, Braille readers must move their fingertips sequentially over Braille characters [9], one Braille character at a time [7]. Their window of perception is limited, which prevents an immediate overview of an expression or equation and requires a gradual build-up.

Looking at the preceding equation from the perspective of Braille readers (as shown in the Dutch mathematical Braille notation in Table 1), they read the equation linearly from left to right, without initially grasping global and local features. Only after mental processing, can they recognize that the two fractions form the central operations sharing the denominator of the square root of three.

The linear nature of Braille, the low redundancy of Braille characters, and the limited window of perception make it difficult to comprehend expressions and equations [10]. In addition, this particular way of processing information poses a challenge for facilitating collaborative and mathematical communication for Braille readers—an essential aspect for the development of their overall mathematical skills [11].

mathematical notation	Dutch mathematical Braille notation
$\frac{(x+2)^2}{\sqrt{3}} = \frac{3(x+2)}{\sqrt{3}}$	⠦⠦⠭ ⢖ ⠣⠴⡬⠣⠌⠎⠟⠗⠞⠦⠩⠴ ⢶ ⠦⠩⠦⠭ ⢖ ⠣⠴⠴⠌⠎⠟⠗⠞⠦⠩⠴ ((x + 2)^2 / s q r t (3) = (3(x + 2))) / s q r t (3)

Table 1: An equation represented in mathematical and Dutch mathematical Braille notation

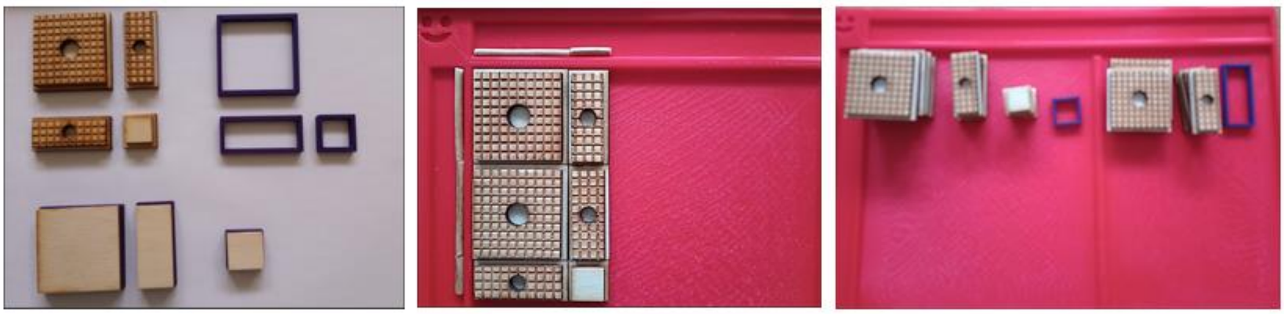
To facilitate the introduction of basic algebra concepts, we combine two traditional models for teaching algebra: algebra tiles for modelling factorization and simplification of expressions [12] and balance for modelling equations and their solutions [13]. The tiles are available in different rectangles, colors and materials with areas representing x^2, x and 1, among others, and allow users to model algebraic expressions. The balance model promotes adding or removing expressions on both sides of the equation, helping students to perform operations on both sides of an equation and maintain equality. Combining balance and tiles, students can solve linear and quadratic equations without having to resort algebraic symbols. The use of these manipulatives supports students in linking informal to formal approaches [14]. This helps students develop a conceptual understanding of algebra [15], and paves the way for later use of symbols in the learning process.

However, integrating algebra tiles and balance into math instruction for Braille readers presents new challenges due to the tactile exploration and manipulation required. Designers need to develop *fields of promoted actions* tailored to Braille readers. This includes attuning the shapes, sizes, and textures to gripping actions, as well as developing audio feedback and linking to accessible symbols. This study presents the initial iteration of a pure tactile interaction system for learning algebra, which we call 'TacTiles'. By designing innovative technologies, we can also better understand how different individuals can comprehend algebra.

METHOD

Our method, design research, follows an iterative approach in which technologies are developed based on theoretical premises and stakeholder needs and further improved through empirical tryouts and collaborative reflection [16]. In this paper, we present the initial phase of the design of a pure tactile technology, consisting of five micro-iterations to refine our design.

TacTiles include tiles of type x^2, x, and 1 with dimensions x by x, x by 1, and 1 by 1, respectively (Figure 1). The following affordances distinguish our design from available commercial products: (1) The TacTiles afford tactile differentiation of constants and variables through a smooth texture on the top of the 1-tile vs. a gridded texture on the top of the x-tile and x^2-tile. (2) The TacTiles each consist of two wooden pieces that are laser-cut and bonded, with the top piece of each tile being smaller than the bottom. This construction eases manipulation and tactile differentiation when counting adjacent pieces. The x-tile dimensions at the bottom are 4.5 by 2 cm, so that 'x' cannot be counted in units of '1'. (3) We introduce 'negative' TacTiles designed to afford operations with negative coefficients. The plastic frames mirror their counterparts but are 'empty' (Figure 1a). If a positive TacTile is placed on top of its additive counterpart, a zero pair is created. (4) 1D-sticks of length one or x have been incorporated to mark the dimensions of the 2D-rectangles (Figure 1b). (5) Finally, a 3D-printed A4-sized, plastic board affords arranging the TacTiles systematically (Figure 1b and 1c). In the center of the board is a raised bar that embodies the equal sign and thus implements the balance model. On the left and upper edges are two raised bars and a slot for 1D sticks, which support the arrangement of the tiles in the corner and the marking of the dimensions, thus factorizing expressions depicted by TacTiles. The size of the board and TacTiles, as well as the possibility of stacking them (Figure 1c), make it easier to touch equations and expressions with both hands at the same time and to quickly create a comprehensive overview. Figure 1b shows a rectangle with dimensions $x + 1$ and $x + 2$, consisting of two x^2 -tiles, three x-tiles and one 1-tile. This represents the factorization of $2x^2 + 3x + 1$ into $(x + 1)(2x + 1)$. Note that only tiles with equal sides are placed next to each other.



a b c

Figure 1: a. Overview of the TacTiles. Top left is the x^2-tile, the x-tile and the 1-tile. The opposite counterparts are at the top right. Below are the zero pairs. b. Model for

$2x^2 + 3x + 1 = (x + 1)(2x + 1)$ The 1D sticks in the slots mark the dimensions of the rectangle.
c. Stacking of Tactiles

Empirical try-out procedure: The first author introduced the TacTiles and board to the students. In a 1.5-hour session, students were tasked with modelling and simplifying expressions, solving linear equations, and factoring linear and quadratic expressions. For each new task, the instructor cleared the board and put up the TacTiles, reading the instructions aloud and asking students to formulate their thought processes.

First, we familiarized the students with different types of tiles and emphasized that the length of x was unknown and could vary. The first task was to determine the perimeter and area of rectangles formed of TacTiles of different dimensions. We then moved on to 'modelling and simplifying expressions,' where more than ten tiles were placed on the board and students had to form stacks of similar tiles. Combining like terms involved creating and deleting zero pairs, with particular attention given to explaining the concept of zero pairs due to its novelty.

The next step focused on modelling and solving linear equations using the balance method. To prevent students from solving problems without using the tiles, numerous tiles were placed on the board. When factoring linear and quadratic expressions, students were asked to form rectangles from tiles and mark their dimensions. The difficulty of the task was tailored to the individual abilities of the student by adjusting the complexity of the expressions by adding or removing TacTiles from the board.

Participants: Two high school graduates who are Braille readers, pseudonymously called Lars and Chris, both right-handed, took part in the study. Lars was congenitally blind and excelled in mathematics, while Chris had a severe visual impairment from birth and demonstrated average mathematical performance. In high school, it was a challenge for Chris to engage in mathematics because he could not study like a sighted person - his residual vision was too low - and did not have the skills to study as a Braille reader. Neither student had prior experience with algebra tiles.

Data collection and analysis: The students hand movements were video-taped; their utterances were recorded and transcribed. The analysis focused on how TacTiles and the board support sensorimotor processes within the mathematical target actions. Each tryout was discussed with our multidisciplinary team to encourage redesign and theoretical reflection.

RESULTS

Both Braille readers found it easy to use TacTiles. They had no problems gripping and counting the tiles. They could easily establish *a new form of action* of removing tiles and as a result grasp equations as balancing tiles. For Chris, this tactile experience even created a feeling that he only now comprehends what equations are, as he reported ‘*The notation of balance now makes sense*’.

Chris noted that while using Braille, he struggled to physically manipulate terms, which made the concept of balance unclear to him. However, the new embodied experiences of working with TacTiles helped clarify the notion of balance. He also pointed out that the tactile nature of the tiles allowed for a more compact representation of equations compared to Braille. However, he also found that removing the TacTiles felt uncomfortable because, unlike Braille, the TacTiles do not store (memorize) the steps involved in solving an equation. Lars did not encounter any difficulties due to his strong competence in math; he also hypothesizes that zero pairs are particularly useful for novices.

Forming rectangles for factorizing appeared to be a difficult task for both students —the Braille readers tended to place tiles chaotically and did not systematically align tiles with equal sides, namely, x-side next to x-side and 1-side next to 1-side. Figure 2 shows an example of incorrect tile alignment. In addition, they consistently created rectangles at the bottom left corner of the board instead of the expected top right corner.

Figure 2: Example of incorrect tile alignment

CONCLUSION AND DISCUSSIONS

TacTiles have provided students with a tactile experience that enhances their comprehension of algebra. The combination of algebra tiles and balance models helped students simplify expressions with positive and negative coefficients and integrate this experience into solving equations by adding and removing tiles. Theoretically, the sensorimotor experience of symmetrically adding or removing tiles supports the mathematical conceptualization of equations and the act of sorting tiles and forming zero pairs supports the understanding of simplifying expressions. However, further improvements are needed to improve the ability to factorize and store steps. Students encountered difficulties arranging the tiles to form rectangles, and often placed them randomly and without a systematic alignment of tiles with equal sides. Providing the tiles with soft edges that allow for correct placement could help students rediscovering factoring rules. In addition, students consistently placed rectangles in the bottom left corner instead of the top right corner as expected. Therefore, a slot should be added to the bottom of the board to allow users to indicate the dimensions below the rectangle. We also used a gridded texture on top of the x^2-tile and the x-tile. After some thought, we assume that it might be tempting for many to use the grid to estimate the value of x, although 'our' students have not done so. Therefore, we want to replace this texture with a more chaotic one.

Unlike Braille, TacTiles does not store the steps taken when solving an equation, which caused some discomfort for one of the students, as he wanted to review his work. This result marks the next phase of our design work on establishing connections between TacTiles and devices that use Braille to store a history of interactions. We are exploring the potential of having the board to store and transmit information about the loaded tiles based on their conductivity. This study represents the first phase of our design research on TacTiles, which we tested with students who have already studied algebra. We expect that automatic feedback about balance will play a critical role in discovering actions for learning equations by novices. Interfacing the board and tiles with Braille displays and sound systems presents key challenges for the field of promoted actions that we are developing.

REFERENCES

[1] Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: Signifying sensorimotor enactment in fields of promoted action. *ZDM Mathematics Education*, 47(2), 295–306.

[2] Lambert, S. G. (2022). A tangible manipulative for inclusive quadrilateral learning. *Journal on Technology and Persons with Disabilities*, 10(1), 66–81.

[3] Stöger, B., & Miesenberger, K. (2015). Accessing and dealing with mathematics as a blind individual: State of the art and challenges. In G. Kouroupetroglou (Ed.), *Proceedings of International Conference on Enabling Access for Persons with Visual Impairment* (pp. 199–203), Athens, Greece: National and Kapodistrian University of Athens Department of Informatics and Telecommunications Speech and Accessibility.

[4] Millar, S. (2003). *Reading by touch*. London, UK: Routledge.

[5] Tobin, M. J., & Hill, E. W. (2015). Is literacy for blind people under threat? Does Braille have a future? *British Journal of Visual Impairment*, 33(3), 239–250.

[6] Millar, S. (1994). *Understanding and representing space: Theory and evidence from studies with blind and sighted children.* Clarendon Press/Oxford University Press.

[7] Millar, S. (1997). Theory, experiment and practical application in research on visual impairment. *European Journal of Psychology of Education*, 12(4), 415–430.

[8] Schneider, E., Maruyama, M., Dehaene, S., & Sigman, M. (2012). Eye gaze reveals a fast, parallel extraction of the syntax of arithmetic formulas. *Cognition*, 125(3), 475–490.

[9] Breidegard, B., Jönsson, B., Fellenius, K., & Strömqvist, S. (2006). Disclosing the secrets of braille reading-computer-aided registration and interactive analysis. *Visual Impairment Research*, 8(3), 49–59.

[10] Van Leendert, A., Boonstra L. G., Doorman, M., Drijvers, P., van der Steen, J., & Pel. J. (2021). An exploratory study to improve reading and comprehending mathematical expressions in Braille. *British Journal of Visual Impairment*, 41(2), 312–327. https://doi.org/10.1177/02646196211044972

[11] Riccomini, P. J., Smith, G. W., Hughes, E. M., & Fries, K. M. (2015). The language of mathematics: The importance of teaching and learning mathematical vocabulary. *Reading & Writing Quarterly*, 31(3), 235–252.

[12] Agrawal, J., & Morin, L. L. (2016). Evidence-based practices: Applications of concrete representational abstract framework across math concepts for students with mathematics disabilities. *Learning Disabilities Research and Practice*, 31(1), 34–44. https://doi.org/10.1111/ldrp.12093

[13] Marschall, G., & Andrews, P. (2015). Polish teachers' conceptions of and approaches to the teaching of linear equations to grade six students: An exploratory case study. *Research in Math. Educ.*, 17(3), 220–238.

[14] Uribe-Flórez, L. J. & Wilkins, J. L. M. (2010). Elementary school teachers' manipulative use. *School Science and Mathematics*, 110(7), 363–371.

[15] Witzel, B & Allsopp, D. (2007). Dynamic concrete instruction in an inclusive classroom. *Mathematics Teaching in the Middle School*, 13(4), 244–248.

[16] Bakker, A. (2018). *Design research in education: A practical guide for early career researchers*. Routledge.