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TriO: A Multiplayer, Immersive, Virtual Environment for Exploring R^3

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Abstract

We present *TriO,* a multiplayer, immersive, virtual environment for exploring R^3. *TriO* was designed and developed by the research team to be an example of a three-dimensional parametric movement experience. In *TriO*, three virtual handles—colored red, green, and blue—control the *x, y,* and *z* coordinates of a small, disk-like object (the *widget*). Through parametrized movement along the three coordinate axes, players work together to navigate the widget to various points in R^3. We describe the design of the environment, consider its affordances for learning, present a multimodal summary of how three secondary mathematics teachers worked together in *TriO* to navigate to each of the vertices of a cube, and reflect on how collaborative, movement-based explorations of R^3 could create new opportunities for mathematical learning.

Keywords Virtual Reality · Mathematics Education · Coordinate Systems

Introduction

Cartesian coordinates are a crucial component of the underlying conceptual architecture that is necessary for abstract mathematical thinking. In two dimensions, a representation of a Cartesian coordinate plane can be fashioned from the orthogonal edges of any flat surface. This is not the case for three-dimensional Cartesian coordinates, where it requires techniques from perspective drawing to show three

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orthogonal axes on a two-dimensional surface. In two dimensions, children can fail to connect coordinate descriptions of space with their intuitive conceptions of location (Barrett & Clements, 2003; Sarama et al., 2003). In three dimensions, the difficulties with Cartesian coordinates are compounded by the representation problems inherent to diagramming three-dimensional figures (Trigueros & Martínez-Planell, 2010). Figure 1 is a paradigmatic illustration of these representation problems: The flattening of three-dimensional figures into two-dimensional diagrams creates a conflict between visual observation and mathematical theory. In Fig. 1, perpendicular lines meet at non-orthogonal angles; congruent segments are represented by different lengths; different locations in space are collapsed into the same area, among other distortions.

Moreover, any specific planar representation of a spatial figure depends on choices, often unstated and beyond the viewer's control, for angle of view and method of projection (Bakó, 2003; Blanz et al., 1999; Dimmel & Milewski, 2019; Edelman et al., 1992; Panorkou & Pratt, 2016). This is significant because different viewing angles could create different vision/theory conflicts. For example, in Fig. 2, the set of points from Fig. 1 is viewed from directly overhead—*i.e.*, the angle of view is perpendicular to the *xy* plane. From this view, it is evident that ABCD is a square; the *x* and *y* axes are perpendicular, and reading the *xy* coordinates of each point is straightforward. But this clarity about the *xy* relationships among the points comes at the price of collapsing the *z* axis. From this view, there is no way to visually observe that ABCD lies in the plane $z = 3$.

The tradeoffs that surface whenever three-dimensional figures are represented in two-dimensional diagrams are captured by the *close/distant* distinction (Parzysz, 1988). Close representations preserve dimension, and thus can resemble the figures they realize, such as a physical model of a cube. Distant representations, by contrast, decrease dimension. In the process, distant representations distort geometrical relationships, such as a cube drawn as an overlapping pair of squares whose

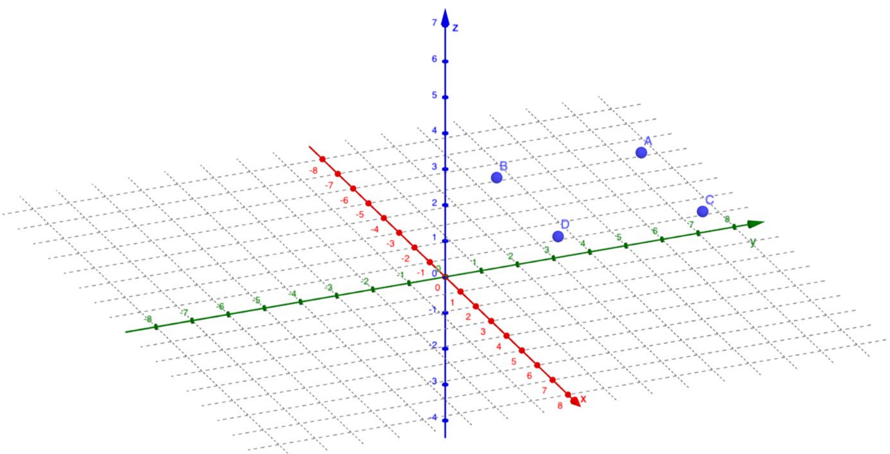


Fig. 1 Four points plotted in R.^{3}

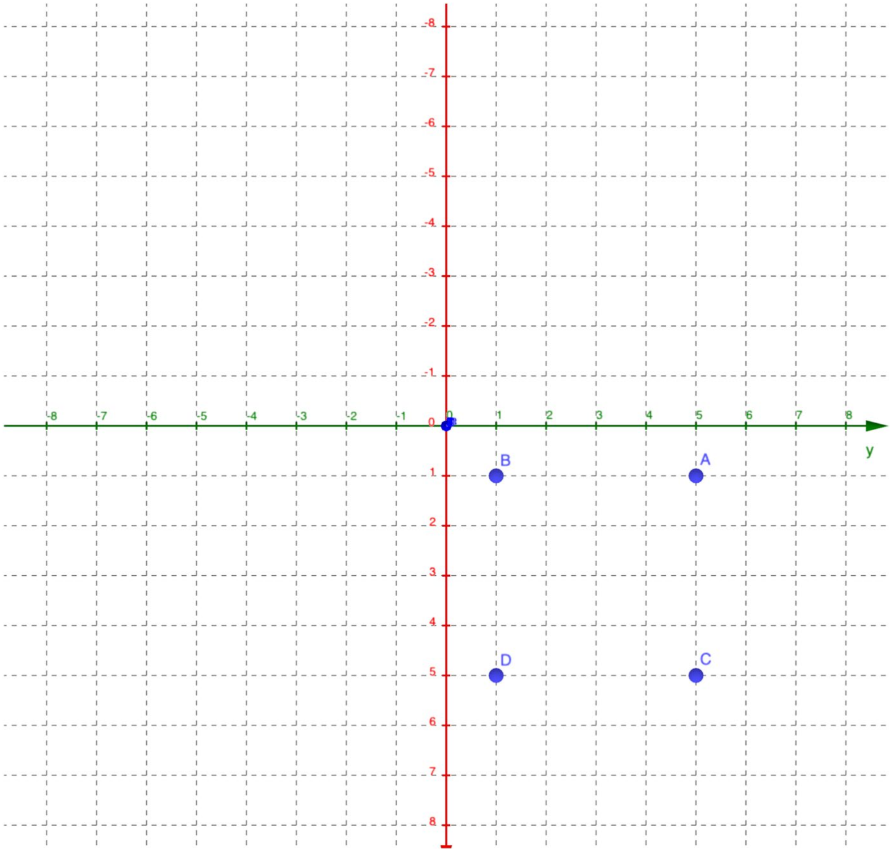


Fig. 2 An overhead view of ABCD. From this viewing angle, it is not possible to visually convey the depth of the diagram

corresponding vertices are connected by oblique segments (Fujita et al., 2020; Palatnik & Abrahamson, 2022).

Historically, representations of three-dimensional figures could be realized *either* through close physical models *or else* through distant diagrams. But with the emergence of extended reality (XR) technologies, such as virtual and augmented reality, we are undergoing a shift in our capabilities for representing and interacting with information. Three-dimensional figures can now be represented as spatially extended, interactable, three-dimensional diagrams (Dimmel & Bock, 2019; Kaufmann & Schmalstieg, 2002; Walkington et al., 2024). Interactable spatial diagrams make it possible to realize *close* diagrammatic representations of three-dimensional figures. When rendered in immersive virtual environments and viewed through stereoscopic, head-mounted displays, diagrams of three-dimensional figures present like spatially extended, worldly objects that can be explored and manipulated using familiar strategies for navigating space—e.g., moving one's head or body to vary the angle of view. At the same time, interactable spatial diagrams are free of material or physical constraints—they can be rendered at any size, in any orientation, and at any position in space, and can thereby realize a more varied set of mathematical concepts than what is practicable with physical models (Dimmel et al., 2021). Interactable spatial diagrams are potentially significant for the learning and teaching of three-dimensional geometry because they combine the visual fidelity of physical models with the continuous transformability of dynamic diagrams (Dimmel, Pandisio, & Bock, 2021). How might interactable spatial diagrams create new opportunities for learning and teaching mathematics?

As an initial move toward investigating this question, the research team designed and developed *TriO*, a multiplayer, immersive virtual environment. *TriO* was designed to host coordination tasks, in which three players work together to navigate an object, which we refer to as the *widget*, through three-dimensional space. Each player controls one degree of freedom of the widget by sliding a virtual handle along the *x, y,* or *z* axes. Our work on *TriO* is part of a multi-year study that is investigating how secondary mathematics teachers imagine teaching with interactable spatial diagrams. We describe *TriO's* design rationale, illustrate its affordances through a multimodal vignette, and reflect on how *TriO* could create opportunities for student mathematical learning.

Theoretical Framework

The underlying theory of learning at the core of *TriO* is enactivism (Varela et al., 2017). Specifically, the movement-based design of *TriO* frames learning as an emergent state of equilibrium within dynamical systems of agents, tools, and environments, wherein motor-control problems are solved through gradual refinements of coordinated actions (Abrahamson, 2014). By *motor-control problem*, we mean a challenge that can be solved through particular coordinated actions. For example, in *TriO,* a task could be: "Move the widget along a diagonal path." It would require simultaneous movement from at least two players to complete this task—e.g., sliding the *x*-axis handle to the right at the same time that the *z*-axis handle is slid up. This is a motor-control problem in the sense that the players need to learn how to move together in order to accomplish the goal. Motor-control problems that can occasion the emergence of mathematical perception are a design feature of other interactive environments, such as the Mathematics Imagery Trainer for Proportion (MITp; Abrahamson & Trninic, 2015).[1] By *emergent mathematical perception*, we mean developing the capacity to apprehend the world—in the sense of Goodwin's professional vision (Goodwin, 2015) – in mathematically specific ways. Through the cultivation of mathematical perception as well as developing normative discourse about

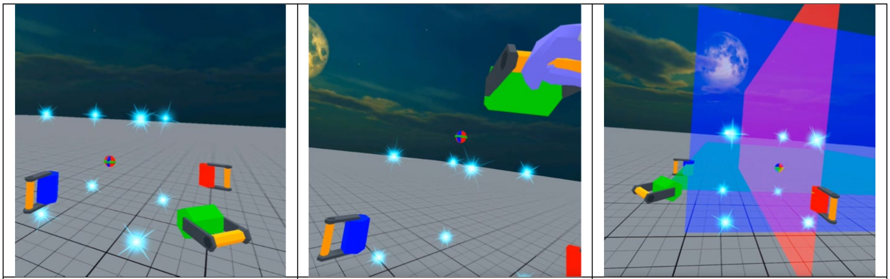
[1] In this design, a player coordinates the up/down movement of their hands to change the color of an interactive screen from red to green—the screen color changes to green, and then stays green, when the hands are moved at speeds that are in a particular goal ratio. If the hands begin on the desk and both move up at the goal ratio of, say, 1:2, then their respective heights above the desk will maintain at a 1:2 ratio.

the perception, using standard semiotic instruments, it becomes possible for learners to recognize how theoretical mathematical structures relate to and describe activities in the world. For example, the MITp is designed to train students to apprehend two contiguous or otherwise mentally associated consistent dynamics as instantiating a proportional transformation. With *TriO*, our goal was to design a collaborative, interactive environment that could motivate the emergence of a perceptual structure that we refer to as *coordinate perception* – i.e., the capacity to recognize that the continuous movement of any particle in three-dimensional space – however whirly or erratic or variable it may seem – can be resolved into the simultaneous movements of three one-dimensional components.

A key precept of the MITp and related designs, which we take up in our work with *TriO,* is that coordinated action is a pathway to mathematization. Or: As one learns how to move one's body to negotiate mathematics-inspired motor-control problems, the attention to, awareness of, and reflection on that movement can create opportunities for emergent mathematical learning (Abrahamson & Trninic, 2015). That learning is produced by cultivating skilled movement is evident in athletic or musical pursuits (Abrahamson & Sánchez-García, 2016). For example, a pitcher learns to throw a baseball with speed and accuracy by developing a kinetic chain that coordinates the movement of their feet, legs, hips, torso, shoulder, elbow, wrist, and fingers. With the embodied turn in cognitive science (Varela et al., 2017), there is an emerging body of research that shows how the cultivation of skilled movement is constitutive of the meanings we learn to express as mathematical concepts (Abrahamson et al., 2014). The design of *TriO* builds on these footings. We hypothesize that the social experience of students working together to achieve collective movement goals in *TriO* will motivate, and accelerate, student conceptual understanding of $\mathbf{R}^3$ as a coordinate system. We hypothesize, further, that the goal of navigating the widget through space will induce students to develop linguistic and, eventually, symbolic resources that will allow them to coordinate and account for their activity in *TriO*. Namely, as players attempt to coordinate enacting together the unfamiliar joint action of tri-axially maneuvering an object in space, we expect they will gradually determine and pace the vectorial distribution of their respective actions along each segment and yet that doing so will require spontaneously generating and socially negotiating a system of multimodal reference to emerging properties of the proximal instruments and their distal effects. That is, we are looking to foster the emergence of a 3D Cartesian discourse. These hypotheses are grounded in prior work with interactive environments that center parametrized movement.

Design Rationale

From classic games like *Labyrinth* and *Etch-A-Sketch* to mathematical installations such as *OЯTHO*(Potega vel Żabik et al., 2024) and *Drawing in Motion* (Nemirovsky et al., 2013), there is a rich history of parametrized designs that allow learners to coordinate horizontal (x-axis) and vertical (y-axis) movements. In these experiences, motor-control problems (e.g., *draw a circle*) can be directly linked to the mathematical idea of component-wise, two-dimensional movement. What would it look like



3a. First-person view of *TriO.* Each handle moves along an axis that controls the x (blue), y (red), or z (green) coordinate of a widget. The widget is formed by the intersection of blue, green, and red disks.	3b. The first author grabs the green handle to change the z-position of the widget.	3c. Guide planes can be toggled on or off to help learners track how their movements of a handle affect the position of the widget. The guide planes visually highlight the degrees of freedom controlled by each handle.

Fig. 3 First-person views of *TriO*

to create a three-dimensional iteration of such an experience? One possible solution to this problem is an experience we call *TriO*: an interactive, multiplayer experience that allows three players to collaboratively explore parametrized three-dimensional movement.

The initial idea for *TriO* was suggested by the second author, as a means to realize an *OЯTHO*-style coordination experience in three dimensions. As with *OЯTHO,* the rudimentary learning goal for *TriO* is the cultivation of *coordinate perception*, which is the capacity to link apprehensions of movement through three-dimensional space to component-wise movements along a set of orthogonal coordinate axes. The underlying hypothesis for the design is that, through coordinating their actions to solve motor-control problems, players will gradually learn to connect the component-based movements along the orthogonal axes to the location of a particle in three-dimensional space. The initial idea was collaboratively refined by the research team and then realized in an immersive virtual environment.[2] By immersive virtual environment, we mean a digitally rendered, three-dimensional world that is accessed via a head-mounted display. It is also an environment in the sense that it is an open, flexible, three-dimensional canvas, in which groups of immersed players can work together to solve a range of problems. Figure 3 shows first-person views of *TriO*.

TriO

To realize a three-dimensional parametrized movement experience, we designed a multiplayer, immersive virtual environment that consisted of (1) a series of three handles, (2) a set of points in *xyz* space (the electric blue starbursts), and a *widget* formed by the intersections of three disks (red, blue, and green; Fig. 4 shows a

[2] The third author was the lead developer of the immersive environment, using the *Unity* engine. *TriO* is compatible with a range of consumer virtual reality headsets, including the Quest (Meta) and Vive (HTC). Please email the corresponding author if you are interested in playing *TriO*.

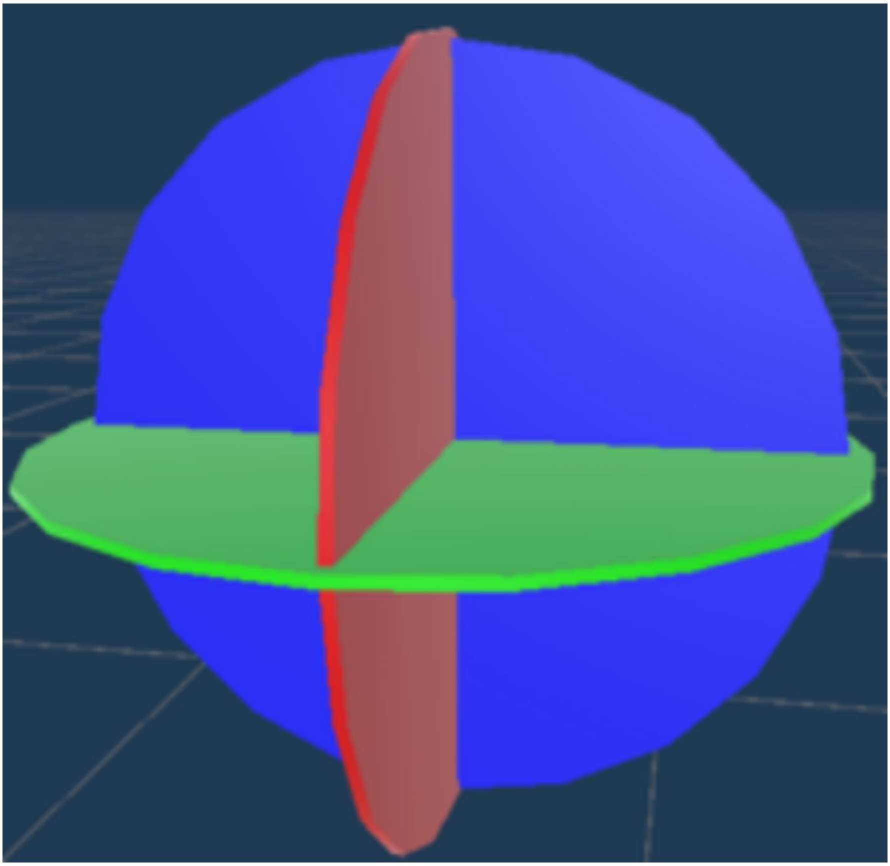


Fig. 4 Congruent red, green, and blue disks intersect orthogonally at one point

detail). Each disk is a subset of planes parallel to the *yz* (blue), *xy* (green), or *xz* (red) planes (Fig. 2c).

The widget was designed to be a visual cue that would help learners connect their movements of the handles to the effects of those movements on the position of the widget. The three-disk intersection also solved a design challenge: We wanted the widget to be large enough to be easily visible and also to be a specific point in *xyz* space. The three intersecting disk design is one solution to this set of constraints.

Each handle can be slid along either the *x, y,* or *z* axes (invisible in Fig. 3) by pushing or pulling actions. By moving these handles, players work together to move the widget in space. The position of each handle along its respective axis controls a corresponding degree of freedom of the widget. The handles can be moved simultaneously, thereby allowing for coordinated action on the widget—it is thus possible to move the widget along any continuous curvilinear path through *xyz* space. The goal of the experience is to move the widget into contact with each of the electric blue starbursts in the space (Fig. 3a–c). The eight points arranged at the vertices of a cube are the initial level of the experience. We chose this cubic configuration of vertices as the opening state in *TriO* because it afforded a variety of possibilities for one-player (edges), two-player (face diagonals), and three-player (interior diagonals) coordinated movement. We are planning to develop other levels, with other movement challenges, similar to the different tracks in *OЯTHO*. As an immersive, multiplayer experience, *TriO* can be played both locally (all players in a shared physical space) or distally (one or more players in different physical locations).

Play Testing

During a three-day workshop (August, 2024), we convened a group of 6 secondary mathematics teachers from four different schools in a rural northeastern state. The teachers participated in a series of focus groups (Liamputtong, 2011), during which they explored immersive virtual environments and reflected on their experiences in a group debrief. The goal of the focus group was to consider how immersive virtual environments could create new opportunities for mathematical learning. The teachers that attended the focus group were positioned as participant co-designers of the *TriO* immersive environment. In this way, *TriO* is neither a curriculum nor a specific activity, but rather an immersive learning platform within which it is possible to design tasks that would be expected to help middle or high school students develop their coordinate perception of *xyz* space. The teachers were crucial to helping us investigate how learning could emerge with such a platform.

We report here a vignette of a group of these teachers, engaging with *TriO* at their level, to highlight its affordances for staging compelling mathematical investigations. We recognize that the specific modes of engagement would be different for students. However, the task that the teachers created and explored is a valuable proof-of-concept that illustrates the possibilities for doing mathematical activities while immersed in *TriO*.

Data

The focus group sessions were audio and video recorded. There were two primary settings for data collection. The first setting was the *playspace,* a roughly 300-square-foot open space (Eklund, 2022). Each of the six teacher participants had a designated area within the playspace. The playspace was recorded from three camera angles. The second setting was the immersive environment participants explored by wearing head-mounted displays. Each participant's immersed, first-person view of the virtual environment was recorded. The different recordings were edited to create composite views of the playspace and the immersive environment from different, simultaneous perspectives (Fig. 5).

Figure 5 shows Mike, Clair, and Kendra wearing Meta Quest 3 head-mounted displays. The three participants are in their respective areas in the playspace. They are also simultaneously immersed in a shared virtual environment. The three images at the bottom of Fig. 4 show each participant's first-person, or headset, view of the

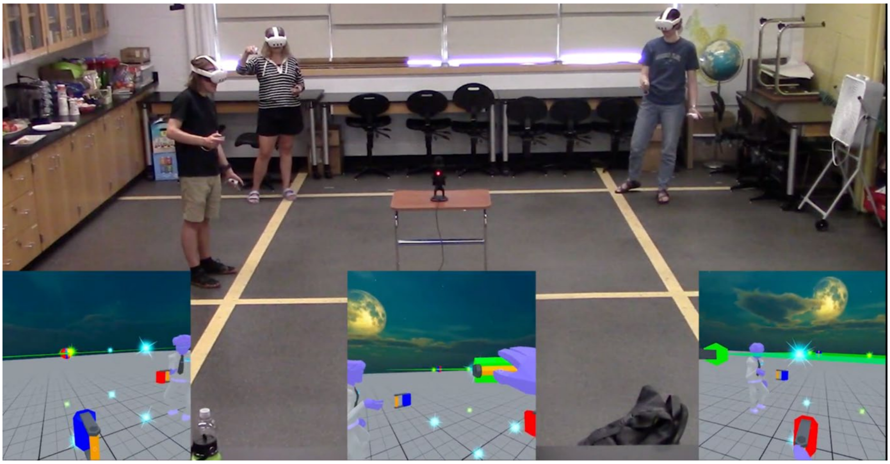


Fig. 5 Composite of the playspace and headset views of three secondary mathematics teachers exploring *TriO:* Mike (left), bottom-left view; Clair (middle), bottom-middle view; and Kendra (right), bottom-right view (pseudonyms)

immersive world. Mike's view is on the bottom left, with the blue handle in the foreground. Clair's view is in the bottom middle, with the green handle in the foreground. Kendra's view is on the bottom right, with the red handle in the foreground. Mike controls the widget's back/forth position (x-axis/blue); Clair controls its up/down position (z-axis/green); and Kendra controls its left/right position (y-axis/red). We use the composite view of the playspace plus the first-person, immersed views to underscore the collaborative co-creation of action in the virtual environment through coordinated movements in the physical world. In addition to the blue, green, and red handles, each player controlled a generic, blue-hued avatar, whose limbs moved roughly with participants' hand and arm movements.

Mike, Clair, and Kendra explored *TriO* during two different sessions. Below, we describe the group's work on a task that was suggested by Mike during the second session. His suggestion emerged organically as the teachers were considering what they could do in the environment that their students would recognize as mathematics. This episode was selected for multimodal transcription because it explored a task that was posed by a teacher participant who offered it specifically as an example of the mathematical activity that is possible in *TriO*. We present this episode through a multimodal summary of its key moments. Throughout the summary, we shift the perspective on the playspace to capture how the environment looked to each immersed participant.

The Diagonal Network Challenge

This episode occurred during the second instance that the teachers explored *TriO*. The teachers worked in two groups of three, and each group had its own playspace, to minimize crosstalk. At the start of this episode, there was an initial discussion about the environment and how it could be used by students to learn mathematics. This developed into a consideration of the opportunities for mathematical learning that are afforded by *TriO's* design. After a brief discussion, Mike suggested an activity for the group.

> Mike: Ok, ok, here's one then: Can we get all of the orbs lit…without traveling on an axis line from node to node, every single time. That is, we can't go, left/right, up/down…you can't just manipulate one at a time.
> Kendra: Oh! [excited] So, like we are only moving across the diagonals [Mike: Yes] to light them all up.
> Mike: Other…other constraint: You cannot go back to a node that you have already been to.
> Kendra: Ohhh!
> Clair: Ok, so where do we want to start, which one?

The group set to work on Mike's task. Kendra begins by pointing to an orb with her right hand controller. From the headset views, we can see that Kendra's avatar is pointing to a specific vertex in the network. The headset views show two different third-person views (Mike/left, Clair/center) and one first-person view (Kendra/right) (Fig. 6).

Clair indicates the starting location:

> Clair: It looks like we are going to start down here [squats down as she lowers the green handle to move the widget to the starting position; Figure 7].

The group considers their first move. Mike moves his right hand to indicate the vertices that are off-limits, given the constraint that they only move the widget along diagonal paths:

> Mike: We can't touch this one [points to upper, front, right vertex; Figure 8], that one [points to lower, front, left vertex], or that one [points to lower, back, right vertex].

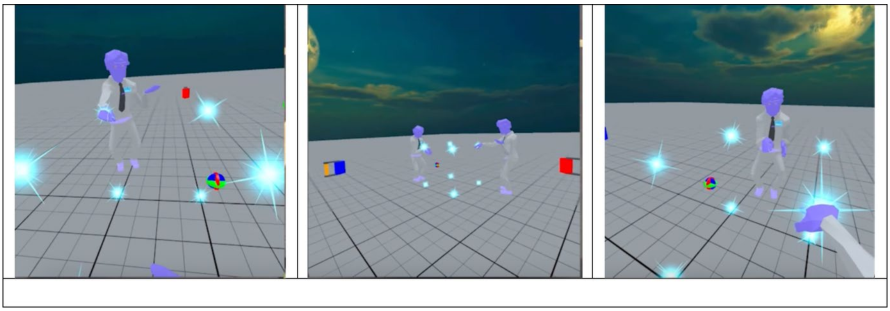


Fig. 6 Close-up views from each headset. In the left panel, Mike is looking across the network at Kendra's avatar. In the middle panel, Clair sees Mike on the left and Kendra on the right. In the right panel, Kendra points to a specific orb to indicate what she thinks their starting point should be as she looks across to Mike's avatar

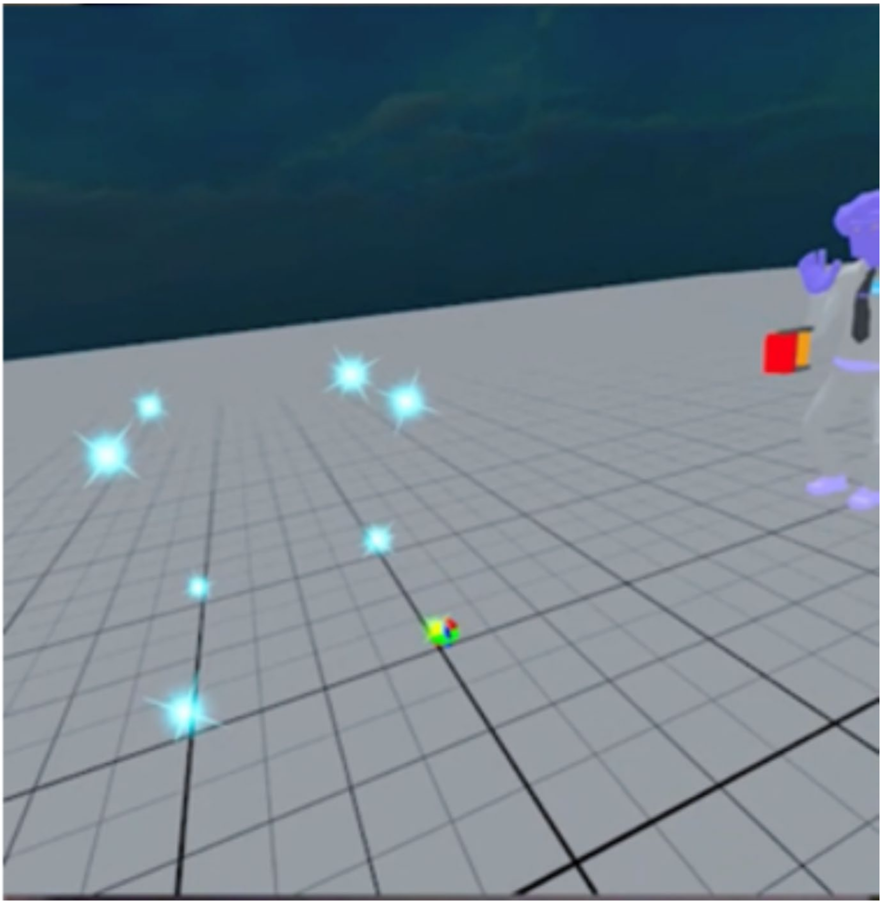


Fig. 7 The starting position for the diagonal network task—the widget is positioned at the lower, front, right vertex (from Mike's perspective)

For their first move, the group decides that they want to "go diagonally on the bottom" (Kendra), which meant that, "blue and red would move" (Mike), while green remained unchanged.[3] Mike and Kendra did not plan their movements ahead of time. Mike and Kendra each controlled the positions of their respective handles with their right hand. Mike moved his right hand across his body, to the left. Kendra moved her right hand away from her body, out farther to the right—these movements were responsive to their specific positions within *TriO*. Their coordinated movement took approximately 5 s. Mike and Kendra executed their respective movements at

[3] Given the constraint about multiplayer (i.e., diagonal) paths, there are four legal moves once the starting vertex has been selected: Three moves along the diagonals of each face that meets at that vertex (i.e., the bottom, front, and right faces of the cube, from Mike's perspective); one move along the great diagonal through the interior of the cube.

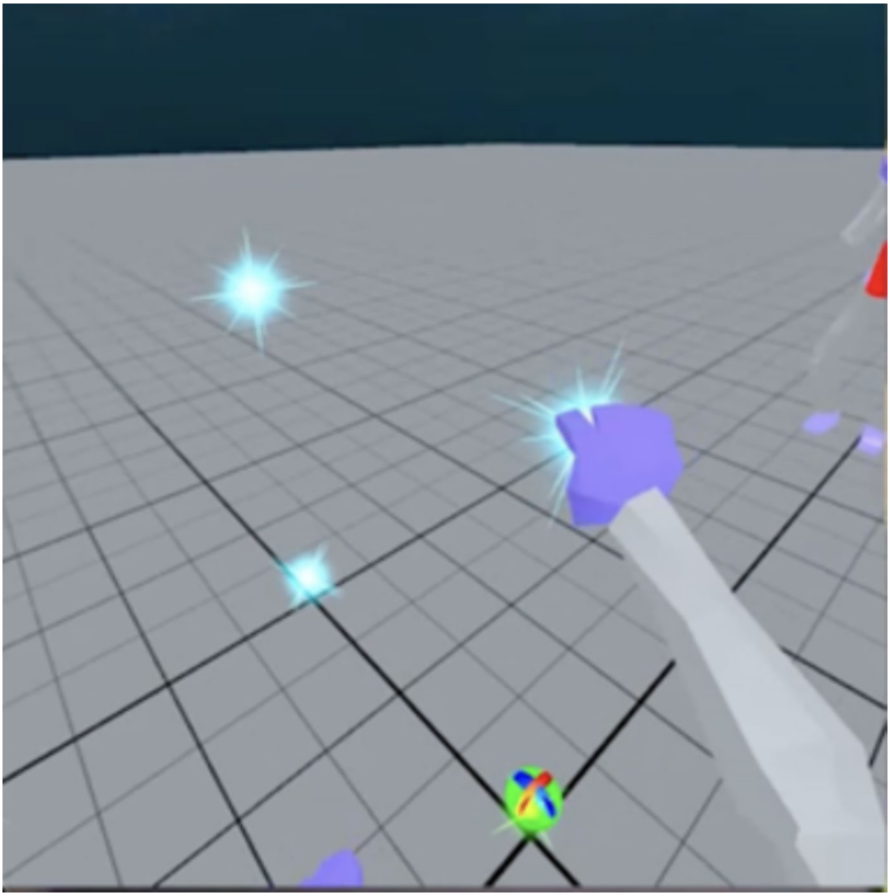


Fig. 8 Mike indicating one of the vertices they need to avoid for their first move

approximately the same speed, which kept the widget on their intended diagonal path. The group did not discuss how their independent rates of movement contributed to the overall movement of the widget, though this would be a natural place for a teacher to ask for a mathematical elaboration. Figures 9, 10, and 11 show still images from the coordinated movements of Mike and Kendra.

For the next move, Kendra suggested that, “Maybe, I could stay here, but blue and green can move up…to the one closest to where mine [points at herself] is.” Following this suggestion, Mike and Clair coordinated their movements to move the widget along a diagonal on the back face of the cube (from Mike’s perspective)—from the back lower left vertex to the back upper right vertex (Fig. 12).

Up to this point, the widget has been moved along two of the cube’s twelve exterior diagonals. The group paused at this juncture to consider their next move:

Kendra: Maybe—[Mike interrupts].
Mike: I think we should move…ahhhh.

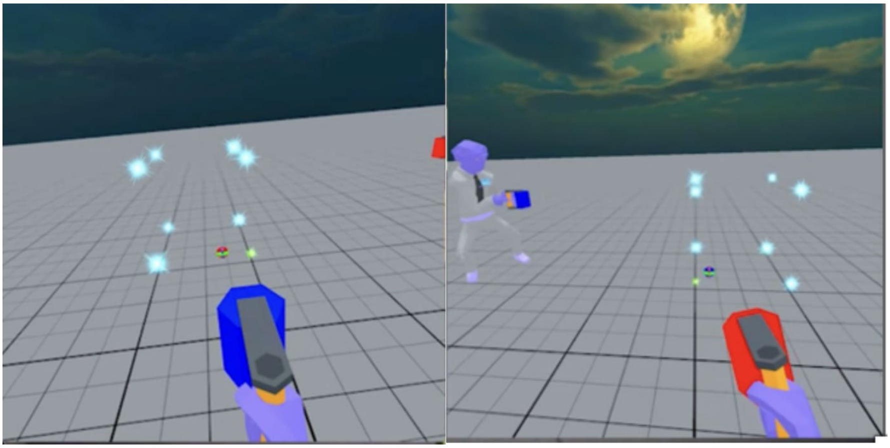


Fig. 9 Mike and Kendra coordinate their movements of the blue (Mike/left) and red (Kendra/right) handle to move the widget along a bottom diagonal of the cube

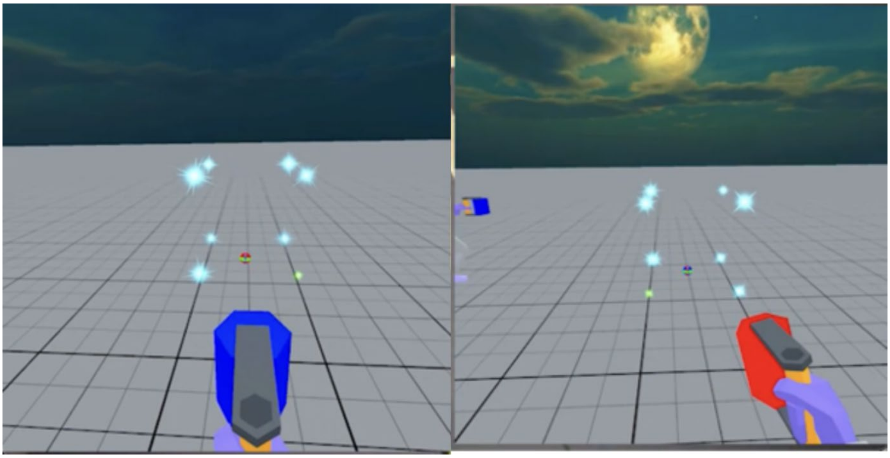


Fig. 10 The widget at its midway point along the diagonal

> Kendra: I'm just thinking, blue hasn't stayed in…the same yet…I don't know if that, logistically helps our goal, I'm just noticing that.
> Mike: Wait…wait…we have to be careful now, because we can lose this game. I just thought of a losing scenario.
> Clair: I think we need to go…
> Mike: We either have to go…We can't go diagonally across the middle yet.
> Kendra: Oh, I didn't even think about the diagonal across the middle.

Mike and Clair consider some of the remaining possible paths through the network that will allow them to win the game—i.e., they need to continue to move the widget along non-axial paths while avoiding any vertices they've already touched.

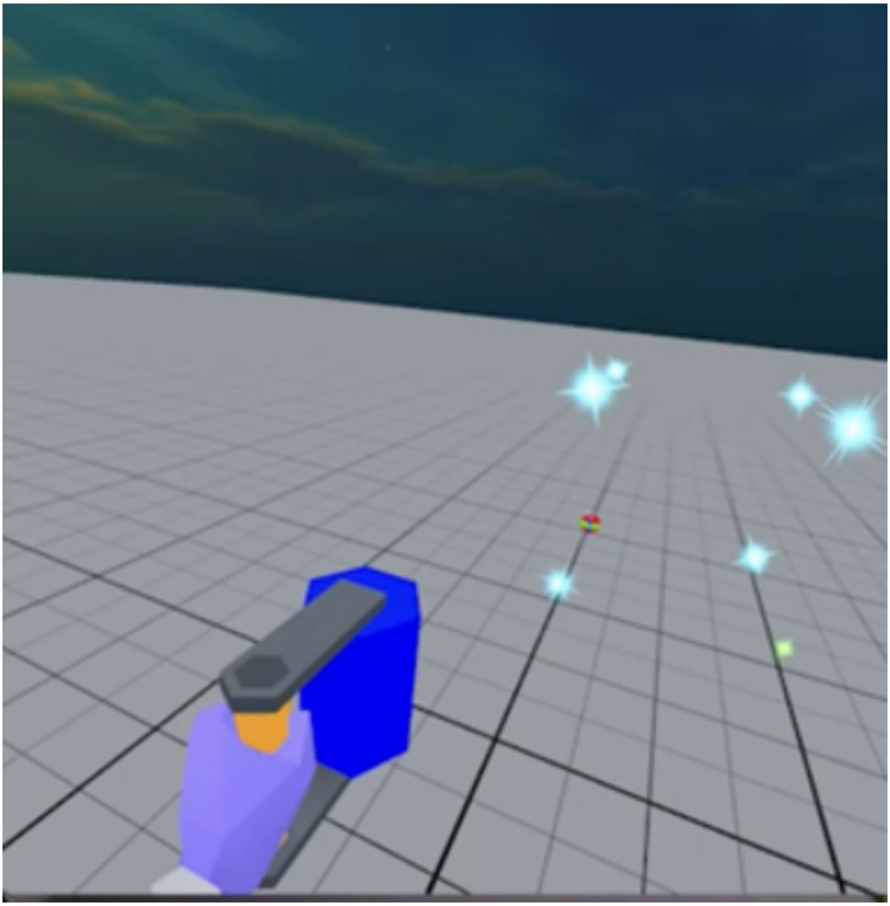


Fig. 11 The widget at the end of the first move, now coincident with the lower, back, left vertex (from Mike's perspective)

Mike realizes that it is possible for the group to lose the game, though he did not elaborate any specific losing scenario.[4] Clair and Mike negotiate these possibilities while gesturing at the points (Fig. 13).

Clair says, "I think I've got the path: doot, doot, doot, doot," where each "doot" coincided with a pointing gesture to indicate a vertex. Mike said, "Yep, those doots will do it." The group accepted Clair's gestured circuit through the remaining points and then began to implement her idea:

Kendra: Ok, so am I moving to the one—[Clair interrupts].

[4] Though Mike did not elaborate, a "losing scenario" would be moving to a vertex that is diagonally linked *only* to other vertices that have already been visited. This would not leave any valid paths to any remaining untouched vertices.

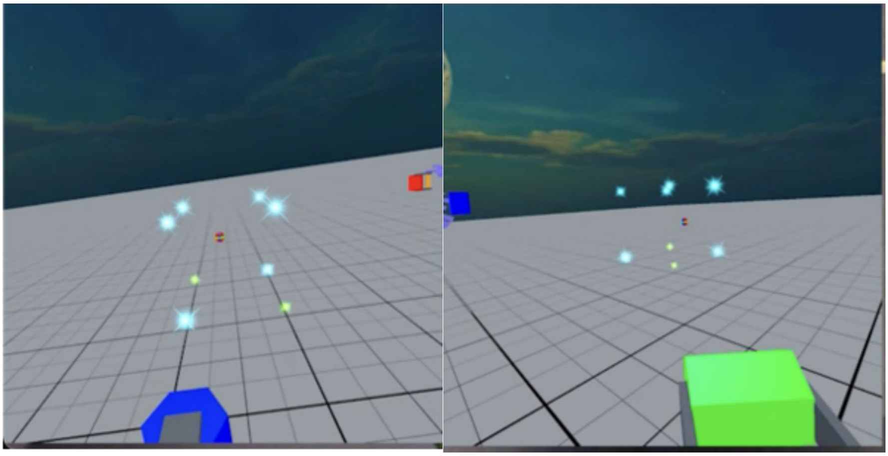


Fig. 12 Clair (right panel) raises the green handle at roughly the same speed that Mike (left panel) slides the blue handle to his right

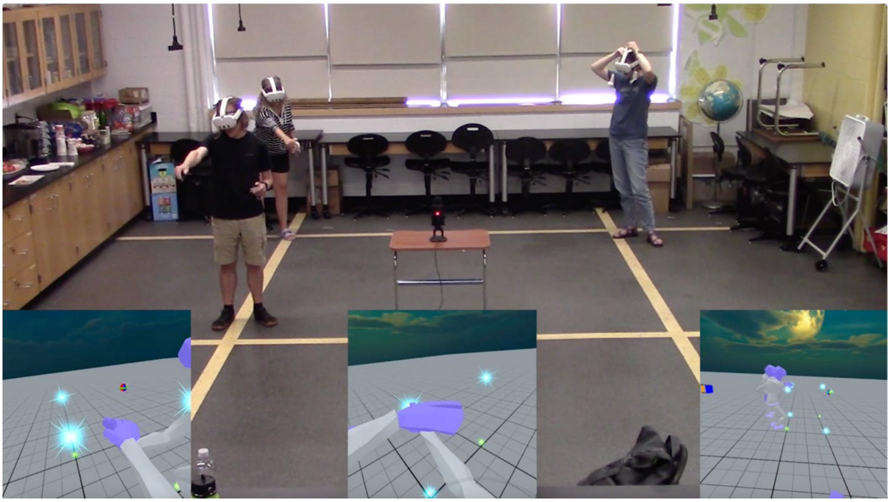


Fig. 13 Mike (left) and Clair (center) point to a possible next destination

Clair: I think it's you guys for this one.
Mike: Red and blue are moving across the top of the cube to the opposite corner [Mike and Kendra move together to move the widget along the top xy diagonal].
Kendra: And then we all three have to do a diagonal, all the way down.
Mike: We're incorporating a lot of specific language in this activity right now.
Researcher: What do you mean by that?
Mike: We have to be incredibly accurate in our speech so we can convey ideas without having to draw it out.

The next move is the first path where all three players are simultaneously coordinating their actions to move the widget, “diagonally through the middle of the cube,” in Mike’s words. Figure 14 shows a still of this maneuver at roughly the halfway point.

When the group completed this first tri-part coordinated movement, Clair exclaimed “ahhhhhh,” as if to mark a shared moment of triumph. The three teachers coordinated their movements without pre-planning, once again appearing to have a shared sense that they could move the widget along the intended diagonal path by moving each of the three components at the same rate. The group continued their way around the vertices of the cube, coordinating action in pairs to bring the widget into contact with each. Figure 15 shows the group on the verge of navigating the widget to the final vertex.

Reflections on Mathematics and Learning in *TriO*

Coordinate Perception

What did the teachers do in *TriO?* The diagonal network challenge created opportunities for Mike, Clair, and Kendra to work together to navigate the vertices of the cube using parametric movement. In doing so, the teachers developed language and gestures that allowed for clear communication in *TriO,* such as Mike posing the task in terms of “axis line[s]” and Kendra rephrasing this as “only moving across the diagonals.” They also had a shared sense that “diagonal movement” would entail not only simultaneous actions but also balanced actions—i.e., sliding the handles at the same rate. The teachers recognized the

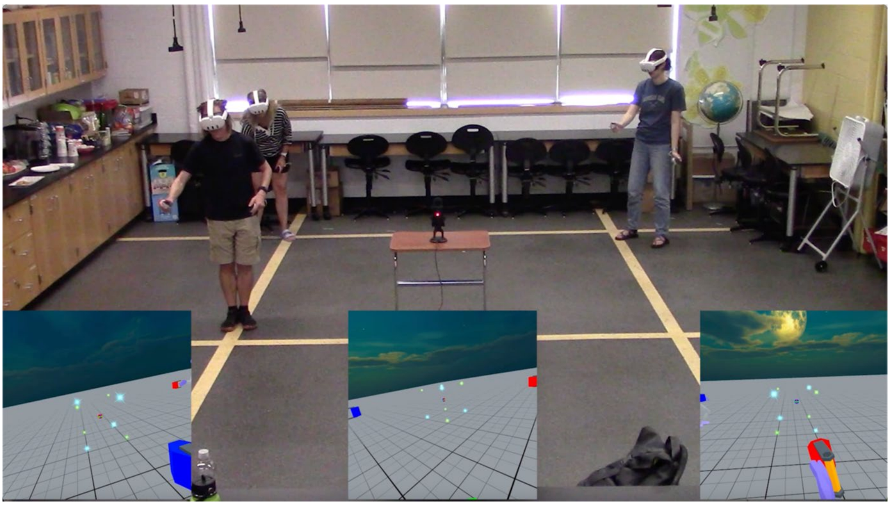


Fig. 14 Moving the widget “diagonally through the middle” with all three participants coordinating their movements

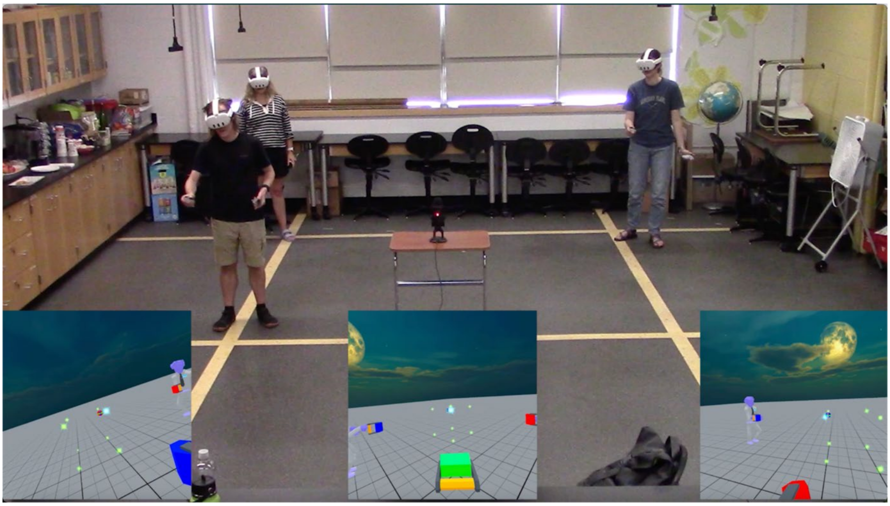


Fig. 15 Mike (left) and Kendra (right) move the widget back across the top *xy* diagonal

need for balanced movements without specifically discussing this; we believe this was because the teachers were already familiar with $\mathbf{R}^3$.

For students who are less familiar with three-dimensional Cartesian coordinates, we would expect coordinated-movement challenges in *TriO* would motivate the emergence of perceptual structures that would help the students grasp how their movements act on the widget. Namely, they will each perceive *distal* goal paths through *xyz* space in terms of anticipating their own respective *proximal* linear movement along the *x, y,* or *z* axes (Abrahamson & Bakker, 2016). By tuning task constraints to create greater demand for coordination, the emergence and practice of these perceptual structures could spur the development of linguistic and (eventually) symbolic semiotic resources that would allow the students to describe *TriO* mathematically. These emergent mathematical descriptions would, in turn, lend greater clarity and precision to their collective movement solutions. The continual development from movement to perceptual structures to linguistic and symbolic description has been observed in other movement-based designs for exploring mathematical relationships, such as the Mathematics Imagery Trainer for Proportion (Abrahamson & Trninic, 2015). In this way, we hypothesize that explorations in *TriO* will create opportunities for students to ground the prospective notions of $\mathbf{R}^3$, basis vectors, ordered tuples, and related mathematical concepts. As we continue our work with teachers as participant co-designers, we plan to curate a collection of teacher-initiated *TriO* tasks that would target specific ideas in three-dimensional mathematics. We plan to conduct a series of teaching experiments that will investigate how these tasks create opportunities for student learning.

Connections to Other Mathematical Ideas

The learning goal that motivated the design of *TriO* was to use coordinated, three-dimensional parametrized movement to cultivate coordinate perception of three-dimensional space. From this initial test with teachers, we remain optimistic that playing *TriO* could afford such opportunities for students. The experience with the teachers during the focus group also surfaced other possibilities. The diagonal network challenge highlighted, for the research team, how *TriO* connects with other areas of mathematics, such as Graph Theory. Graphs are abstract structures wherein a set of vertices is connected by a set of edges. Mike's statement of the "Diagonal Network Challenge" could be understood as a graph theory problem, where the graph is the eight vertices and its diagonal edges (twelve external, four internal). Mike's problem then becomes a problem of finding a Hamiltonian path through this graph—i.e., a path that touches each vertex in the graph exactly once. In *TriO,* this is a path for which coordinated movement by at least two players is necessary to complete each of its segments.

This task is a portal into a range of other tasks that could have graph-theoretic and also algebraic dimensions. For example, suppose we use {1, 2, 3} to index the degree of a curve that can be traced in *TriO*, where *degree* indicates the number of players that would need to simultaneously move to enact the curve. For paths that consist entirely of line segments, we could also consider the *rank* of a path, where rank is defined as the sum of the degrees of each of its segments. Our use of *degree* and *rank* to mathematize the varieties of paths that can be generated through coordinated movement represents one possibility for how mathematical analysis could emerge from group activity in *TriO*.

The teachers' work on the diagonal network challenge and our own ongoing reflections on the mathematical affordances of *TriO* lead us to believe that we are just scratching the surface for how an open, collaborative, movement-based interactable spatial diagram could create opportunities for learning and exploration. The representational infrastructure for spatial geometry is shifting, as immersive, movement-based virtual environments are moving out of research labs and into mathematics classrooms. We offer *TriO* as an example of the kinds of mathematical activities that are possible on this new frontier. In turn, witnessing our participant teachers explore possibilities to implement in *TriO* activities that foster rudimentary coordinate perception as well as activities that explore advanced subject matter suggests that this environment could host a "lifelong" span of geometry learning. Partnering with teachers enables us to trace these potential curricular trajectories with an eye on mandated standards and perhaps beyond. Namely, as design-based researchers, our purview includes vigilant evaluation of whether these standards, which express prevalent beliefs about *what* students should know and *when* they might be able to learn it, may be inadvertently under-ambitious, because beliefs about students' learning capacities are unwittingly constrained by available media wherein learning is implicitly expected to take place. Consider the historical transition from Roman to Hindu–Arabic numerals, which radically transformed human arithmetic horizons, so that the challenging feat of multiplication became within the reach of elementary school students, not only post-graduates. Analogously, new forms of digital technological media enable designers to "restructurate" traditional concepts (Wilensky & Papert, 2010), reconstituting familiar mathematical phenomena in the form of more cognitively ergonomic semiotic modes. Whereas immersive VR is a relatively young playing field for mathematics educators, we tentatively hazard that *TriO* might transform students' 3D geometric horizons, rendering Hamiltonian curves accessible to elementary school students.

Author Contribution J.D. and D.A. wrote the main manuscript text; M.P. developed the immersive virtual environment and worked with J.D. to prepare the figures. All authors reviewed the manuscript.

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Data Availability No datasets were generated or analysed during the current study.

Virtual environment availability Please contact the corresponding author if you would like to explore the immersive virtual environment that is the setting for the activities reported in this manuscript. The immersive virtual environment is compatible with Meta Quest 2/3 headsets and the HTC Vive and Vive Pro.

Declarations

Competing interests The authors declare no competing interests.

References

Abrahamson, D. (2014). Rethinking probability education: Perceptual judgment as epistemic resource. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 75–92). Springer. https://edrl.berkeley.edu/publications/abrahamson-d-2014-rethinking-probability-education-perceptual-judgment-as-epistemic-resource/. Accessed 3 Mar 2025.

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. In N. Newcombe & S. Weisberg (Eds.), Embodied cognition and STEM learning [Special issue]. *Cognitive Research: Principles and Implications, 1*(1), 1–13. https://doi.org/10.1186/s41235-016-0034-3

Abrahamson, D., Lee, R. G., Negrete, A. G., & Gutiérrez, J. F. (2014). Coordinating visualizations of polysemous action: Values added for grounding proportion. *ZDM, 46*, 79–93. https://doi.org/10.1007/s11858-013-0521-7

Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *The Journal of the Learning Sciences, 25*(2), 203–239. https://doi.org/10.1080/10508406.2016.1143370

Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: Signifying sensorimotor enactment in fields of promoted action. *ZDM Mathematics Education, 47*, 295–306. https://doi.org/10.1007/s11858-014-0620-0

Bakó, M. (2003). Different projecting methods in teaching spatial geometry. In M. A. Mariotti (Ed.), *Proceedings of the Third Conference of the European Society for Research in Mathematics Education*. Edizione Plus, Pisa University Press.

Barrett, J. E., & Clements, D. H. (2003). Quantifying path length: Fourth-grade children's developing abstractions for linear measurement. *Cognition and Instruction, 21*(4), 475–520. https://doi.org/10.1207/s1532690xci2104_4

Blanz, V., Tarr, M. J., & Bülthoff, H. H. (1999). What object attributes determine canonical views? *Perception, 28*(5), 575–599. https://doi.org/10.1068/p2897
Dimmel, J.K., and Bock, C.G. (2019). Dynamic mathematical figures with immersive spatial displays: The case of HandWaver. In: G. Aldon and J. Trgalová (Eds.), *Digital Technology to Teach, Learn and Assess Mathematics: Featuring extended selected papers of ICTMT* 13 (pp. 99–122). Berlin: Springer. https://doi.org/10.1007/978-3-030-19741-4_5
Dimmel, J. K., & Milewski, A. M. (2019). Scale, perspective, and natural mathematical questions. *For the Learning of Mathematics,39*(3), 34–40.
Dimmel, J. K., Pandiscio, E. A., & Bock, C. G. (2021). The geometry of movement: Encounters with spatial inscriptions for making and exploring mathematical figures. *Digital Experiences in Mathematics Education,7*(1), 122–148.
Edelman, S., & Bülthoff, H. H. (1992). Orientation dependence in the recognition of familiar and novel views of three-dimensional objects. *Vision Research, 32*(12), 2385–2400.
Eklund, V. (2022). Maximizing the VR play space by using procedurally generated impossible spaces: Research on VR play spaces and their impact on game development. [Master's Thesis, Blekinge Institute of Technology]. DiVa: https://www.diva-portal.org/smash/get/diva2:1664805/FULLTEXT01.pdf. Accessed 3 Mar 2025.
Fujita, T., Kondo, Y., Kumakura, H., Kunimune, S., & Jones, K. (2020). Spatial reasoning skills about 2D representations of 3D geometrical shapes in grades 4 to 9. *Mathematics Education Research Journal, 32*, 235–255. https://doi.org/10.1007/s13394-020-00335-w
Goodwin, C. (2015). Professional vision. In *Aufmerksamkeit: Geschichte-theorie-empirie* (pp. 387–425). Springer Fachmedien Wiesbaden.
Kaufmann, H., & Schmalstieg, D. (2002). Mathematics and geometry education with collaborative augmented reality. In *ACM SIGGRAPH 2002 Conference Abstracts and Applications* (pp. 37–41).
Liamputtong, P. (2011). *Focus group methodology: Principle and practice*. SAGE Publications.
Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372–415. https://doi.org/10.5951/jresematheduc.44.2.0372
Palatnik, A., & Abrahamson, D. (2022). Escape from Plato's cave: An enactivist argument for learning 3D geometry by constructing tangible models. *Twelfth Congress of the European Society for Research in Mathematics Education* (CERME12), Bozen-Bolzano, Italy. ffhal-03751500f.
Panorkou, N., & Pratt, D. (2016). Using Google SketchUp to develop students' experiences of dimension in geometry. *Digital Experiences in Mathematics Education,2*, 199–227. https://doi.org/10.1007/s40751-016-0021-9
Parzysz, B. (1988). "Knowing" vs. "seeing": Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics,19*(1), 79–92. https://doi.org/10.1007/BF00428386
Potega vel Żabik, K., Abrahamson, D., & Iłowiecka-Tańska, I. (2024). It takes two to OЯTHO: A tabletop action-based embodied design for the Cartesian system. *Digital Experiences in Mathematics Education, 10*(2), 189–201. https://doi.org/10.1007/s40751-024-00139-8
Sarama, J., Clements, D. H., Swaminathan, S., McMillen, S., & González Gómez, R. M. (2003). Development of mathematical concepts of two-dimensional space in grid environments: An exploratory study. *Cognition and Instruction, 21*(3), 285–324. https://doi.org/10.1207/S1532690XCI2103_03
Trigueros, M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two-variable functions. *Educational Studies in Mathematics, 73*, 3–19.
Varela, F. J., Thompson, E., & Rosch, E. (2017). *The embodied mind, revised edition: Cognitive science and human experience*. MIT press.
Walkington, C., Nathan, M. J., Washington, J., Hunnicutt, J., Darwin, T., Daughrity, L., & Schenck, K. (2024). Comparing learning geometry using a tablet to head-mounted display augmented reality: How and when dimensionality matters. *Education and Information Technologies*, 1–30.
Wilensky, U., & Papert, S. (2010). Restructurations: Reformulations of knowledge disciplines through new representational forms. *Constructionism, 17*(2010), 1–15.