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ORIGINAL RESEARCH

Walking the number line: towards an enactive understanding of integer arithmetic

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Abstract

Early mathematics education presents middle-school students with the challenge of adding and subtracting negative integers. This paper reports on results from the experimental implementation of a proposed educational design for integer arithmetic that utilized the number-line (NL) form as a resource for students to enact simple addition and subtraction problems under two conditions: (1) a body-scale floor-based NL, where arithmetic operations are enacted by walking; and (2) a regular desk-based NL supplemented with an action-figure for re-enacting the floor-based solutions. This design is the first iteration of a design-based research project and was developed based on the experience of the first author's five years teaching in this topic. 15 Grade 7 students participated in the project's pilot study that centered on how students coordinate procedurally analogous calculation activities across the large and small NL. The activity elicited students' implicit confusions surrounding integer subtraction, thus creating opportunities for corrective intervention. Analyses also generated operative inferences shaping the subsequent design iteration. Implications are drawn more broadly for enactive mathematics pedagogy, particularly through the lens of comparing students' egocentric orientations toward immersive instantiations of cultural–historical mathematical forms to their allocentric perceptual orientations toward the normative forms of the same concepts. As Extended Reality (XR, e.g. virtual reality, augmented reality) experiences enter mathematics classrooms, it may become vital to develop pedagogical methodologies in support of coordinating conceptually complementary perceptual perspectives.

Keywords Integer operations · Embodied cognition · Number line · Perspective

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Design problem

Basic number sense is a high predictor of students' future mathematical achievement (Lyons & Beilock, 2011; Varma & Schwartz, 2011). In elementary school, students develop number sense through their interaction with the world around them. Counting fingers, toys, or other concrete objects enables children to develop a sense of the value, cardinality, and order of integers. Studies show that students develop a linear, spatial–numerical mental number line as early as Grade 2 (Mock et al., 2019). However, students often struggle to expand their mental number line to include negative values (Bofferding & Hoffman, 2014).

Negative integers are conceptually challenging, because, as compared to positive integers, they are not as easily modeled on real life situations—negative numbers index the absence of enumerable entities, so that inherently there is "nothing there" to count. Even when engaging with everyday contexts selected to model negative integers, such as elevators and temperature, students have been shown to struggle to accurately conceptualize negative integers (Bofferding & Farmer, 2018). Consequently, once the curricular content of negative numbers is introduced, many students experience challenges assimilating these obscure mathematical entities in performing basic arithmetic such as addition and subtraction. Indeed, standardized mathematics test scores become further stratified as students reach middle school, when negative numbers are usually introduced (Juvonen et al., 2004; NCES, 2022). Notwithstanding, by the end of middle school, it is vital that students will have mastered basic arithmetic operations with positive and negative integers to meaningfully participate in high-school mathematics classrooms.

This design-based research study arose from the first author's five years teaching experience in both general and special education mathematics, where she noticed that current instructional resources meant to teach integer arithmetic were often unsuccessful and would require deeper reconsideration with an emphasis on operations involving negative integers. All students benefit from instructional resources that enable them to develop deep understanding of curricular concepts. The purpose of this study is to propose instructional resources that would allow all students to build on their mathematical knowledge to develop fluency with adding and subtracting positive and negative integers. Whereas our proposed activity leverages elements from prior design work, (as we shall elaborate), our activity introduces several new resources to enhance the realization of those elements in practice.

Previous solutions

Many teachers face difficulty when attempting to concretely model the addition and subtraction of positive and negative integers (Bofferding & Farmer, 2018). Bossè et al. (2016) surveyed the variety of concrete representations which teachers employ to teach integer operations. The researchers found that these representations fall under three categories: (1) isomorphic blocks; (2) colored counters; and (3) number lines, as we now elaborate.

Isomorphic blocks refer to any manipulatives in which the absolute value of each unit is represented by an item identical to all others in the set. For example, base-10 blocks or other counters are often used to teach basic counting and ordering. Isomorphic blocks are helpful to concretely teach students about addition and subtraction when none of the integers involved in the arithmetic are negative. However, consider the arithmetic expression, "4–5." If you have four blocks and want to *take away* five blocks, there is not a clear way

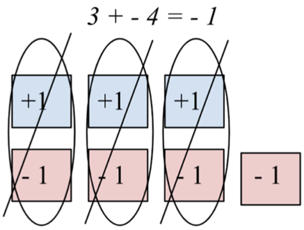


Fig. 1 Example of colored counters

to represent the difference of negative one, since all of the blocks are identical. Therefore, isomorphic blocks are usually used to teach operations involving positive or zero solutions only; they are less suitable for subtraction problems in which the subtrahend is greater than the minuend.

Colored counters are similar to isomorphic blocks in that each unit is represented by one item, but one color represents positive integers and another color represents negative integers. Colored counters are also referred to as *cancellation models* (Nurnberger-Haag, 2018), because each colored unit is said to "cancel out" one unit of the other color (see Fig. 1). Cancellation models can help students develop procedural fluency with integer arithmetic, but they bear conceptual tradeoffs. Colored counters tokenize negative integers as concrete entities that can be manipulated and moved in the same way as positive integers. This can be confusing, as negative integers should represent the absence of substance or, more generally, quantity. Furthermore, colored counters do not represent the ordinal continuum of positive and negative numbers across zero.

Finally, the *number line* is a linear representation of numbers, where all integer marks are spaced evenly apart, often with zero at the center; positive numbers extend to one side, while negative numbers extend to the other. Of note, number lines (henceforth, NL) differ from manipulable tokens: whereas tokens can move freely in space and still represent a constant cardinality (i.e. the set's collective "how-much-ness"), NL hash marks are bound to fixed spatial locations whose symbolic meaning derives from their uniformly unitized spatial distances from zero. As such, NLs do not portray negative integers as concrete, countable 'things.' Rather, NLs highlight the ordinal and spatial nature of numbers (Saxe et al., 2013). Moreover, the NL's equispaced structuration of cardinality affords perceptual judgments of proportional relations as well as considering non-integer numbers that fall between the marks. For example, Saxe et al. (2013) studies how elementary school students form "agreements" with researchers on the ordinal locations of zero and non-whole numbers on the NL. These researchers argued that the NL could constitute a vital educational resource for learning fractions because, similar to the case of negative integers, students generally struggle to conceptualize the relative value of fractions in the absence of a linear model.

NLs can be presented in a variety of configurations corresponding to various metaphors supporting students' grounding of negative quantities (Nurnberger-Haag, 2018). For example, teachers may invite students to perceive *vertical* NLs as thermometers marking positive and negative temperatures or as elevators that travel up and down to floors above or below ground level (i.e., zero). Teachers have also used *horizontal* NLs to represent an agent traveling "east" and "west" of some designated zero point. Many students who evidence comfort with NLs when engaging in positive integer arithmetic nevertheless struggle to utilize NLs in understanding both the placement and arithmetic of negative integers (Bofferding & Farmer, 2018). This general failure of the NL as a pedagogical support for basic arithmetic with positive and negative numbers, however, may be due not to inherent properties of the NL resource itself as much as due to the instructional activities it has served, which may have not drawn on students' naturalistic orientations and competencies (i.e., they were not "body syntonic," per Papert, 1980). Moreover, students' embodied and enactive experiences with the NL should then be recruited and coordinated within normative semiotic practices of the mathematics discipline, such as inscribing and manipulating symbolic notations that are taken as expressing the same ideas. As we will later explain, our project's focal design sought to leverage children's basic walking and orientation skills by having them enact addition and subtraction as stepwise movement back and forth along a floor-based NL, followed by a reenactment of their bodily movements mimicked by a small figurine. The goal of this design was to allow students the opportunity to coordinate their bodily experience on a walking NL with a traditional, static NL task.

Cognitive domain analysis

Students generally succeed in mental arithmetic involving small *positive* integers by relying on their past experiences of manipulating enumerable objects, such as blocks or just their fingers. These mental routines, however, are not readily conducive to the case of *negative* integers, whereupon students often require support in the form of dedicated semiotic resources, such as material instantiations of the NL, with which they are guided to assimilate this new class of mathematical entities into their existing routines. As such, students are liable to encounter at once a "double whammy"—they are to understand the NL both as a means of learning the idea of a negative number *and* to perform arithmetic with negative numbers. Mock et al. (2019) found that, while 6th grade students did visualize a mental spatial–ordinal representation of positive numbers, they did not yet similarly conceptualize negative numbers as continuing "to the left of zero," symmetrically reflecting the positive side. Rather, the mental NL and other mathematical representations of integers that students might visualize are grounded in embodied experiences of counting, measuring, and ordering, experiences that do not clearly bear on the case of negative numbers and, in any case, do not appear to lend themselves to conceptual expansion from positive to negative quantities.

Conjecture

The theory of embodied cognition recognizes that the mind and body are not divided, but that cognition relies heavily on the experience of embodiment, that is, of physically existing in the world (Abrahamson, 2020). Therefore, thinking and learning only through manipulating symbols written on paper according to a set of rules does not allow students to make sense of mathematical concepts. Students' perceptions of an object, such as a mathematics learning resource, are formulated through having previously interacted with the object in an attempt to perform some motor-control task (Abrahamson, 2020). Indeed, Mock et al. (2019) showed that students who utilized whole-body movements developed stronger spatial–numerical associations than those who did not. These sensorimotor experiences enable students to ground otherwise nebulous mathematical notions, such as negative-number operations. Similarly, students who experience negative numbers by walking along a NL demonstrate a higher level of proficiency as students who learn the same content with colored counters (Nurnberger-Haag, 2018). Typically, these students abandoned the metaphors previously used to describe NLs (i.e. floors on an elevator, temperature) after their experience with whole-body movement. However, once seated back at their desks, the vast majority of these students still struggled to accurately complete addition and subtraction problems involving negative numbers. In other words, these students struggled to bridge the connection between their bodily experiences and symbolic mathematics tasks.

Embodied designs (such as the walking NL) typically enable students to enact problems from an egocentric perspective. In an *egocentric* perspective, objects and actions are experienced or described with respect to one's own body, while in an *allocentric* perspective, objects are experienced or described with respect to other objects (Tversky & Hard, 2009). For example, when we walk in a city, we perceive streets egocentrically, but we would perceive these same streets allocentrically as viewed from a drone filming from high above the city. Students who use their whole body to move and learn in new ways, such as by enacting arithmetic on the NL, are often not required to coordinate this perspective with an allocentric perspective on similar displays, for example, by *re*enacting their full-body NL experience on a regular small NL printed on paper.

The results from Mock et al. (2019) and Nurnberger-Haag (2018), we propose, implicate this egocentric-to-allocentric experiential disconnect as critically compromising the pedagogical and cognitive potential inherent to embodied design. Pedagogically, the purpose of many embodied designs is to occasion opportunities for students to coordinate an egocentric perspective on a situated phenomenon with an allocentric perspective on a normative mathematical model of that same phenomenon (Abrahamson, 2009, 2012). We, as researchers and teachers, are looking to facilitate the cognitive coordination between primitive ecological behaviors such as walking, and sophisticated cultural practices, such as using mathematical instruments to support arithmetic reasoning. At the same time, we recognize the vital function of educational resources and teacher facilitation in guiding students' re-invention of these cultural–historical practices (e.g., Gravemeijer, 1994). Our tasks are designed to guide this re-invention by way of staging opportunities for students to appreciate how these cultural practices, in particular the available symbolic artifacts, offer solutions to emergent problems students incur in the course of attempting to perform their assigned task. From the perspective of cultural–historical psychology (Vygotsky, 1978), we look to support students' cognitive reconfiguration of their "real" (primitive, evolutionary) egocentric form of engaging the environment into the culturally instrumented "ideal" form. Complementarily, the cognitive-developmental perspective of genetic epistemology (Piaget, 1975) suggests that "real" forms of engaging the environment—situated know-how—are amenable to change when they are perturbed beyond a student's capacity for tacit assimilation–accommodation adjustment, whence this know-how is surfaced and undergoes major reconfiguration (*reflective abstraction*). The re-presentation of egocentric experience in allocentirc form is a unique human capacity that evolved as a society's collective solution to the problem of coordinating joint action (Donald, 2010; Saxe, 2012) and often utilizes heterogenous semiotic contributions (Goodwin, 2013) organized in consensual epistemic forms (Collins & Ferguson, 1993). These phylogenetic achievements, per the pedagogical notion of guided re-invention, can be expediently reenacted as students' ontogenetic achievements by encouraging their re-presentation of egocentric experience in allocentric form (Benally et al., 2022). In particular, we conjecture, students may better avail of enacting arithmetic on a large NL when they are guided to coordinate this experience with scaled-down actions on a regular desk-top paper NL.

Teachers and educational designers seek to foster the educational conditions for students to adopt the normative NL as an instrument for mathematical reasoning. According to Vérillion and Rabardel (1995), doing so requires a nontrivial cognitive effort. In particular, for students to *instrument* themselves, with this new artifact, that is, extend their

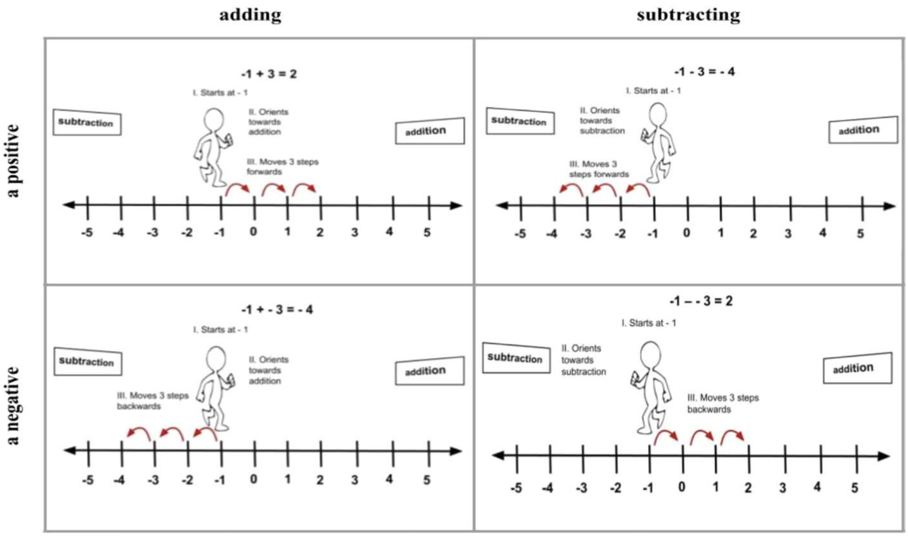


Fig. 2 Walking number line (egocentric perspective)

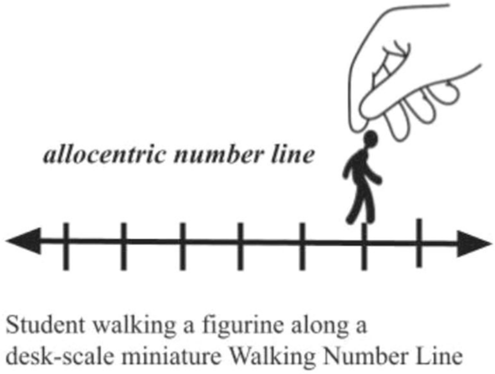
sensorimotor capacity, they need to exercise its underlying conceptual system (Abrahamson & Bakker, 2016).

Design solution

In order to ground the symbolic notation of integer arithmetic in physical walking, students begin by standing and walking along a big NL (BNL; egocentric orientation). In order to solve each integer addition or subtraction problem, students: (a) stand on the first number in the problem; (b) orient their body towards the positive side of the NL for addition or the negative side of the NL for subtraction; (c) walk forwards if the second number in the problem (i.e., the addend or subtrahend) is positive, and backwards if the second number is negative; as many steps as (d) the amount specified by the absolute value of that second number (see Fig. 2). For example, if the student were solving the problem "− 1 + 3," (see top left box in Fig. 2), they would: (a) begin by standing on − 1; (b) would then orient towards *addition*; then (c)walk *forwards*…(d) *three* steps, arriving at 2. However, if the student were solving the problem "− 1–(− 3)" (see bottom right box in Fig. 2), the student would: (a) again start by standing on *-1*; but would then (b) orient towards *subtraction;* before (c) walking *backwards*… *three* paces arriving at + 2. Students thus enact the arithmetic operation (adding or subtracting) as a shift in orientation, and the polarity of the second number (positive or negative) as direction of movement, either forwards or backwards. This designed functional decoupling of directionalities—orientation vs. movement—disambiguates the polysemy of the '−' sign, which can be challenging for students (Abrahamson et al., 2014; Mamolo, 2010). Figure 2 shows teachers' instructions and students' actions with the four basic moves corresponding to adding or subtracting positive or negative numbers on the BNL (student egocentric orientation).

Teaching students procedural rules, such as how to carry out arithmetic operations with positive and negative integers, does not provide them with opportunities to develop conceptual understanding (Kamii & Dominick, 1998; Thompson, 2013). In particular, parroting

Fig. 3 Small number line (allocentric perspective)



"minus plus is minus, plus minus is minus, minus minus is plus" is a form of knowledge devoid of systemic logical coherence. Anecdotally, many teachers will offer the mnemonic that *minus* (left-hand index pointing horizontally toward the right) *minus* (right-hand index pointing horizontally to the left, opposite the other index finger) makes *plus* (index fingers move towards each other, right-hand index rotates to vertical orientation, intersecting left-hand index, thus forming a plus symbol). Instead, by systematically deploying strings of arithmetic symbols as situated rules of location, orientation, and movement, our design offers a consistent and coherent model, where the operative meaning of sign composites, namely $+(-k)$, $-(+k)$, and $-(-k)$, emanates from transparently carrying out the succession of rules rather than from opaquely abiding with symbolic subjugation.

After several sessions of enacting solutions to arithmetic problems via an egocentric perspective on the floor-based NL, students will simulate their full-body actions by manipulating a "mini-me" figure on a desk-scale NL (allocentric orientation; see Fig. 3). Note that whereas the figurine is reenacting an egocentric experience, scaled down from the student's prior experience walking the BNL (i.e., the figurine is looking forward along the NL), the student who is now puppeteering the figurine sees the NL from an allocentric perspective. It is expected that this blended bi-perspectival activity configuration will support students in learning the content through figuring out how to coordinate the two perspectives (Benally et al., 2022), ultimately grounding a normative allocentric use of the standard NL in their egocentric experience walking the body-scale NL. In this design, we hypothesize, the conceptually critical cognitive process of symbol grounding plays out as the perceptual work of coordinating egocentric and allocentric perspectives for operating on the NL, leading to students developing integer arithmetic fluency.

Finally, it is important to note that this study continues the initial phase of a longer design-based research (DBR) project. DBR is an iterative process, where contributions to scholarship emerge from reflections on each cycle of the design (diSessa & Cobb, 2004). Therefore, the purpose of this study is to glean theoretical insights explicitly from the documented and analyzed *shortcomings* of the design, so that we can both improve our theory and, in so doing, figure out how to improve the design, moving forward (see Anton et al., 2024 for a further iteration of the project). We are eager to share our insights and new theoretical directions developed based on our deep analysis of this early study.

Research questions

Design-based research studies often launch not from a classical research question, as in for example cognitive psychology, but more so from a design problem (Bakker, 2018). To the extent that these studies engage with research questions, these questions emerge in the course of either the design process or, more frequently, the data analysis (see also Edelson, 2002). The following research questions guided this study:

1. What are the different affordances of body-scale vs. desk-scale number-line models for task-based embodied interaction?
2. How do students coordinate ambulatory actions on the body-scale number line with manual actions on desk-scale models, and how is this coordination supportive of learning?

Methods

Participants

For this first iteration of the experimental design, participants included 15 students enrolled in a public, Title I middle school in San Diego, California. All participants were 7th grade students in a general education inclusive Math class, and all were pre-identified by their teacher as experiencing difficulty specifically with the topic of adding and subtracting positive and negative integers. Three students (20%) identified as male and twelve (80%) identified as female. Four of the participants (27%) had an Individualized Education Plan (IEP) for disability services, including ADHD (n = 1), OHI (n = 1), and autism (n = 2).

Materials

The materials used included the body-scale number line (BNL), which was made of a long roll of paper (20’ × 4’) displaying a horizontal NL from − 5 to + 5. The distance between consecutive number markings on the NL was approximately one child step. A small sign with the word “addition” was placed at the right (positive) end of the BNL, and a small sign with the word “subtraction” was placed at the left (negative) end of the BNL. See Fig. 2 for an illustration of the BNL. The small number line (SNL) consisted of a typical (8.5″ × 11″) sheet of paper with an identical, scaled down NL (ranging from − 5 to + 5). A small puppy figurine (1″ tall) was placed on top of the SNL. See Fig. 3 for an illustration of the SNL.

Procedure

Participants completed the activity individually during one 50 min class period. First, the student and the researcher introduced themselves to each other and chatted informally about past experiences with NLs and in math classrooms (see Interview Protocol in Appendix A). Next, the researcher shared the four steps to solving problems on the walking NL: (1) start by standing on the first number in the problem, facing orthogonally to the line; (2) turn to face the addition sign (positive direction) for addition problems, or turn to face the subtraction sign (negative direction) for subtraction problems; (3) prepare to walk forwards if the second number (the addend or subtrahend) is positive and backwards if the second number is negative; and (4) walk the amount of steps of the absolute value of that second number. Then, students engaged in solving simple addition and subtraction problems provided by the researcher.

After about 25 min enacting addition and subtraction problems on the BNL, students shifted to the SNL. Students were instructed to use the puppy figurines to mimic their movements on the BNL. Again, students completed basic addition and subtraction problems provided by the researcher. Throughout the process, the researcher conducted a semi-structured interview (Ginsburg, 1997) in order to gain a better understanding of students' thinking and shifting perspectives (see Interview Protocol in Appendix A).

Data gathered

Data were collected during 50 min class periods over three different days. The first 6 students completed the activity on the first day, the next 4 completed the activity on the second day, and the final 5 completed the activity on the third day. While all students had written parental consent for participation in the study, only nine of the fifteen participants (60%) gave assent to be video and audio recorded. As such, the researcher video and audio recorded those nine participants and took detailed, in-the-moment field notes on the remaining six participants. The researcher added to the field notes directly following each session.

Data analysis

Data analysis for this study followed a case study design, focusing on moments from individual students' experiences (Yin, 2006). The primary researcher examined the recordings and field notes and identified instances of tension, misunderstanding, and productive discourse. In this way, the researcher was able to triangulate (Yin, 2006) multiple forms of data (e.g., actions on the NL, responses to semi-structured interview questions, past performance in mathematics class, etc.) to build an account of events and, in particular, to hypothesize how students were making sense of positive and negative numbers given the available resources. The researcher utilized grounded theory (Creswell, 1998) and thematic analysis (Lester et al., 2020) to surface across the entire data corpus moments that the team judged as key to developing a systematic account of all study participants' conceptual struggles and resolutions. Once we had completed our tentative general account of students' experience with our design, we selected three students to exemplify the range of these experiences with respect to several key parameters. During the research team's collaborative microgenetic analysis, these three case episodes gradually surfaced as paradigmatic of the range and types of challenges students faced through engaging with the activity (Siegler, 2006).

Our design and data analysis draw on several theoretical frameworks, notably instrumental genesis (Vérillion & Rabardel, 1995) and coordinating perceptual perspectives (Benally et al., 2022). In seeking both theoretical and practical coherence, our paper proposes how these theoretical resources could be interleaved to offer a richer, more nuanced understanding of childrens' learning process within these instrumented learning environments.

Results

While implementing the educational design with students, key themes emerged. When walking on the BNL, most of the participants (n = 12) experienced no difficulty in enacting the arithmetic problems. The remaining 3 participants experienced few moments of confusion, and were able to correct their mistakes with support. However, when the students were reenacting their full-body movement through manual manipulation of a figurine on the SNL, only 4 of the participants were successfully able to coordinate their egocentric experience with this allocentric display, resulting in no errors. Many participants (n = 9) experienced breakdown in their attempts to accomplish this perspectival coordination on the SNL, in particular committing errors when presented with problems involving subtracting a negative number. Other errors on the SNL included subtracting a positive number from a negative (n = 4) and adding a negative number to a positive (n = 1). Specific episodes of student thinking (described below) are shown in Table 1. These episodes were chosen as representative samples of the dataset as a whole and provide insight into students' confusions surrounding NLs and integer arithmetic.

Episode 1. Zero is nothing

Description. After a session on the body-scale number line (BNL), the students transitioned to the small number line (SNL) and were given similar addition and subtraction problems along with a small figurine. Students were instructed to use the figurine on the SNL to mimic their bodily movement on the BNL. The teacher presented the problem "0–(− 4)" (see Table 1, Episode 1). The student began by placing the figurine on the 0 mark, facing the negative (subtraction) direction. The student then moved the figurine immediately (without stepping) to − 4 on the NL, whereas the correct response would have landed the figurine at + 4. Importantly, this student had previously solved the problems "− 1–(− 3)" and "− 1–3" correctly on the BNL, where she was able to adequately explain her process and solutions to both problems. After the incorrect performance on the SNL, the researcher prompted:

Researcher (R): What would the answer to 0 minus 4 be?
Student (S): Also negative 4.
R: So these two [points to "0–− 4" and "0–4" problem cards] are the same?
S: Yeah.
R: But these two [points to "− 1–− 3" and "− 1–3" problem cards] are not the same?
S: Yeah, because zero represents like nothing, so …

Table 1 Selected episodes of student thinking

	Episode 1: Zero is Nothing	Episode 2: Unlearning Rules	Episode 3: Returning to the BNL
Teacher: *Presents problem*	0–(− 4)	4 + (− 7)	2–(− 2)
Student:	"You just stay at the negative 4 because the zero is nothing."	"Well, it's going to be negative."	*Moves to BNL.* "I would go backwards to 4!"

R: But what would this subtraction sign mean? [circles subtraction sign on the "0–− 4" problem card]

S: Well… I think of zero as sort of just like a placeholder, because if it's nothing, then I don't really have to add or subtract anything.

Two other students took the same approach to this very same problem of "0–(− 4)". One student claimed, "You just stay at the negative 4, because the zero is nothing." The second student asked, "How do I take out negative 4 from zero?" and then said, "I started at zero, I turned and faced subtraction….. But, it was zero, so it was going to be negative 4 anyways, right?".

It is also important to note here that while enacting arithmetic on the BNL, these three students had not claimed that zero should be ignored; rather, they had correctly solved problems like "0–2" and "0–(− 2)." *Something about the act of walking appears to have disrupted their previous notions that zero was nothing*. In contrast, the SNL's familiar desk setting appears to have reinforced prior habits (see Papert, 2004, on the alleged ills of "paper math").

These students followed the first two of the procedural steps correctly on the SNL (Step 1: start at the first number; Step 2: face either the addition or subtraction direction), but faltered when it came to Step 3 (prepare to walk either forwards or backward). Instead, these students were focused on the *value* of zero in terms of personally familiar contexts (e.g., having zero dollars in a bank account, that is, the cardinal property of zero) rather than on the *function* of zero in the particular context of the NL (an ordinal marker along a succession of equidistant spatial intervals).

Discussion. Children typically begin to learn and use NLs with positive numbers and zero only. In common early metaphors regarding addition and subtraction, zero is indeed described as nothing—if you have three dollars, and I take away three dollars, then you have nothing left. In contrast, when solving addition and subtraction problems on the NL, the *value* of each integer is not as important as its ordinal place in the number line continuum. These students stepped on the zero hash-mark on the BNL just as if it was any other integer. However, when solving problems on the SNL, students became concerned with and distracted by the value of zero and did not rely on their just-previous movement-based strategies employed on the BNL. In other words, students failed to coordinate their egocentric experience of walking on the NL with the allocentric experience of moving a figurine along a smaller NL and, as such, did not instrumentalize the SNL in the same way as the BNL.

Episode 2. Unlearning rules

Description. Similar to the previous episode, this episode, too, concerns students who had moved their whole bodies on the BNL and then transitioned to moving a small figurine on the SNL. Having completed a few BNL problems correctly, a student was posed with the item "2 + (− 5)." Before beginning the problem, the student said, "Well, it's going to be negative." When prompted how she knew this, the student explained "because a positive and a negative will be a negative." While in this situation she was correct that "2 + (− 5)" does indeed result in a negative number, the researcher then proposed the problem "− 4 + 7." After following the steps to solve this problem correctly, the student showed evident surprise and said, "Wow! So not always."

When presented with the item “(− 4) + (− 1),” another student claimed that a negative and a negative cancel out to a positive (“Well normally, I don’t really think about the positives or the negatives, I normally do the actual equation, and then just depending on if it’s, like, ‘a negative and a negative,’ I make it positive or, like, depending on the sign”). However, after correctly solving the problem on the BNL, she exclaimed, “I had always thought that if it was ‘a negative and a negative’ then it’s positive so I don’t know… but the number line makes it more clear.” Again, the student had not used this strategy on the BNL, but reverted to old habits when presented with the SNL.

A third student, when solving the item “− 1–3,” claimed that the answer was + 2. When prompted to elaborate on her strategy, she said, “So, basically, we learned this strategy in Math class, that if you’re subtracting a positive number from a negative number it’s like adding.” In this case, it seems that her teacher was referring to the fact that the absolute value of both integers added together will equal the absolute value of the solution, for example, − 5–2 = − 7, and 5 + 2 = 7.

Discussion. These episodes, thus, present further cases of students perceiving the meaning of a numerical symbol in a way that was incongruent with the instructor’s perception, which, by contrast, was normative to the context of this mathematical practice. These students had clearly heard an instructor state a rule for a type of problem and had stuck to their understanding of that rule, even if it did not align with their emergent conceptual understanding of negative integers gained from walking on the BNL. When further prompted, these students revealed that they clung to these rules in a fear of being unsuccessful in their attempts to solve problems involving negative numbers on their own (cf. Erlwanger, 1973). These students have not yet *instrumented* (Vérillion & Rabardel, 1995) themselves with the NL as a cultural tool bearing utility for engaging in mathematical practice (Menary & Gillett, 2022). Instead, it was simply a piece of paper evoking the confusion and frustration they had previously experienced in math classrooms (Papert, 2004). This sudden transition of the SNL from a meaningful cognitive tool to a meaningless piece of paper occurred after the students had already successfully used the SNL many times. We hypothesize that their success on the first few problems was a direct result of coordinating perspectives from the egocentric experience on the BNL to the allocentric experience of moving a figurine on the SNL. In other words, the enactive experience of solving the problems on the BNL, which had interrupted the old erroneous ways of thinking, initially guided them on the SNL. However, when a problem came up on the SNL where they were unable to rely on the BNL physical experience, either because too much time had passed or because the chosen problem reminded them of a past erroneous strategy, students fell back to their prior strategies and relied on those to get them through the problem.

Episode 3. Returning to the BNL

Description. This case concerns students who, while solving problems on the SNL, became stuck and returned temporarily to the BNL as a resource for solving the SNL problem. One student switched to using the BNL when presented with the item “2–(− 2)”:

S: Wait, it’s like taking two from two. But 2–2 is that [points at 0] but 2–(− 2)... It’s either 4 or zero.

R: OK, what can we do to help us decide whether it’s 4 or zero?

S: Well it’s gonna be facing subtraction, and I’m taking 2 out, so it’s zero I think.

R: What do you mean by that?

S: If it was just 2, positive 2, it would be zero, but it's negative 2 so I can't tell if I'm going to 4 or zero.

R: Do you want to try it on the walking one? Would that be easier?

S: Sure! [walks to BNL] OK so 2 [stands on 2] minus… [turns towards subtraction… slowly walks to zero] But, when I go this way [walks to 4] I'm taking negatives out… Oh no! I would go backwards to 4!

This student was unable to decide which answer was correct on the SNL and, like students in the previous episodes, struggled with the direction of movement. However, she was able to use the BNL as a resource to help her decide on the direction of movement, which she could then apply on the SNL.

Similarly, a second student claimed that they got multiple answers when solving "3–(− 2)" on the SNL (he moved the figurine both backwards to positive 5 and forwards to positive 1). However, when asked which answer made the most sense, the student said: "positive 5… because that's how I did it on the big number line." In addition, two other students failed to solve a problem on the SNL, stood up to check their work on the BNL, and returned with the correct answer.

Discussion. This episode highlights students misinterpreting the operative meanings of the polarity symbol semiotic resource. However, these students utilized the availability of the BNL to reenact the problem and thus bridge their egocentric and allocentric experiences. Moreover, by asking to use the BNL when they were stuck on problems, students were proactively negotiating between two different spatial perspectives. Benally et al. (2022) differentiate among consequences of perspectival coordination: in perspectival substitution, one perspective replaces the other; in perspectival mutuality, individuals use an alternative perspective to validate their own; in perspectival synergy, a combination of two perspectives results in one that is greater than each perspective alone. In this case, students had achieved perspectival mutuality, and were validating their thinking on the SNL by utilizing their perspective on the BNL. Eventually, students may negotiate between these two perspectives to achieve perspectival synergy.

In summary, these three episodes demonstrated conceptual, strategic, and perspectival semiotic breakdowns related to adding and subtracting positive and negative integers. The ambulatory actions on the body-scale BNL were successful in disrupting students' previous misconceptions and erroneous strategies used in traditional mathematics classroom settings. However, in order to leverage their embodied experience on the BNL, students had to shift their perspective to a smaller scale, the SNL. This shift requires coordinating their BNL walking experience with the SNL experience of manipulating an object. In other words, the purpose of this design was to allow for students to practice negotiating between various spatial perspectives in order to make sense of a semiotic artifact which might serve them in future mathematics practices. The aim of the study was to further understand how students coordinate egocentric and allocentric perspectives in the service of learning mathematics. The act of shifting perspectives from egocentric (BNL) to allocentric (SNL) afforded the students the opportunity to achieve perspectival synergy and develop a perspective that is greater than the sum of its parts (Benally et al., 2022).

Conclusions

Within the complex practice of design-based research, this paper reports on the first iteration cycle of a design, through which the team became cognizant of implicit elements in students' perceptual experiences as they engaged with the tools to perform the assigned task. Drawing on, and interleaving theories from the cognitive sciences that treat the role of perceptual perspectives (Benally et al., 2022) and instrumentalization (Vérillion & Rabardel, 1995) in everyday coping, we put forth a proposed ontological innovation (diSessa & Cobb, 2004) that highlights critical perspectival aspects of artifact instrumentalization in mathematics learning.

This study led to three general conclusions. First, non-canonical enaction of arithmetic solution procedures productively disrupts students' problematic routines, including their mathematically inappropriate heuristics. Procedure-oriented mathematics instruction, with little attention to conceptual grounding, has been found to foster feelings of frustration and failure. In response, some students develop idiosyncratic procedural strategies that may not align with mathematical theory (Erlwanger, 1973). Activities such as the walking NL, which differ so drastically from typical classroom experiences, enable students to dishabituate from their idiosyncratic erroneous coping mechanisms and, instead, re-engage meaningfully with the core concepts. This is shown by the relatively few students who experienced difficulty on the BNL.

Second, students can learn mathematical concepts through coordinating first-person situated enactment of procedural instantiations, such as walking a body-scale NL, and third-person canonical diagrammatic operations using conventional media, such as using a paper-based NL. More generally, practicing mathematical skills across two or more media may deepen conceptual understanding through eliciting and correcting any medium-specific heuristics that do not generalize. These designs also allow teachers to become aware of students' ways of thinking which were previously invisible, but which have become publicly inspectable through enaction.

Third, and by way of summary, whole-body movement and the coordination of multiple perspectives serve as enculturation opportunities by way of appropriating a tool into the cognitive system (cf., Menary & Gillett, 2022). In this study, the NL started out for most students as a simple tool which they had previously used, sometimes successfully, sometimes unsuccessfully. However, through their participation in the design, the NL became more than just a symbolic representation; for some students, the design became an *instrument*, that is, an "object which the subject associates with his action in order to perform a task, … [ultimately developing an *instrument utilization scheme,* which] enable[s] the repeatability of action" (Vérillion & Rabardel, 1995, pp. 84–87). Therefore, students who instrument themselves with the BNL as a new cognitive tool for carrying out integer arithmetic are developing the necessary cognitive schemes for taking on the SNL.

Implications

All students deserve to be able to rely on embodied experiences involving negative integers. This design allows students the opportunity to cement the abstract notion of a negative integer in concrete action (Varma & Schwartz, 2011). Embodied designs such as this one can be utilized to teach challenging concepts at a deep level. However, this design showed that students must be given scaffolded opportunities to coordinate between perspectives when learning curricular content in an experiential situation. Merely shifting unidirectionally from the BNL to a figurine on the SNL was often not enough; students must be given additional opportunities and supports that might help them when blending perspectives.

In particular, the following insights will be carried over to the next iteration of the design:

In the future, students should spend multiple sessions on the BNL making sense of their movements, before advancing to the SNL.

A longer NL (with a greater numerical range) would allow for more opportunities for students to express their mathematical thinking.

Students need more support in coordinating their egocentric and allocentric perspectives. Switching back and forth between the BNL and SNL, or teaching a peer how to walk on the BNL, might help students navigate this challenge.

Limitations

This study took place within a limited time frame and under the constraints of a typical school day schedule. The limited amount of time in which the researcher was permitted to work with each group of students limited data collection and, more importantly, the study participants' opportunities to train adequately with the new resources. Furthermore, this paper reports on a pilot study with only fifteen students. In the future, more participants will be required in order to validate our generalizations.

In addition, some students claimed that they reached their solutions on the BNL or the SNL because they could not move themselves (or their figurine) past -5 and $+5$ (which marked the material boundaries of our NLs). While the arrows at either end of the NL technically imply that the line extends infinitely in both directions, students did not consider solutions outside of that materially designated range.

Future work

Future work should redesign the procedure of this study so that the protocol allocates more time on each instructional step. In addition, future research should analyze the role of shifting perspectives in various mathematical and cognitive tasks beyond NL and design additional interactions which might facilitate the coordination of perspectives. Ryokai et al. (2022) found that virtual reality allows for students to utilize egocentric and allocentric perspectives in the service of a task. Future designs might utilize this technology to help students coordinate experiences on the BNL and SNL (Anton et al., 2024).

Furthermore, future research could investigate the impact of the facilitator providing students with the rules of movement. One could investigate student learning under the condition that the facilitator leads them through a *guided reinvention* of the rules (Freudenthal, 2012). In engaging students in generating the rules of mathematical movement, the students themselves could have agency over why they might turn and walk in a certain way, thus grounding these movements in their understandings of the symbolic arithmetic display.

Appendix A

Semi-structured interview protocol

What we say/ do	Why we say/ do it	Possible responses	How to respond to these responses
Hi, I'm from the University of California, Berkeley. But, I used to be an 8th grade Math teacher. Thanks for participating in this activity. I'm hoping that it will help me understand how to be a better teacher. Remember, I'm looking for clues to what you are thinking, not necessarily for a 'right' or 'wrong' answer. Do you have any questions?	Introduce myself/ build trust	I have a friend who goes to Cal. Do you know_____?	How fun!
You'll notice that I am video recording this session. This is just so that I can remember all of the cool conversations that we had later. If at any point you are uncomfortable with this, I can stop the filming- just let me know!	Build trust; make student comfortable; make sure they know that they can opt out at any time	Will I be on TV?	Probably not, but this video may be shown to some professors at Cal!
What is your name? How old are you?	Gather information on the child; build trust	I'm _______ and I'm ___ years old	Nice to meet you!
[set up walking number line on the ground] This is what I call the "walking number line"—it's a number line that you can actually walk on. What do you know about number lines so far?	Gather information on previous experiences with number lines; introduce the big number line (BNL)	I've used them before. I know that the positive numbers are on the right and the negative numbers are on the left	Great! Seems like you're pretty much an expert!

What we say/ do	Why we say/ do it	Possible responses	How to respond to these responses
Let's practice using the walking number line to solve some Math problems. For example, let's take the problem 3 +—4. There are four steps to solving these problems: Step1: start standing on the first number. Step 2: face the direction of the operation-that means adding or subtracting. So, to add 2, where should I face? (*repeat with addition and subtraction problems)*	Give instruction on the BNL	Which way do I face again for adding a negative number?	When you are adding, you always face the positive direction (to the right) towards the sign that says "add"
OK, now onto the last two steps. Step 3: Decide which way to walk. If the number is positive, you walk forwards. If the number is negative, you walk backwards. Step 4: walk the amount of spaces in the second number. (*repeat for multiple examples*)	Give instruction on the BNL	Which way do I walk again if I am subtracting 6?	When you are subtracting, you always face the negative direction (to the left). But, since you are subtracting *positive* 6 (and not negative 6), walk forwards
Can you show me how to solve (*insert integer operation problem here*)	Student demonstrates understanding of the rules of the BNL	Sure, where do I start again?	Always start standing on the first number in the problem
How did you know which direction to face?	Student demonstrates understanding of the rules of the BNL	If the problem says to add, then I face the adding sign I always face the right	Great! Cool, that makes sense. Let's see if it works with this problem: 2–3
How did you know which direction to walk?	Student demonstrates understanding of the rules of the BNL	I walk forwards for positive numbers and backwards for negative numbers I always walk forwards	Great! Cool, that makes sense. Let's see if it works with this problem: 1 + − 3
How do you remember the rules on the walking number line?	Gain insight into student's connections between bodily movements and perception	I look for the addition and the subtraction signs, and then I think about the problem	Great! What specifically about the problem are you thinking?
Do you think these rules provide you with accurate answers for these math problems? Why or why not?	Gain insight into student's mathematical thinking	Yes, because I am getting the right answers	How do you know that they are the right answers? What makes an answer wrong?

What we say/ do	Why we say/ do it	Possible responses	How to respond to these responses
[set up small number line with puppy figurine] Ok, now we are going to transition to using the small number line. What similarities and differences do you notice about this number line versus the walking one?	Introduce student to small number line (SNL)	This one is the same, just smaller	Great! Anything else?
Now see the puppy? Imagine that you are the puppy. Let's walk through some addition and subtraction problems like we did before, and you can move the puppy in the same way that you walked on the walking number line (*provide several addition and subtraction problems, highlighting all four types*)	Observe student's perceptual change and coordination from the BNL to the SNL	OK, I'm ready	Awesome! Let's get started
Can you show me how you did that problem?	Determine whether or not the procedural and conceptual understandings of the number line applied after the perceptual shift	The problem was -3–– 4. So, the puppy started at -3 and faced the subtract (negative) direction. Then the puppy took 4 steps backwards since it's a negative 4. They ended at positive 1	Great! Would you have arrived at the same answer using the walking number line? Why or why not?
How do you know how to move the puppy?	Gain further insight into the student's actions on the small number line	I know how to move them based on the problem	Could you tell me more? Walk me through whatever problem you would like!
If you add a negative number, will the puppy always end up in the negative area? Why or why not?	Gain insight into student's conceptual understanding of number magnitude	No, it depends on where you start Yes, because adding a negative means you go backwards towards the negative area	Cool! What do you mean by where you start? Can you give me an example? I see. Why do you go backwards? Can you show me on the problem 4 + – 2?

What we say/ do	Why we say/ do it	Possible responses	How to respond to these responses
How would you show someone who has never used a number line before how to move the puppy so that they get the correct answer?	See if the student has a better conceptual understanding of number lines after the activity	The puppy starts at the first number, and then you turn the puppy to face either add or subtract, depending on the problem. The puppy walks the number of spaces of the second number, forwards if it is positive, and backwards if it is negative	Do these rules lead you to the same answer that you would get when using a calculator? Why or why not?
How is your experience moving the puppy different from walking on the walking number line? How is it the same?	Investigate student's perceptual coordination between the BNL & SNL	On the walking number line I got to move myself. On the small number line I move the puppy	Cool! Can you think of any other differences? How did you feel on both? What could you see?
Would you rather solve problems on the walking number line or the small number line? Why?	Elicit more insight into the two experiences	Probably the small number line since it would be easier to carry around	That makes sense. Did you feel like problems were more challenging or easier on one version of the number line?
How might you have moved the puppy if you had never walked on the walking number line? Can you show me?	Investigate student's perceptual coordination between the BNL & SNL	Well, I used to use number lines like this (*shows the typical student drawing on number lines*)	That's cool! If I gave you a new problem and told you to solve it on the number line in any way, how would you do it?
What did you like about this activity? What didn't you like? How might I make it better?	Get feedback from the student!	I liked walking on the number line but I didn't like learning the rules	That makes sense- thank you so much for your feedback! You'll help make this more enjoyable for future students

Declarations

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